EXERCISES ON EXPANDERS

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We recall notation of the lecture: Let G be a d-regular connected graph with vertex set V and edge set E. Let n denote the cardinality of V, $n = |V|$. Let $\ell^2(V) \simeq \mathbb{R}^n$ denote the space of real-valued functions on V. Denote by A its adjacency matrix, defined by

$$Af(x) = \sum_{y \sim x} f(y)$$

where the sum runs over all neighbours y of x (with multiplicity if G is not simple, that is, has multiple edges between two vertices. Read the symbol $\sim$ as "adjacent to"). A is symmetric so that there exists an orthonormal basis of $\ell^2(V)$ that are eigenfunctions of A.

The $\ell_0^2(V)$ be the subspace of $\ell^2(V)$ for which $\sum_{x \in V} f(x) = 0$. Let λ_2 be the second largest eigenvalue of A. (And we call $d - \lambda_2$ the spectral gap of G.)

Let $\langle f, g \rangle = \sum_x f(x)g(x)$ denote the inner product on $\ell^2(V)$ and $\| \cdot \|_2$ the induced norm.

For a subset $S \subset V$, we define its boundary

$$\partial S = E(S, S^c) = \{(x, y) \in E : x \in S, y \in Y\}.$$

The Cheeger constant of G is

$$h(G) = min \left\{ \frac{|\partial S|}{|S|} : S \subset V, |S| \leq n/2 \right\}.$$

The Laplacian $\Delta = d - A$ satisfies $\langle \Delta f, f \rangle = \sum_{(x,y) \in E} (f(x) - f(y))^2$.

Exercise 1. Prove the following formula:

$$\lambda_2 = \max\{\langle Af, f \rangle : f \in \ell_0^2(V),\ \|f\|_2 = 1\}.$$

Exercise 2. G is called bipartite if there exists a partion $L \sqcup R = V$ such that all edges of G run between L and R (that is, E can be thought of as subset of $L \times R$). Prove that G is bipartite if and only if $-d$ is an eigenvalue.

Exercise 3. Complete the proof of the upper bound of Cheeger's inequality,

$$\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2)}$$

Recall that if $\tilde{f} \in \ell_0^2$ with $A\tilde{f} = \lambda_2 \tilde{f}$, we denoted $f(x) = \max(\tilde{f}(x), 0)$. We may assume that $V^+ = \operatorname{Supp} f$ has cardinality less $\leq n/2$. We also put an orientation on G by letting $(x, y) \in E$ if $x \sim y$ and $f(x) \geq f(y)$. We defined $B_f = \sum_{(x,y) \in E} (f(x)^2 - f(y)^2)$ (and can be thought as "gradient" of f^2).

Prove that:

- $\langle \Delta f, f \rangle \leq (d - \lambda_2) \|f\|_2^2$

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- $B_f^2 \le 2d\langle \Delta f, f\rangle \|f\|_2^2$

In the lecture we saw that $h(G)\|f\|_2^2 \le B_f$. The above two steps complete the upper bound of Cheeger's inequality.

Project. Implement one of the graphs we discussed in class using SageMath. Run experiments on bounds for the spectral gap and Cheeger constant. Draw nice pictures. (Contact me if interested)

Define

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

and

$$\mathrm{ASL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{bmatrix} : a, b, c, d, x, y \in \mathbb{Z}, ad - bc = 1 \right\}.$$

Let $\sigma = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\tau = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\rho = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ be elements in $\mathrm{SL}_2(\mathbb{Z})$.

Exercise 4. Show that σ and τ generate $\mathrm{SL}_2(\mathbb{Z})$. For this, you have to understand how σ and τ act on rows and column and apply Euclid's algorithm to reduce $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ to, say, $\left[\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right]$.

Exercise 5. Check that $\mathrm{ASL}_2(\mathbb{Z}) \simeq \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ and that the elements considered in the Gaber-Galil construction generate $\mathrm{ASL}_2(\mathbb{Z})$.

Exercise 6. Show that the graph G_p with $V = \mathbb{P}^1(\mathbb{F}_p) = \{0, \dots, p-1, \infty\}$ and $E = \{(x, y) \in V^2 : \text{ either } xy = -1 \text{ or } x = y \pm 1\}$ is a Schreier graph of $\mathrm{SL}_2(\mathbb{Z})$.

Exercise 7. Let $G(V, E)$ be a d-regular tree, i.e. an infinite graph without any cycles such that each vertex has d edges connected to it. Note that if $d = 2k$, then G is the Cayley graph of a free group of k generators. Let $A : \ell^2(V) \to \ell^2(V)$ the corresponding adjancency operator, i.e. for $Af(x) = \sum_{y\sim x} f(y)$. Note that $\ell^2(V)$ is now a infinite-dimensional Hilbert space (which we may take to be over the complex numbers), and A is no longer a matrix acting on $\mathbb{R}^n$. Let $\|A\|$ denote the operator norm and $\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not invertible}\}$ its spectrum. A is called self-adjoint if $\langle Af, g\rangle = \langle f, Ag\rangle$ and $\|A\| < \infty$. For such operators, one can show that in fact $\sigma(A) \subset [-\|A\|, \|A\|]$.

Show that

- A is self-adjoint.

  Using ideas from the proof of the Gabber-Galil construction and its postdiscussion:
- $\|A\| = 2\sqrt{d-1}$ (i.e. $\ge$ and $\le$).

  Construct approximate eigenfunctions for all $\lambda \in [-\|A\|, \|A\|]$ to strengthen to:
- $\sigma(A) = [-\|A\|, \|A\|]$.

Exercise 8. ([1]) Let Γ be the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by $a = \sigma^2$ and $b = \tau^2$.

- If $A = \{(x, y) : \mathbb{R}^2 : |x| < |y|\}$ and $B = \{(x, y) : \mathbb{R}^2 : |x| > |y|\}$ show that $a^n A \subset B$ and $b^n B \subset A$ for any nonzero integer n.
- Show that Γ is a free group of two generators. Use that by the above, any reduced word of a and b, that begins or ends with $a^{\pm 1}$ or begins and ends with $b^{\pm 1}$, is not equal to the identity.
- Show that Γ has finite index in $\mathrm{SL}_2(\mathbb{Z})$: Show that any given element of $\mathrm{SL}_2(\mathbb{Z})$ with columns $v, w \in \mathbb{Z}^2$, one can multiply this element on the left by some word in a, b to minimise the magnitude $|\langle v, w\rangle|$, until one reaches a point where $|\langle v, w\rangle| \leq \|v\|^2, \|w\|^2$. Now use Lagrange identity $|\langle v, w\rangle|^2 + 1 = \|v\|^2\|w\|^2$ to conclude that v, w have bounded size.

Exercise 9. ([2]) The n citizens of the "Land of Make Believe" have elected a new president among two candidates. The electronic voting system claimed that the winner got at least 3/4 of the votes, but unfortunately lost all other data, including the winner's name! Time being money, it has been decided to chose the president as follows:

Let $G(V, E)$ be a d-regular graph with $|V| = n$ and $\bar{\lambda} = \lambda_2/d$

- Identify the citizens with the node set V.
- Chose a citizen v at random.
- The candidate getting the majority of votes in $N(v)$, the set of the d neighbors of v, will be the new president!

Prove that the probability of error is bounded by $4\bar{\lambda}^2$.

Exercise 10. We construct a family of expanders $\mathrm{LD}_{p,k}$ on $n = p^{k+1}$ vertices and degree $d = p^2$. Consider the (abelian) group $\mathbb{F}_p^{k+1}$ and let $S = \{(a, ab, ab^2, \ldots, ab^k) : a, b \in \mathbb{F}_p\}$. Then $\mathrm{LD}_{p,k}$ is the Cayley graph $\mathrm{Cay}(\mathbb{F}_p^{k+1}, S)$ defined by $\mathbb{F}_p^{k+1}$ and S. Show that $|\lambda_2| \leq kp$! For this, given $x, y \in \mathbb{F}_p^{k+1}$, let $\chi_x(y) = \exp \frac{2\pi i}{p}\langle x, y\rangle$ be the xth character of $\mathbb{F}_p^{k+1}$ and check that χ_x is an eigenvector of the adjacency operator of $\mathrm{LD}_{p,k}$ and bound the eigenvalue by kp if $x \neq (0, \ldots, 0)$.

The *Low Degree* graph $\mathrm{LD}_{29,7}$ was used as base graph for an iterative expander construction using the Zig-Zag product. For varying $k = \lfloor \frac{p}{2} \rfloor$, and p running over primes the family $\mathrm{LD}_{p,\lfloor \frac{p}{2} \rfloor}$ defines an expander family of *Logarithmic Degree.*

Exercise 11. For the eigenvalue bound for the Zig-Zag product of two graphs G_1 and G_2 in the lecture, we in fact assumed that $\alpha \in L_0^2(G_1 \text{ⓩ} G_2)$. In particular, it may happen that that the eigenvalue 1 has multiplicity! Show that the Zig Zag product is not always connected, by choosing the small graph G_2 wisely and a bad labelling for G_1.

We have shown Property (T) for $\mathrm{SL}_3(\mathbb{R})$. It is possible to deduce Property (T) for the discrete subgroup $\mathrm{SL}_3(\mathbb{Z})$ from this. Assuming therefore Property (T) for $\mathrm{SL}_3(\mathbb{Z})$, show that for the standard unipotent elementary matrices $S = S^{-1}$ in $\mathrm{SL}_3(\mathbb{Z})$, the Caylay graph $(\mathrm{SL}_3(\mathbb{Z}/p\mathbb{Z}), S)$ is an expander family for p any odd prime. Here, $\mathrm{SL}_3(\mathbb{Z}/p\mathbb{Z})$ is the finite group obtained by mapping $A \in \mathrm{SL}_3(\mathbb{Z})$ to $A \mod p$ (coordinatewise) and

[1]Taken from Ex 2.3.1 "Expansion in Finite Simple Groups of Lie Type, T. Tao"

[2]Taken from Mathematics of Computer Science taught by M. Couchand at ETH.

in fact agrees with group of 3×3-matrices with coordinates in $\mathbb{Z}/p\mathbb{Z}$ and determinant $1 \in \mathbb{Z}/p\mathbb{Z}$ (which can be seen by noticing that the elementary unipotents generate this group).

Exercise 12. Let (Q, ε) be the uniform spectral gap parameters of $\mathrm{SL}_3(\mathbb{Z})$. That is, $Q \subset \mathrm{SL}_3(\mathbb{Z})$ is a compact (thus finite) subset and $\varepsilon > 0$ a positive number such that there are no (Q, ε)-almost invariant eigenvectors in any unitary representation of $\mathrm{SL}_3(\mathbb{Z})$. Now let $\mathrm{SL}_3(\mathbb{Z})$ act on $L^2_0(\mathrm{SL}_3(\mathbb{Z}/p\mathbb{Z}))$, let A_p denote the adjacency operator with respect S, $A_p f(x) = \sum_{s \in S} f(s \cdot x)$. Show that there is $\epsilon' > 0$ such that for any function $f \in L^2_0(\mathrm{SL}_3(\mathbb{Z}/p\mathbb{Z}))$ of norm one, $\sup_{s \in S} \|s \cdot f - f\| > \epsilon'$ (or equivalently, (S, ϵ') are uniform spectral gap parameters) and deduce an upper bound for the operator norm of A_p.

Exercise 13. Similar to the above, if $\mathrm{SL}_2(\mathbb{R})$ had a uniform spectral gap, then so did $\mathrm{SL}_2(Z)$. Recall from Exercise 8 that $\mathrm{SL}_2(Z)$ free finite index subgroup of two generators. Show therefore that $\mathrm{SL}_2(\mathbb{R})$ does not have Property (T) by showing that the free subgroup of Exercise 8 cannot have Property (T), by first showing that $\mathbb{Z}^k$ cannot have Property (T) and then deducing that a free subgroup on k-generators cannot have Property (T).

Exercise 14. We have proven a lemma Of Burger, which can be stated as follows: Let $F \subset \mathrm{SL}_2(\mathbb{R})$ be a finite set of transformations acting on $\mathbb{R}^2 \setminus \{0\}$, define

$$\alpha(F) = \inf_{\mu \in \mathcal{M}^1(\mathbb{R}^2 \setminus 0)} \sup_{B \subset \mathbb{R}^2 \setminus \{0\}} \max_{\gamma \in F} |\mu(\gamma B) - \mu(B)|$$

where $\mathcal{M}^1(\mathbb{R}^2 \setminus \{0\})$ denotes the space of probability measures on $\mathbb{R}^2 \setminus \{0\}$ and B runs over the Borel subsets in $\mathbb{R}^2 \setminus \{0\}$. Then

$$\alpha(\Sigma) \geq \frac{1}{4}$$

for $\Sigma = \{\left[\begin{smallmatrix} 1 & \pm 1 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ \pm 1 & 1 \end{smallmatrix}\right]\}$. Show that

$$\alpha(\Sigma_2) = \frac{1}{2}$$

for $\Sigma_2 = \{\left[\begin{smallmatrix} 1 & \pm 2 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ \pm 2 & 1 \end{smallmatrix}\right]\}$.

Exercise 15. One can show that $\mathrm{SL}_3(\mathbb{Z})$ has bounded generation: Let M denote the set of unipotent elementary matrices in $\mathrm{SL}_3(\mathbb{Z})$, i.e. having ones on the diagonal, one entry equal to one off-diagonal and all others zero. There exists $k \in \mathbb{N}$ such that any element $g \in \mathrm{SL}_3(\mathbb{Z})$ can be written as $g = u_1^{\ell_1} \dots u_k^{\ell_k}$ for $u_i \in M$. Using relative property (T) of $(\mathrm{ASL}_2(\mathbb{Z}), \mathbb{Z}^2)$, show that $\mathrm{SL}_3(\mathbb{Z})$ has property (T) assuming bounded generation. Note how this exercise is a cousin of Exercise 12.

A mapping $f : X \to Y$ where (X, ρ) and (Y, σ) are metric spaces is called a D-embedding if there exists $r > 0$ such that for any $x, y \in X$,

$$r\rho(x, y) \leq \sigma(f(x), f(y)) \leq Dr\rho(x, y)$$

The infimum of D such that f is a D-embedding is called the distortion of f.

Exercise 16. [3] Consider the two 4-point examples the square and the star; prove that they cannot be isometrically embedded into $\ell^2(\mathbb{R}^2)$. Can you determine the minimum necessary distortion for embedding into $\ell^2(\mathbb{R}^2)$?

Exercise 17. Prove that a bijective mapping f between metric spaces is a D-embedding if and only if $\|f\|_{\mathrm{Lip}}\|f^{-1}\|_{\mathrm{Lip}} \leq D$.

[3]Taken from Matousek's book