1. Gita is a software engineer. She has a task that needs to be done in the next two hours, and knows she could do it by hand in that amount of time. She also knows she could write code to automate it in less than 30 minutes. The code itself will take some additional time to run. The script involves 8 tasks. Each must be done one after the other. Because of how the data is being processes, the time to perform two adjacent tasks is half as long as it takes to do the next task. The first task will take 2 seconds. The fourth will take 20 seconds.

   (a) Write a system of equations describing the length of each task.

   Solution:

$$\left(\begin{array}{cccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 20 \\ 1 & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -\frac{1}{2} & 0 \end{array}\right)$$

   (b) She plugs this into her computer and gets the matrix

$$\left(\begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 20 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 56 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 152 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 416 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1136 \end{array}\right)$$

   Given that it takes no more than 30 minutes to write the script. Will Gita be able to complete the task in under two hours? Explain.

   Solution:

$$2+2+8+20+56+152+416+1136 = 1792 \text{ seconds} \approx 29.9 \text{ minutes.}$$

2. (courtesy A. Raymond)

   (a) **Use Gauss-Jordan elimination** to find the general solution for the following system of linear equations:

$$\begin{array}{rcrcrcrl} & & z_2 & + & 3z_3 & - & z_4 & = 0 \\ -z_1 & - & z_2 & - & z_3 & + & z_4 & = 0 \\ -2z_1 & - & 4z_2 & + & 4z_3 & - & 2z_4 & = 0 \end{array}$$

   Solution: The matrix in echelon form is:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{array}\right)$$

Thus, z_4 is free. Let $z_4 = s \in \mathbb{R}$. Our solution set is all vectors of the form $\begin{bmatrix} s \\ -s/2 \\ s/2 \\ s \end{bmatrix}$.

(b) Give an example of a solution to the previous system of linear equations.

Solution: Set $s = 2$ and we have $(2, -1, 1, 2)$ is one of many possible solutions.

(c) The points $(1, 0, 3)$, $(1, 1, 1)$, and $(-2, -1, 2)$ lie on a unique plane $a_1x_1 + a_2x_2 + a_3x_3 = b$. **Using your previous answers**, find an equation for this plane. (Hint: think about the relationship between the previous system and the one you would need to solve in this question.)

Solution: This system corresponds to solving

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & -1 & 0 \\ 1 & 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & -1 & 0 \end{array}\right)$$

This is the matrix from the system in part (a) if we switch col2 with col1 and multiply row2 by -1 and row4 by 2. The column switch makes us switch a_2 and a_1, but otherwise changes nothing. Thus, our answer from (b) with the first and second entries switched is fine to use. This gives the equation

$$-x_1 + 2x_2 + x_3 = -2.$$

3. Consider the following linear system,

$$\begin{array}{rcrcrl} y_1 & - & 3y_2 & + & 5y_3 & = b \\ 2y_1 & + & 4y_2 & & & = 10 \\ 3y_1 & + & 7y_2 & + & ay_3 & = 3b. \end{array}$$

We see that the system involves two unknown constants, a and b. You enter this into a computer and find that the system in echelon form is:

$$\begin{bmatrix} 1 & -3 & 5 & : & b \\ 0 & 1 & -1 & : & 1 - \frac{b}{5} \\ 0 & 0 & \frac{a+1}{16} & : & \frac{b}{5} - 1 \end{bmatrix}.$$

(a) For what values of a and b does the system have no solution?

Solution: The matrix in echelon form is There will be no solution when the bottom row makes the system inconsistent. This will occur when $\frac{a+1}{16} = 0$ but $\frac{b}{5} - 1 \neq 0$. So the corresponding set of values is

$$a = -1 \text{ and } b \neq 5.$$

(b) For what values does the system have infinitely many solutions?

Solution: Using the same echelon form, we will have infinitely many solutions when our system is consistent with a free variable. This occurs precisely when the bottom row of our echelon form matrix is all 0s. For this to happen we need

$$a = -1 \text{ and } b = 5.$$

(c) Give an example of a pair of values a and b for which the system has exactly one solution, and find that solution.

Solution: Any other pair of values for a and b will yield exactly one solution. Choosing $a = 15$ and $b = 5$ yields the augmented matrix

$$\begin{bmatrix} 1 & -3 & 5 & : & 5 \\ 0 & 1 & -1 & : & 0 \\ 0 & 0 & 1 & : & 0 \end{bmatrix}$$

Back-substitution yields the solution $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$.