# PROPERTIES OF TRIANGLES



**Q01.** In a triangle ABC, if a = 18, b = 24 and c = 30, find cos A, cos B and cos C.

**Sol.** We have a = 18, b = 24 and c = 30

$\because \cos A = \frac{b^2+c^2-a^2}{2bc} \Rightarrow \cos A = \frac{24^2+30^2-18^2}{2\times 24\times 30} = \frac{4}{5}$

Similarly, $\cos B = \frac{3}{5}$ and $\cos C = 0$.

**Q02.** In a $\Delta ABC$, if a = 18, b = 24 and c = 30, find sin A, sin B and sin C.

**Sol.** We have a = 18, b = 24 and c = 30

$\because \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow \frac{\sin A}{18} = \frac{\sin B}{24} = \frac{\sin C}{30} = k$ (say)

$\Rightarrow \sin A = 18k,\ \sin B = 24k$ and $\sin C = 30k$.

Since $30^2 = 18^2 + 24^2$ that means $\Delta ABC$ is a right angled triangle such that $\angle C = 90^o$, which is the angle opposite to the biggest side c.

$\therefore\ \sin C = \sin 90^o = 30k \Rightarrow 1 = 30k \Rightarrow k = \frac{1}{30}$

So, $\sin A = 18\times\frac{1}{30} = \frac{3}{5},\ \sin B = 24\times\frac{1}{30} = \frac{4}{5}$.

**Q03.** For any triangle ABC, prove that $\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$.

**Sol.** See Q17 (e).

**Q04.** For any triangle ABC, prove that $\frac{a-b}{c} = \frac{\sin\left(\frac{A-B}{2}\right)}{\cos\frac{C}{2}}$.

**Sol.** LHS : $\frac{a-b}{c} = \frac{k\sin A - k\sin B}{k\sin C}$ [By Sine rule

$\Rightarrow \quad = \frac{\sin A - \sin B}{\sin C} = \frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{\sin C}$

$\Rightarrow \quad = \frac{2\cos\left(90^o - \frac{C}{2}\right)\sin\frac{A-B}{2}}{2\sin\frac{C}{2}\cos\frac{C}{2}} = \frac{\sin\frac{C}{2}\sin\frac{A-B}{2}}{\sin\frac{C}{2}\cos\frac{C}{2}}$ [$\because A+B+C = 180^o$

$\Rightarrow \quad = \frac{\sin\left(\frac{A-B}{2}\right)}{\cos\frac{C}{2}} = \text{RHS}$.

**Q05.** For any triangle ABC, prove that $\sin\left(\frac{B-C}{2}\right) = \frac{b-c}{a}\cos\frac{A}{2}$.

**Sol.** See Q17 (h).

**Q06.** For any triangle ABC, prove that $a\left(b\cos C-c\cos B\right)=b^2-c^2$.

**Sol.** See Q17 (o).

**Q07.** For any triangle ABC, prove that $a\left(\cos C-\cos B\right)=2\left(b-c\right)\cos^2\frac{A}{2}$.

**Sol.** See Q17 (j).

**Q08.** For any triangle ABC, prove that $\frac{b^2-c^2}{a^2}=\frac{\sin\left(B-C\right)}{\sin\left(B+C\right)}$.

**Sol.** See Q17 (f).

**Q09.** For any triangle ABC, prove that $(b+c)\cos\frac{B+C}{2}=a\cos\frac{B-C}{2}$.

**Sol.** LHS : $(b+c)\cos\frac{B+C}{2}=a\cos\frac{B-C}{2}\Rightarrow(b+c)\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)=a\cos\frac{B-C}{2}$ [$\because A+B+C=\pi$

$\Rightarrow a\cos\left(\frac{B-C}{2}\right)=\left(b+c\right)\sin\frac{A}{2}$. Now see Q17 (k).

**Q10.** For any triangle ABC, prove that $a\cos A+b\cos B+c\cos C=2a\sin B\sin C$.

**Sol.** LHS : $a\cos A+b\cos B+c\cos C$

$=k\sin A\cos A+k\sin B\cos B+k\sin C\cos C$ [By Sine rule

$=\frac{k}{2}\left(2\sin A\cos A+2\sin B\cos B+2\sin C\cos C\right)$

$=\frac{k}{2}\left(\sin 2A+\sin 2B+\sin 2C\right)$

$=\frac{k}{2}\left(2\sin(A+B)\cos(A-B)+2\sin C\cos C\right)$

$=k\left(\sin(\pi-C)\cos(A-B)+\sin C\cos C\right)$

$=k\sin C\left(\cos(A-B)+\cos[\pi-(A+B)]\right)$

$=k\sin C\left(\cos(A-B)-\cos(A+B)\right)$

$=k\sin C\left(-2\sin A\sin(-B)\right)$

$=2\left(k\sin A\right)\sin B\sin C$

$=2a\sin B\sin C=\text{RHS}$.

**Q11.** For any triangle ABC, prove that $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=\frac{a^2+b^2+c^2}{2abc}$.

**Sol.** See Q17 (p).

**Q12.** For any triangle ABC, prove that $\left(b^2-c^2\right)\cot A+\left(c^2-a^2\right)\cot B+\left(a^2-b^2\right)\cot C=0$.

**Sol.** See Q17 (v).

**Q13.** For any triangle ABC, prove that $\frac{b^2-c^2}{a^2}\sin 2A+\frac{c^2-a^2}{b^2}\sin 2B+\frac{a^2-b^2}{c^2}\sin 2C=0$.

**Sol.** See Q17 (s).

**Q14.** A tree stands vertically on a hill side which makes an angle of $15^\circ$ with the horizontal. From a point on the ground 35m down hill from the base of the tree, the angle of elevation of the top of the tree is $60^\circ$. Find the height of the tree.

**Sol.** Let PQ = h (in metres) be the tree on the hill QR.
In triangle AQR ,

$\frac{QR}{AQ}=\sin 15^\circ\Rightarrow QR=35\sin 15^\circ$

and, $AR=AQ\cos 15^\circ=35\cos 15^\circ$.

In triangle APR,

$\frac{PR}{AR} = \tan 60^o \Rightarrow PR = \sqrt{3}AR$

$\Rightarrow PQ + QR = \sqrt{3}\left(35\cos 15^o\right)$

$\Rightarrow h + 35\sin 15^o = \sqrt{3}\left(35\cos 15^o\right)$

$\Rightarrow h = 35\left(\sqrt{3}\cos 15^o - \sin 15^o\right)$

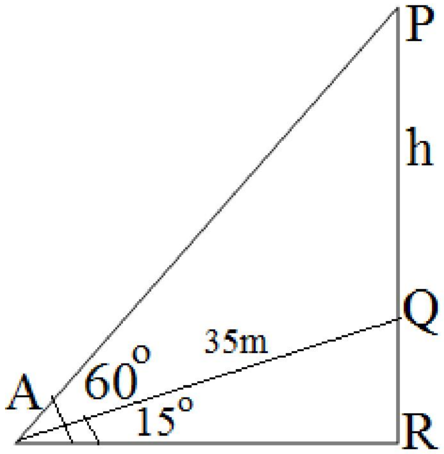
$\Rightarrow h = 35\left(\sqrt{3}\cos[45^o - 30^o] - \sin[45^o - 30^o]\right)$

$\Rightarrow h = 35\left(\sqrt{3}\times\frac{\sqrt{3}+1}{2\sqrt{2}} - \frac{\sqrt{3}-1}{2\sqrt{2}}\right)$

$\Rightarrow h = 35\left(\frac{3+\sqrt{3}-\sqrt{3}+1}{2\sqrt{2}}\right)$

$\Rightarrow h = 35\sqrt{2}$.

Hence the height of the tree is $35\sqrt{2}$m.




**Q15.** Two ships leave a port at the same time. One goes 24kmph in the direction N$45^o$ E and other goes 32kmph in the direction S$75^o$ E. Find the distance between the ships at the end of 3hours.

**Sol.** Let two ships A and B start from the port O with speed 24kmph and 32kmph in the direction of OA and OB respectively.

When OA = N$45^o$ E and OB = S$75^o$ E.

$\angle AOB = \angle AOE + \angle EOB = 45^o + 15^o = 60^o$.

The distance raveled by ship A in 3 hours

$\Rightarrow \quad = 3\times 24 = 72\text{km} = OA$

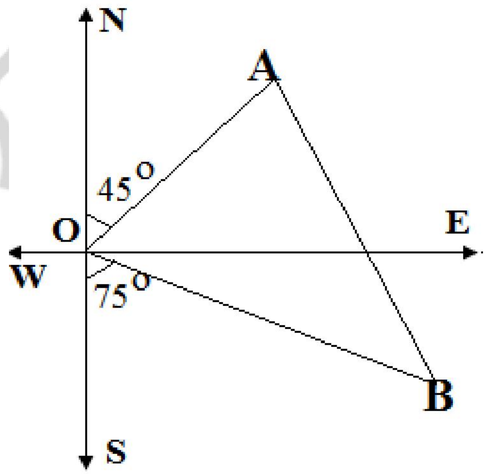
The distance raveled by ship B in 3 hours

$= 3\times 32 = 96\text{km} = OB$

In $\Delta OAB$, $\cos O = \frac{OA^2 + OB^2 - AB^2}{2.OA.OB}$

$\Rightarrow \cos 60^o = \frac{72^2 + 96^2 - AB^2}{2\times 72\times 96}$

$\Rightarrow AB = 86.5\text{km}$.




**Q16.** Two trees A and B are on the same side of a river. From a point C in the river, the distance between the trees A and B is 250m and 300m respectively. If the angle C is $45^o$, find the distance between the trees. [Use $\sqrt{2} = 1.414$].

**Sol.** As shown in the figure, A and B are the trees and C is the point in the river.

So AC = 250m, BC = 300m and $\angle C = 45^o$.

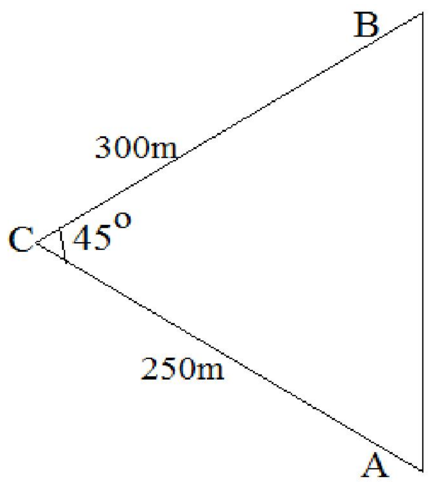
In $\Delta ABC$, $\cos C = \frac{BC^2 + CA^2 - AB^2}{2.BC.CA}$

$\Rightarrow \cos 45^o = \frac{300^2 + 250^2 - c^2}{2\times 300\times 250}$

$\Rightarrow 300^2 + 250^2 - c^2 = \frac{2\times 300\times 250}{\sqrt{2}}$

$\Rightarrow c = 215.49\text{m}$

$\therefore$ The distance between the two tress is 215.5m (Approx.)



**Q17.** In any $\Delta ABC$, prove that :

(a) $\dfrac{a^2-c^2}{b^2}=\dfrac{\sin(A-C)}{\sin(A+C)}$

(b) $b\cos B+c\cos C=a\cos(B-C)$

(c) $a\sin A-b\sin B=c\sin(A-B)$

(d) $\dfrac{a^2+b^2}{a^2+c^2}=\dfrac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}$

(e) $\dfrac{a+b}{c}=\dfrac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$

(f) $\dfrac{b^2-c^2}{a^2}=\dfrac{\sin(B-C)}{\sin(B+C)}$

(g) $\dfrac{a-b}{a+b}=\dfrac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}$

(h) $\sin\left(\dfrac{B-C}{2}\right)=\dfrac{b-c}{a}\cos\dfrac{A}{2}$

(i) $\dfrac{c}{a-b}=\dfrac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\tan\frac{A}{2}-\tan\frac{B}{2}}$

(j) $a(\cos C-\cos B)=2(b-c)\cos^2\dfrac{A}{2}$

(k) $a\cos\left(\dfrac{B-C}{2}\right)=(b+c)\sin\dfrac{A}{2}$

(l) $2\left(b\cos^2\dfrac{C}{2}+c\cos^2\dfrac{B}{2}\right)=a+b+c$

(m) $(a-b)^2\cos^2\dfrac{C}{2}+(a+b)^2\sin^2\dfrac{C}{2}=c^2$

(n) $\dfrac{c-b\cos A}{b-c\cos A}=\dfrac{\cos B}{\cos C}$

(o) $a(b\cos C-c\cos B)=b^2-c^2$

(p) $\dfrac{\cos A}{a}+\dfrac{\cos B}{b}+\dfrac{\cos C}{c}=\dfrac{a^2+b^2+c^2}{2abc}$

(q) $a^2=(b+c)^2-4bc\cos^2\dfrac{A}{2}$

(r) $(b-c)\cot\dfrac{A}{2}+(c-a)\cot\dfrac{B}{2}+(a-b)\cot\dfrac{C}{2}=0$

(s) $\dfrac{b^2-c^2}{a^2}\sin 2A+\dfrac{c^2-a^2}{b^2}\sin 2B+\dfrac{a^2-b^2}{c^2}\sin 2C=0$

(t) $a(\sin B-\sin C)+b(\sin C-\sin A)+c(\sin A-\sin B)=0$

(u) $2(bc\cos A+ca\cos B+ab\cos C)=a^2+b^2+c^2$

(v) $(b^2-c^2)\cot A+(c^2-a^2)\cot B+(a^2-b^2)\cot C=0$

**Solution :**

(a) LHS : $\dfrac{a^2-c^2}{b^2}=\dfrac{k^2\sin^2 A-k^2\sin^2 C}{k^2\sin^2 B}$ [By Sine rule

$\Rightarrow \quad =\dfrac{\sin^2 A-\sin^2 C}{\sin^2 B}$ $\quad\Rightarrow\quad =\dfrac{\sin(A-C)\sin(A+C)}{\sin^2[\pi-(A+C)]}$

$\Rightarrow \quad =\dfrac{\sin(A-C)\sin(A+C)}{\sin^2(A+C)}$ $\quad\Rightarrow\quad =\dfrac{\sin(A-C)}{\sin(A+C)}=\text{RHS}$.

(b) LHS : $b\cos B+c\cos C=k\sin B\cos B+k\sin C\cos C$ [By Sine rule

$\Rightarrow \quad =\dfrac{k}{2}[2\sin B\cos B+2\sin C\cos C]$ $\quad\Rightarrow\quad =\dfrac{k}{2}[\sin 2B+\sin 2C]$

$\Rightarrow \quad =\dfrac{k}{2}\left[2\sin\dfrac{2B+2C}{2}\cos\dfrac{2B-2C}{2}\right]$ $\quad\Rightarrow\quad =k[\sin(B+C)\cos(B-C)]$

$\Rightarrow \quad = k\left[\sin(\pi - A)\cos(B - C)\right]$ $\quad\Rightarrow \quad = k\sin A\cos(B - C)$

$\Rightarrow \quad = a\cos(B - C) = \text{RHS}.$

(c) LHS : $a\sin A - b\sin B = k\sin A\sin A - k\sin B\sin B$ [By Sine rule

$\Rightarrow \quad = k[\sin^2 A - \sin^2 B]$ $\quad\Rightarrow \quad = k[\sin(A + B)\sin(A - B)]$

$\Rightarrow \quad = k\sin(\pi - C)\sin(A - B)$ $\quad\Rightarrow \quad = k\sin C\sin(A - B)$

$\Rightarrow \quad = c\sin(A - B) = \text{RHS}.$

(d) RHS : $\dfrac{1 + \cos(A - B)\cos C}{1 + \cos(A - C)\cos B} = \dfrac{1 + \cos(A - B)\cos(\pi - (A + B))}{1 + \cos(A - C)\cos(\pi - (A + C))}$

$\Rightarrow \quad = \dfrac{1 - \cos(A - B)\cos(A + B)}{1 - \cos(A - C)\cos(A + C)}$ $\quad\Rightarrow \quad = \dfrac{1 - [\cos^2 A - \sin^2 B]}{1 - [\cos^2 A - \sin^2 C]}$

$\Rightarrow \quad = \dfrac{1 - \cos^2 A + \sin^2 B}{1 - \cos^2 A + \sin^2 C}$ $\quad\Rightarrow \quad = \dfrac{\sin^2 A + \sin^2 B}{\sin^2 A + \sin^2 C}$

$\Rightarrow \quad = \dfrac{k^2\sin^2 A + k^2\sin^2 B}{k^2\sin^2 A + k^2\sin^2 C}$ [Multiplying Nr & Dr by $k^2$

$\Rightarrow \quad = \dfrac{a^2 + b^2}{a^2 + c^2} = \text{LHS}.$

(e) LHS : $\dfrac{a + b}{c} = \dfrac{k\sin A + k\sin B}{k\sin C}$ [By Sine rule

$\Rightarrow \quad = \dfrac{k[\sin A + \sin B]}{k\sin C}$ $\quad\Rightarrow \quad = \dfrac{2\sin\frac{A + B}{2}\cos\frac{A - B}{2}}{2\sin\frac{C}{2}\cos\frac{C}{2}}$

$\Rightarrow \quad = \dfrac{\sin\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\frac{A - B}{2}}{\sin\frac{C}{2}\cos\frac{C}{2}}$ $\quad\Rightarrow \quad = \dfrac{\cos\frac{C}{2}\cos\frac{A - B}{2}}{\sin\frac{C}{2}\cos\frac{C}{2}}$

$\Rightarrow \quad = \dfrac{\cos\frac{A - B}{2}}{\sin\frac{C}{2}} = \text{RHS}.$

(f) LHS : $\dfrac{b^2 - c^2}{a^2} = \dfrac{k^2\sin^2 B - k^2\sin^2 C}{k^2\sin^2 A}$ [By Sine rule

$\Rightarrow \quad = \dfrac{k^2[\sin^2 B - \sin^2 C]}{k^2\sin^2 A}$ $\quad\Rightarrow \quad = \dfrac{\sin(B + C)\sin(B - C)}{\sin^2[\pi - (B + C)]}$

$\Rightarrow \quad = \dfrac{\sin(B + C)\sin(B - C)}{\sin^2(B + C)}$ $\quad\Rightarrow \quad = \dfrac{\sin(B - C)}{\sin(B + C)} = \text{RHS}.$

(g) LHS : $\dfrac{a - b}{a + b} = \dfrac{k\sin A - k\sin B}{k\sin A + k\sin B}$

$\Rightarrow \quad = \dfrac{\sin A - \sin B}{\sin A + \sin B}$ $\quad\Rightarrow \quad = \dfrac{2\cos\frac{A + B}{2}\sin\frac{A - B}{2}}{2\sin\frac{A + B}{2}\cos\frac{A - B}{2}}$

$\Rightarrow \quad = \cot\dfrac{A + B}{2}\tan\dfrac{A - B}{2}$ $\quad\Rightarrow \quad = \dfrac{\tan\left(\frac{A - B}{2}\right)}{\tan\left(\frac{A + B}{2}\right)} = \text{RHS}.$

(h) RHS : $\dfrac{b-c}{a}\cos\dfrac{A}{2}=\dfrac{k\sin B-k\sin C}{k\sin A}\cos\dfrac{A}{2}$

$\Rightarrow \quad =\dfrac{\sin B-\sin C}{\sin A}\cos\dfrac{A}{2}$ $\qquad \Rightarrow \quad =\dfrac{2\cos\dfrac{B+C}{2}\sin\dfrac{B-C}{2}}{2\sin\dfrac{A}{2}\cos\dfrac{A}{2}}\cos\dfrac{A}{2}$

$\Rightarrow \quad =\dfrac{\cos\left(\dfrac{\pi}{2}-\dfrac{A}{2}\right)\sin\dfrac{B-C}{2}}{\sin\dfrac{A}{2}}$ $\qquad \Rightarrow \quad =\dfrac{\sin\dfrac{A}{2}\sin\dfrac{B-C}{2}}{\sin\dfrac{A}{2}}$

$\Rightarrow \quad =\sin\dfrac{B-C}{2}=\text{LHS}.$

(i) RHS : $\dfrac{\tan\dfrac{A}{2}+\tan\dfrac{B}{2}}{\tan\dfrac{A}{2}-\tan\dfrac{B}{2}}=\dfrac{\dfrac{\sin\dfrac{A}{2}\cos\dfrac{B}{2}+\sin\dfrac{B}{2}\cos\dfrac{A}{2}}{\cos\dfrac{A}{2}\cos\dfrac{B}{2}}}{\dfrac{\sin\dfrac{A}{2}\cos\dfrac{B}{2}-\sin\dfrac{B}{2}\cos\dfrac{A}{2}}{\cos\dfrac{A}{2}\cos\dfrac{B}{2}}}$

$\Rightarrow \quad =\dfrac{\sin\dfrac{A}{2}\cos\dfrac{B}{2}+\sin\dfrac{B}{2}\cos\dfrac{A}{2}}{\sin\dfrac{A}{2}\cos\dfrac{B}{2}-\sin\dfrac{B}{2}\cos\dfrac{A}{2}}$ $\qquad \Rightarrow \quad =\dfrac{\sin\left(\dfrac{A}{2}+\dfrac{B}{2}\right)}{\sin\left(\dfrac{A}{2}-\dfrac{B}{2}\right)}$

$\Rightarrow \quad =\dfrac{\sin\left(\dfrac{\pi}{2}-\dfrac{C}{2}\right)}{\sin\left(\dfrac{A-B}{2}\right)}$ $\qquad \Rightarrow \quad =\dfrac{\cos\dfrac{C}{2}}{\sin\left(\dfrac{A-B}{2}\right)}$

$\Rightarrow \quad =\dfrac{2\sin\dfrac{C}{2}\cos\dfrac{C}{2}}{2\sin\dfrac{C}{2}\sin\left(\dfrac{A-B}{2}\right)}$ $\qquad \Rightarrow \quad =\dfrac{\sin C}{2\sin\left(\dfrac{\pi}{2}-\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)}$

$\Rightarrow \quad =\dfrac{\sin C}{2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)}$ $\qquad \Rightarrow \quad =\dfrac{\sin C}{\sin A-\sin B}$

$\Rightarrow \quad =\dfrac{k\sin C}{k\sin A-k\sin B}$ $\qquad$ [Multiply Nr & Dr by $k$

$\Rightarrow \quad =\dfrac{c}{a-b}=\text{LHS}.$

(j) LHS : $a(\cos C-\cos B)=k\sin A(\cos C-\cos B)$

$\Rightarrow \quad =k\sin A\left(-2\sin\dfrac{C+B}{2}\sin\dfrac{C-B}{2}\right)$

$\Rightarrow \quad =k\left(2\sin\dfrac{A}{2}\cos\dfrac{A}{2}\right)\left[-2\sin\left(\dfrac{\pi}{2}-\dfrac{A}{2}\right)\sin\dfrac{C-B}{2}\right]$

$\Rightarrow \quad =2k\left(-2\sin\dfrac{A}{2}\cos\dfrac{A}{2}\cos\dfrac{A}{2}\sin\dfrac{C-B}{2}\right)$

$\Rightarrow \quad = 2k\cos^2\dfrac{A}{2}\left(-2\sin\left(\dfrac{\pi}{2}-\dfrac{C+B}{2}\right)\sin\dfrac{C-B}{2}\right)$

$\Rightarrow \quad = -2k\cos^2\dfrac{A}{2}\left(2\cos\dfrac{C+B}{2}\sin\dfrac{C-B}{2}\right)$

$\Rightarrow \quad = -2k\cos^2\dfrac{A}{2}(\sin C-\sin B)$

$\Rightarrow \quad = 2\cos^2\dfrac{A}{2}(k\sin B-k\sin C)$

$\Rightarrow \quad = 2(b-c)\cos^2\dfrac{A}{2} = \text{RHS}.$

(k) RHS : $(b+c)\sin\dfrac{A}{2} = (k\sin B+k\sin C)\sin\dfrac{A}{2}$

$\Rightarrow \quad = k(\sin B+\sin C)\sin\dfrac{A}{2}$

$\Rightarrow \quad = k\left(2\sin\dfrac{B+C}{2}\cos\dfrac{B-C}{2}\right)\sin\dfrac{A}{2}$

$\Rightarrow \quad = k\cos\dfrac{B-C}{2}\left[2\sin\left(\dfrac{\pi}{2}-\dfrac{A}{2}\right)\sin\dfrac{A}{2}\right]$

$\Rightarrow \quad = k\cos\dfrac{B-C}{2}\left[2\cos\dfrac{A}{2}\sin\dfrac{A}{2}\right]$

$\Rightarrow \quad = k\cos\dfrac{B-C}{2}[\sin A]$

$\Rightarrow \quad = [k\sin A]\cos\dfrac{B-C}{2}$

$\Rightarrow \quad = a\cos\dfrac{B-C}{2} = \text{LHS}.$

(l) LHS : $2\left(b\cos^2\dfrac{C}{2}+c\cos^2\dfrac{B}{2}\right) = b\left(2\cos^2\dfrac{C}{2}\right)+c\left(2\cos^2\dfrac{B}{2}\right)$

$\Rightarrow \quad = b(1+\cos C)+c(1+\cos B)$

$\Rightarrow \quad = b+b\cos C+c+c\cos B$

$\Rightarrow \quad = b+c+b\cos C+c\cos B$ [By Projection formula, $a = b\cos C+c\cos B$

$\Rightarrow \quad = a+b+c = \text{RHS}.$

(m) LHS : $(a-b)^2\cos^2\dfrac{C}{2}+(a+b)^2\sin^2\dfrac{C}{2}$

$\Rightarrow \quad = a^2\cos^2\dfrac{C}{2}+b^2\cos^2\dfrac{C}{2}-2ab\cos^2\dfrac{C}{2}+a^2\sin^2\dfrac{C}{2}+b^2\sin^2\dfrac{C}{2}+2ab\sin^2\dfrac{C}{2}$

$\Rightarrow \quad = a^2\left(\cos^2\dfrac{C}{2}+\sin^2\dfrac{C}{2}\right)+b^2\left(\cos^2\dfrac{C}{2}+\sin^2\dfrac{C}{2}\right)-2ab\left(\cos^2\dfrac{C}{2}-\sin^2\dfrac{C}{2}\right)$

$\Rightarrow \quad = a^2(1)+b^2(1)-2ab\cos C$

$\Rightarrow \quad = c^2 = \text{RHS}.$

(n) LHS : $\dfrac{c-b\cos A}{b-c\cos A} = \dfrac{a\cos B+b\cos A-b\cos A}{c\cos A+a\cos C-c\cos A}$ [By Projection formulae

$\Rightarrow \quad = \dfrac{a\cos B}{a\cos C} = \dfrac{\cos B}{\cos C} = \text{RHS}.$

(o) LHS : $a(b\cos C-c\cos B) = ab\cos C-ac\cos B$

$\Rightarrow \quad = ab\left(\dfrac{a^2+b^2-c^2}{2ab}\right) - ac\left(\dfrac{a^2+c^2-b^2}{2ac}\right)$

$\Rightarrow \quad = \dfrac{a^2+b^2-c^2}{2} - \dfrac{a^2+c^2-b^2}{2}$

$\Rightarrow \quad = \dfrac{a^2+b^2-c^2}{2} - \dfrac{a^2+c^2-b^2}{2}$

$\Rightarrow \quad = \dfrac{a^2+b^2-c^2-a^2-c^2+b^2}{2}$

$\Rightarrow \quad = \dfrac{2(b^2-c^2)}{2}$

$\Rightarrow \quad = (b^2-c^2) = \text{RHS}$ .

(p) LHS : $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{1}{a}\left(\dfrac{c^2+b^2-a^2}{2bc}\right) + \dfrac{1}{b}\left(\dfrac{a^2+c^2-b^2}{2ac}\right) + \dfrac{1}{c}\left(\dfrac{a^2+b^2-c^2}{2ab}\right)$

$\Rightarrow \quad = \dfrac{c^2+b^2-a^2}{2abc} + \dfrac{a^2+c^2-b^2}{2abc} + \dfrac{a^2+b^2-c^2}{2abc}$

$\Rightarrow \quad = \dfrac{c^2+b^2-a^2+a^2+c^2-b^2+a^2+b^2-c^2}{2abc}$

$\Rightarrow \quad = \dfrac{a^2+b^2+c^2}{2abc} = \text{RHS}$ .

(q) RHS : $(b+c)^2 - 4bc\cos^2\dfrac{A}{2} = b^2+c^2+2bc-4bc\cos^2\dfrac{A}{2}$

$\Rightarrow \quad = b^2+c^2-2bc\left(2\cos^2\dfrac{A}{2}-1\right)$

$\Rightarrow \quad = b^2+c^2-2bc\cos A$

$\Rightarrow \quad = a^2 = \text{LHS}$ .

(r) Consider, $(b-c)\cot\dfrac{A}{2} = (k\sin B - k\sin C)\cot\dfrac{A}{2}$

$\Rightarrow \quad = k\left(2\cos\dfrac{B+C}{2}\sin\dfrac{B-C}{2}\right)\cot\dfrac{A}{2} \qquad \Rightarrow \quad = k\left[2\cos\left(\dfrac{\pi}{2}-\dfrac{A}{2}\right)\sin\dfrac{B-C}{2}\right]\cot\dfrac{A}{2}$

$\Rightarrow \quad = k\left[2\sin\dfrac{A}{2}\sin\dfrac{B-C}{2}\right]\dfrac{\cos\frac{A}{2}}{\sin\frac{A}{2}} \qquad \Rightarrow \quad = k\left[2\sin\dfrac{B-C}{2}\cos\left(\dfrac{\pi}{2}-\dfrac{B+C}{2}\right)\right]$

$\Rightarrow \quad = k\left[2\sin\left(\dfrac{B}{2}-\dfrac{C}{2}\right)\sin\left(\dfrac{B}{2}+\dfrac{C}{2}\right)\right] \qquad \Rightarrow \quad = -k\left[-2\sin\left(\dfrac{B}{2}-\dfrac{C}{2}\right)\sin\left(\dfrac{B}{2}+\dfrac{C}{2}\right)\right]$

$\Rightarrow \quad = -k\left[\cos\dfrac{B}{2}-\cos\dfrac{C}{2}\right] \qquad \Rightarrow \quad = k\left[\cos\dfrac{C}{2}-\cos\dfrac{B}{2}\right]$

So, $(b-c)\cot\dfrac{A}{2} = k\left[\cos\dfrac{C}{2}-\cos\dfrac{B}{2}\right]\ldots$(i)

Similarly $(c-a)\cot\dfrac{B}{2} = k\left[\cos\dfrac{A}{2}-\cos\dfrac{C}{2}\right]\ldots$(ii)

And, $(a-b)\cot\dfrac{B}{2} = k\left[\cos\dfrac{B}{2}-\cos\dfrac{A}{2}\right]\ldots$(iii)

Now adding (i), (ii) & (iii), we get :

$$(b-c)\cot\frac{A}{2}+(c-a)\cot\frac{B}{2}+(a-b)\cot\frac{B}{2}=k\left[\cos\frac{C}{2}-\cos\frac{B}{2}\right]+k\left[\cos\frac{A}{2}-\cos\frac{C}{2}\right]$$
$$+k\left[\cos\frac{B}{2}-\cos\frac{A}{2}\right]$$

$$\Rightarrow \quad =k\left[\cos\frac{C}{2}-\cos\frac{B}{2}+\cos\frac{A}{2}-\cos\frac{C}{2}+\cos\frac{B}{2}-\cos\frac{A}{2}\right]$$

$$\therefore\ (b-c)\cot\frac{A}{2}+(c-a)\cot\frac{B}{2}+(a-b)\cot\frac{B}{2}=0.$$ Hence Proved.

(s) Consider, $\frac{b^2-c^2}{a^2}\sin 2A=\frac{b^2-c^2}{a^2}[2\sin A\cos A]$

$$\Rightarrow \quad =2(ak)\left(\frac{b^2-c^2}{a^2}\right)\left(\frac{b^2+c^2-a^2}{2bc}\right)$$

$$\Rightarrow \quad =k\left(\frac{(b^2-c^2)(b^2+c^2)-a^2b^2+a^2c^2}{2abc}\right)$$

*i.e.*, $\frac{b^2-c^2}{a^2}\sin 2A=k\left(\frac{b^4-c^4-a^2b^2+a^2c^2}{2abc}\right)$ …(i)

Similarly, $\frac{c^2-a^2}{b^2}\sin 2B=k\left(\frac{c^4-a^4-c^2b^2+a^2b^2}{2abc}\right)$ …(ii)

And, $\frac{a^2-b^2}{c^2}\sin 2C=k\left(\frac{a^4-b^4-c^2a^2+c^2b^2}{2abc}\right)$ …(iii)

Now adding (i), (ii) & (iii), we get :

$$\frac{b^2-c^2}{a^2}\sin 2A+\frac{c^2-a^2}{b^2}\sin 2B+\frac{a^2-b^2}{c^2}\sin 2C=k\left(\frac{b^4-c^4-a^2b^2+a^2c^2}{2abc}\right)$$
$$+k\left(\frac{c^4-a^4-c^2b^2+a^2b^2}{2abc}\right)+k\left(\frac{a^4-b^4-c^2a^2+c^2b^2}{2abc}\right)$$

$$\Rightarrow \quad =\frac{k}{2abc}\left[b^4-c^4-a^2b^2+a^2c^2+c^4-a^4-c^2b^2+a^2b^2+a^4-b^4-c^2a^2+c^2b^2\right]$$

$$\Rightarrow \quad =\frac{k}{2abc}[0]=0$$

$$\therefore\ \frac{b^2-c^2}{a^2}\sin 2A+\frac{c^2-a^2}{b^2}\sin 2B+\frac{a^2-b^2}{c^2}\sin 2C=0.$$ Hence Proved.

(t) LHS : $a(\sin B-\sin C)+b(\sin C-\sin A)+c(\sin A-\sin B)$

$\Rightarrow \quad =a\sin B-a\sin C+b\sin C-b\sin A+c\sin A-c\sin B$

$\Rightarrow \quad =k\sin A\sin B-k\sin A\sin C+k\sin B\sin C-k\sin B\sin A+k\sin C\sin A-k\sin C\sin B$

$\Rightarrow \quad =0=\text{RHS}$.

(u) LHS : $2\left(bc\cos A+ca\cos B+ab\cos C\right)$

$$\Rightarrow \quad =2\left(bc\left[\frac{b^2+c^2-a^2}{2bc}\right]+ca\left[\frac{a^2+c^2-b^2}{2ac}\right]+ab\left[\frac{b^2+a^2-c^2}{2ba}\right]\right)$$

$\Rightarrow \quad =b^2+c^2-a^2+a^2+c^2-b^2+b^2+a^2-c^2$

$\Rightarrow \quad =a^2+b^2+c^2=\text{RHS}$.

(v) Consider, $\left(b^2-c^2\right)\cot A=\left(b^2-c^2\right)\frac{\cos A}{\sin A}$

$$\Rightarrow \quad = \left(b^2 - c^2\right)\frac{1}{ka}\left(\frac{b^2 + c^2 - a^2}{2bc}\right)$$

$$i.e., \left(b^2 - c^2\right)\cot A = \frac{b^4 - c^4 - a^2b^2 + a^2c^2}{2kabc} \quad \ldots\text{(i)}$$

Similarly, $\left(c^2 - a^2\right)\cot B = \dfrac{c^4 - a^4 - c^2b^2 + a^2b^2}{2kabc} \quad \ldots\text{(ii)}$

And, $\left(a^2 - b^2\right)\cot C = \dfrac{a^4 - b^4 - c^2a^2 + c^2b^2}{2kabc} \quad \ldots\text{(iii)}$

Now adding (i), (ii) & (iii), we get :

$$\left(b^2 - c^2\right)\cot A + \left(c^2 - a^2\right)\cot B + \left(a^2 - b^2\right)\cot C = \frac{b^4 - c^4 - a^2b^2 + a^2c^2}{2kabc} + \frac{c^4 - a^4 - c^2b^2 + a^2b^2}{2kabc} + \frac{a^4 - b^4 - c^2a^2 + c^2b^2}{2kabc}$$

$$\Rightarrow \quad = \frac{b^4 - c^4 - a^2b^2 + a^2c^2 + c^4 - a^4 - c^2b^2 + a^2b^2 + a^4 - b^4 - c^2a^2 + c^2b^2}{2kabc}$$

$$\Rightarrow \quad = \frac{0}{2kabc}$$

$\therefore \left(b^2 - c^2\right)\cot A + \left(c^2 - a^2\right)\cot B + \left(a^2 - b^2\right)\cot C = 0$. Hence Proved.



❖ Dear Student,

I would urge you for a little favour. Please notify me about any error(s) you notice in this (or other Maths) work. It would be beneficial for all the future learners of Maths like us. Any constructive criticism will be well acknowledged. Please find below my contact info when you decide to offer me your valuable suggestions. I'm looking forward for a response.

Also I would wish if you **inform your friends about my efforts for Maths** so that they may also benefit.

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☞ For any clarification(s), please contact :

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