Math 230, Exam 1 Name: ____________________

February 6, 2017

Instructions.

- You are allowed one side of handwritten notes
- No calculators.
- It is fine to leave your answers unsimplified.
- There are 7 problems on 4 pages. Make sure your exam is complete.

Page	Points	Score
1	14	
2	12	
3	12	
4	12	
Total:	50	

1. Roll two dice. Let X be the sum.

[3 points] (a) Let $p_k = P(X = k)$. The possible dice rolls are given. Fill in the table for p_k.

1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

p_2	
p_3	
p_4	
p_5	
p_6	
p_7	
p_8	
p_9	
p_{10}	
p_{11}	
p_{12}	

[4 points] (b) Let $A = \{X \text{ is even}\}$ and $B = \{X \text{ is divisible by } 3\}$. Are A and B independent? Justify.

Solution:
$P(A) = (1/36)(1+3+5+5+3+1) = \frac{18}{36} = \frac{1}{2}$, and $P(B) = (1/36)(2+5+4+1)) = \frac{12}{36} = \frac{1}{3}$.
We see that $P(A \cap B) = P(X \in \{6, 12\}) = (1/36)(5+1) = \frac{1}{6} = P(A)P(B)$.

[4 points] (c) Let $C = \{X \leq 3\}$ and $D = \{X = 3\}$. Are C and D independent? Justify.

Solution: We have $P(C \cap D) = P(D) \neq P(C)P(D)$. So they are dependent.

[3 points] 2. Suppose that 6 people are each assigned a whole number $1 \leq x \leq 40$. Each number is equally likely to be assigned. What is the probability that at least two people are assigned the same number? (You can leave your answer unsimplified.)

Solution: $1 - \frac{40 \cdot 39 \cdot 38 \cdot 37 \cdot 36 \cdot 35}{40^6}$.

[4 points] 3. Two independent events have probabilities $\frac{1}{4}$ and $\frac{1}{3}$. What is the probability *exactly one* occurs? Hint: The answer is *not* $1/12$.

Solution: Call this first event A and the second B. There are two ways to do this. One is to observe that $(A \cup B) - (A \cap B)$ is the event we want. Since these are disjoint we have

$$P((A \cup B) - (A \cap B)) = P(A \cup B) - P(A \cap B) = P(A) + P(B) - P(A \cap B) - P(A \cap B).$$

Plug in the relevant values to get $\frac{1}{4} + \frac{1}{3} - 2\frac{1}{12} = 5/12$.

Here's another solution: We are interested in

$$P((A \cap B^c) \cup (B \cap A^c)).$$

These are disjoint events. So we can sum their two probabilities:

$$P(A \cap B^c) = \frac{1}{4}\frac{2}{3} = \frac{2}{12},$$

$$P(B \cap A^c) = \frac{1}{3}\frac{3}{4} = \frac{3}{12}.$$

This gives

$$\frac{2}{12} + \frac{3}{12} = \frac{5}{12}.$$

4. In the game Dreidel a 4-sided top is spun. The four equally likely outcomes are:

$$\{\text{Do nothing, Lose \$1, Gain \$2, Gain \$4}\}.$$

Let X be the amount of money exchanged after one spin.

[3 points] (a) What is EX?

Solution:

$$EX = \frac{1}{4}2 + \frac{1}{4}4 - \frac{1}{4}1 = \frac{6}{4} - \frac{1}{4} = \frac{5}{4}.$$

[5 points] (b) What is $\text{var}(X)$?

Solution:

$$EX^2 = \frac{1}{4}2^2 + \frac{1}{4}4^2 + \frac{1}{4}1 = 1 + 4 + 1/4 = 21/4$$

So

$$\text{var}(X) = (21/4) - (5/4)^2 = \frac{59}{16}.$$

5. Suppose Rachel flips a fair coin until *either* she gets a head, *or* she has flipped the coin 3 times.

[3 points] (a) Write all outcomes that have positive probability, and write their probability.

> **Solution:** The possible outcomes are H, TH, TTH, and TTT. These occur with probability 1/2, 1/4, 1/8, and 1/8 respectively.

[3 points] (b) What is the expected number of heads she sees?

> **Solution:** Summing the number of tails in each times the probability gives
>
> $$\text{expected \# of tails} = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{1}{8} = \frac{7}{8}.$$

6. You flip a fair coin and win \$1 if it is heads, or lose \$1 if it is tails. Let $X_1, X_2, \ldots, X_n$ be random variables with $X_i = 1$ if heads is flipped and $X_i = -1$ if tails is flipped.

$$\text{Set } Y = \sum_{i=1}^n X_i \qquad \text{and} \qquad Z = nX_1.$$

[3 points] (a) What are $\text{var}(Y)$ and $\text{var}(Z)$? You may use the fact that $\text{var}(X_i) = 1$ for $1 \leq i \leq n$.

> **Solution:** $\text{var}(nX_1) = n^2 \text{var}(X_1) = n^2$ and $\text{var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \text{var}(X_i) = n$.

[3 points] (b) Explain why the two variances you calculated are different. Give some intuition. Don't just say "because the calculation says so."

> **Solution:** $1000X_1$ only takes the extreme values of 1000 or -1000. The random variable $\sum_{i=1}^n X_i$ spreads more uniormly across values in $[-1000, 1000]$, so ought to have less volatility.

7. Ash is playing Pokémon Go. There are 151 different pokémon. In a special area all pokémon are equally likely to be encountered with the rule that: **the same pokémon is never encountered twice in a row**. For example, if Pikachu was just encountered, then Pikachu will not appear in your next encounter. Instead, you are equally likely to meet any of the other **150 pokémon**.

[2 points] (a) Suppose Ash has encountered i distinct pokémon, and Y_i is the number of pokémon she encounters until she has seen $i+1$ distinct pokémon. Explain why Y_i is a geometric random variable.

> **Solution:** Each box opening is an independent trial with the same probability of successfully finding a new coupon.

[3 points] (b) What is the parameter for Y_0 and Y_i for $1 \leq i \leq 150$? (Hint: $Y_i \neq X_i$ from MAPs)

Y_0 : Y_i :

> **Solution:** 1, and $\frac{151-i}{150}$

[2 points] (c) Let N be the (random) number of encounters needed for Ash to go from seeing 0 to all 151 pokémon. Write a formula for N in terms of the Y_i.

> **Solution:** $N = X_0 + X_1 + \cdots + X_{151}$.

[2 points] (d) What is the expected value of N? It is okay to leave your answer as a sum.

> **Solution:** $1 + \sum_{i=1}^{n} \frac{150}{151-i}$.

[3 points] (e) Suppose that the "never twice in a row" rule isn't in place. So, each encounter is equally likely to be any of the 151 pokémon. Let M be the number of encounters needed to see all 151 pokémon.

Without doing any calculations, explain which ought to be larger between EM and EN.

> **Solution:** $EM > EN$. This is because we are more likely to have repeats when we have a chance of encountering the pokémon last seen.