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عناوین فرآیندهای تصادفی

Stochastic Processes Topics

مؤلف:

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”این کتاب را تقدیم می‌کنم به پسرم آرتیمان خسروتاش“

«آینده»

مکانی نیست که به آنجا می‌رویم،

جایی است که آن را به وجود می‌آوریم.

راه‌هایی که به «آینده» ختم می‌شوند یافتنی نیستند، بلکه

ساختنی‌اند و ساختن آن، هم سازنده و هم مقصد را دگرگون می‌کند.

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یک باور بکارید

سبز بودن، نباید به معنی کاری دشوار و ایجاد کردن تغییرات چشمگیر در زندگی باشد! کارهای کوچک و ساده هم می‌توانند موثر باشند.

محتوای این کتاب کوچک به جهت معرفی مدل‌های تصادفی مورد نیاز در مسایل ریاضیات مالی و انواع پدیده‌های تصادفی همراه با نویز است.

نظرات شما را صمیمانه خواهم شنید.

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introduction

In the mathematics of probability, a **stochastic process** is a random function. In practical applications, the domain over which the function is defined as a time interval (*time series*) or a region of space (*random field*).

Familiar examples of **time series** include stock market and exchange rate fluctuations, signals such as speech, audio and video; medical data such as a patient'sEKG, EEG, blood pressure or temperature; and random movement such as Brownian motion or random walks.

Examples of **random fields** include static images, random topographies (landscapes), or composition variations of an inhomogeneous material.

Author hope it be useful.

darya khosrotash
(Phd candidate of applied mathematics)

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Basic affine jump diffusion

In probability theory, a **basic affine jump diffusion (basic AJD)** is a stochastic process Z of the form

$$dZ_t = \kappa(\theta - Z_t)dt + \sigma\sqrt{Z_t}\,dB_t + dJ_t$$
$$t \geq 0, Z_0 \geq 0$$

where B is a standard Brownian motion, and J is an independent compound Poisson process with constant jump intensity l and independent exponentially distributed jumps with mean μ . For the process to be well defined, it is necessary that $\kappa\theta \geq 0$ and $\mu \geq 0$. A basic AJD is a special case of an affine process and of a jump diffusion. On the other hand, the Cox–Ingersoll–Ross (CIR) process is a special case of a basic AJD.

moment generating function

$$m(q) = \mathrm{E}\left(e^{q\int_0^t Z_s\,ds}\right), \qquad q \in \mathbb{R}$$

characteristic function

$$\varphi(u) = \mathrm{E}\left(e^{iu\int_0^t Z_s\,ds}\right), \qquad u \in \mathbb{R},$$

Bernoulli process

discrete-time processes with two possible states.

A **Bernoulli process** is a finite or infinite sequence of independent random variables $X_1, X_2, X_3, \ldots$, such that

- For each *i*, the value of X_i is either ۰ or 1;
- For all values of *i*, the probability that $X_i = 1$ is the same number *p*.

In other words, a Bernoulli process is a sequence of independent identically distributed Bernoulli trials.

Independence of the trials implies that the process is memoryless. Given that the probability *p* is known, past outcomes provide no information about future outcomes. (If *p* is unknown, however, the past informs about the future indirectly, through inferences about *p*.)

If the process is infinite, then from any point the future trials constitute a Bernoulli process identical to the whole process, the fresh-start property.

- Bernoulli schemes: discrete-time processes with *N* possible states; every stationary process in *N* outcomes is a Bernoulli scheme, and vice versa.

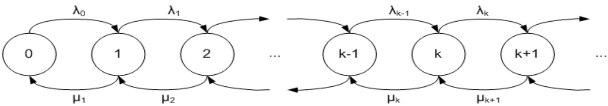
Birth-death process

The **birth–death process** is a special case of continuous-time Markov process where the state transitions are of only two types: «births», which increase the state variable by one and «deaths», which decrease the state by one. The model›s name comes from a common application, the use of such models to represent the current size of a population where the transitions are literal births and deaths. Birth–death processes have many applications in demography, queueing theory, performance engineering, epidemiology or in biology. They may be used, for example to study the evolution of bacteria, the number of people with a disease within a population, or the number of customers in line at the supermarket.

When a birth occurs, the process goes from state n to $n + 1$.

When a death occurs, the process goes from state *n* to state *n* − 1. The process is specified by birth rates $\{\lambda_i\}_{i=0\ldots\infty}$ and death rates $\{\mu_i\}_{i=1\ldots\infty}$

$$\{\lambda_i\}_{i=0\ldots\infty}$$



Branching process

In probability theory, a branching process is a Markov process that models a population in which each individual in generation *n* produces some random number of individuals in generation *n* + 1, according, in the simplest case, to a fixed probability distribution that does not vary from individual to individual.[1] Branching processes are used to model reproduction; for example, the individuals might correspond to bacteria, each of which generates 1, 0, or 2 offspring with some probability in a single time unit. Branching processes can also be used to model other systems with similar dynamics, e.g., the spread of surnames in genealogy or the propagation of neutrons in a nuclear reactor.

The most common formulation of a branching process is that of the Galton–Watson process. Let Z_n denote the state in period n (often interpreted as the size of generation n), and let $X_{n,i}$ be a random variable denoting the number of direct successors of member i in period n, where $X_{n,i}$ are independent and identically distributed random variables over all $n \in \{0, 1, 2, ...\}$ and $i \in \{1, ..., Z_n\}$. Then the recurrence equation is

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i}$$

with $Z_0 = 1$. Alternatively, one can formulate a branching process as a random walk. Let S_i denote the state in period i, and let X_i be a random variable that is iid over all i. Then the recurrence equation is

$$S_{i+1} = S_i + X_{i+1} - 1 = \sum_{j=1}^{i+1} X_j - i$$

Branching random walk

In probability theory, a **branching random walk** is a stochastic process that generalizes both the concept of a random walk and of a branching process. At every generation (a point of discrete time), a branching random walk's value is a set of elements that are located in some linear space, such as the real line. Each element of a given generation can have several descendants in the next generation. The location of any descendant is the sum of its parent's location and a random variable.

Brownian bridge

A **Brownian bridge** is a continuous-time stochastic process *B*(*t*) whose probability distribution is the conditional probability distribution of a Wiener process *W*(*t*) (a mathematical model of Brownian motion) given the condition that *B*(1) = 0. More precisely:

$$B_t = (W_t \mid W_1 = 0), t\in[0,1]$$

The expected value of the bridge is zero, with variance $t(1 - t)$, implying that the most uncertainty is in the middle of the bridge, with zero uncertainty at the nodes. The covariance of $B(s)$ and $B(t)$ is $s(1 - t)$ if $s < t$.

The increments in a Brownian bridge are not independent.

- If *W*(*t*) is a standard Wiener process (i.e., for *t* ≥ ۰, *W*(*t*) is normally distributed with expected value ۰ and variance *t*, and the increments are stationary and independent), then $B(t) = W(t) - tW(1)$ Conversely,if *B*(*t*) is a Brownian bridge and *Z* is a standard normal random variable independent of *B*, then the process $W(t) = B(t) + tZ$

$$W(t) = B\left(\frac{t}{T}\right) + \frac{t}{\sqrt{T}} Z$$

Another representation of the Brownian bridge based on the Brownian motion is, for *t* ∈ [1, 0]

$$B(t) = (1-t)W\left(\frac{t}{1-t}\right)$$

Brownian motion

Brownian motion or **pedesis** is the random motion of particles suspended in afluid (a liquid or a gas) resulting from their collision with the quick atoms or molecules in the gas or liquid. Wiener Processrefers to the mathematical model used to describe such Brownian Motion, which is often called a particle theory.

In mathematics, Brownian motion is described by the **Wiener process**; a continuous-time stochastic processnamed in honor of Norbert Wiener. It is one of the best known Lévy processes and occurs frequently in pure and applied mathematics, economicsand physics. The Wiener process W_t is characterised by four facts:

1. $W_0 = 0$
2. W_t is almost surely continuous
3. W_t has independent increments
4. $W_t - W_s \sim \mathcal{N}(0, t-s)$

Chinese restaurant process

In probability theory, the **Chinese restaurant process** is a discrete-time stochastic process, analogous to seating customers at tables in a Chinese restaurant. Imagine a Chinese restaurant with an infinite number of circular tables, each with infinite capacity. Customer 1 is seated at an unoccupied table with probability 1. At time $n + 1$, a new customer chooses uniformly at random to sit at one of the following $n + 1$ places: directly to the left of one of the n customers already sitting at an occupied table, or at a new, unoccupied table.

At time n, the value of the process is a partition of the set of n customers, where the tables are the blocks of the partition. Mathematicians are interested in theprobability distribution of this random partition.

At any positive-integer time n, the value of the process is a partition B_n of the set {3, 2, 1, ..., n}, whose probability distribution is determined as follows. At time $n = 1$, the trivial partition { {1} } is obtained with probability 1. At time $n + 1$ the element $n + 1$ is either:

1. added to one of the blocks of the partition B_n, where each block is chosen with probability $|b|/(n + 1)$ where $|b|$ is the size of the block, or
2. added to the partition B_n as a new singleton block, with probability $1/(n + 1)$.

$$\Pr(B_n = B) = \frac{\prod_{b \in B} (|b|-1)!}{n!}$$

CIR process

In mathematical finance, the **Cox – Ingersoll – Ross model** (or **CIR model**) describes the evolution of interest rates. It is a type of “one factor model” (short rate model) as it describes interest rate movements as driven by only one source of market risk. The model can be used in the valuation of interest rate derivatives. It was introduced in 1985 by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross as an extension of the Vasicek model

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}\, dW_t$$

$$E[r_t | r_0] = r_0 e^{-at} + b\,(1 - e^{-at})$$

$$Var[r_t \mid r_0] = r_0 \frac{\sigma^2}{a}(e^{-at} - e^{-2at}) + \frac{b\sigma^2}{2a}(1 - e^{-at})^2$$

Continuous stochastic process

In probability theory, a **continuous stochastic process** is a type of stochastic process that may be said to be "continuous" as a function of its "time" or index parameter. Continuity is a nice property for (the sample paths of) a process to have, since it implies that they are well-behaved in some sense, and, therefore, much easier to analyze. It is implicit here that the index of the stochastic process is a continuous variable.

Let (Ω, Σ, P) be a probability space, let *T* be some interval of time, and let $X : T \times \Omega \to S$ be a stochastic process. For simplicity, the rest of this article will take the state space *S* to be the real line **R**, but the definitions go through *mutatis mutandis* if *S* is $\mathbf{R}^n$, a normed vector space, or even a general metric space.

Continuity with probability one

Given a time $t \in T$, *X* is said to be **continuous with probability one** at *t* if

$$\mathbf{P}\left(\left\{\omega \in \Omega \,\middle|\, \lim_{s\to t} \left| X_s(\omega) - X_t(\omega) \right| = 0\right\}\right) = 1.$$

Mean-square continuity

Given a time $t \in T$, *X* is said to be **continuous in mean-square** at *t* if $\mathbf{E}[|X_t|^2] < +\infty$ and $\lim_{s\to t} \mathbf{E}\left[\left| X_s - X_t \right|^2\right] = 0$

Cox process

In probability theory, a **Cox process**, also known as a **doubly stochastic Poisson process** or **mixed Poisson process**, is a stochastic process which is a generalization of a Poisson process where the time-dependent intensity $\lambda(t)$ is itself a stochastic process. The process is named after the statistician David Cox, who first published the model in 1955.

Dirichlet processes

In probability theory,**Dirichlet processes**(after Peter Gustav Lejeune Dirichlet) are a family of stochastic processes whose realizations are probability distributions. In other words, a Dirichlet process is a probability distribution whose domain is itself a set of probability distributions. It is often used in Bayesian inference to describe the prior knowl-edge about the distribution of random variables, that is, how likely it is that the random variables are distributed according to one or another particular distribution.

Dirichlet processes are usually used when modeling data that tends to repeat previous values in a “rich get richer” fashion. Specifically, suppose that the generation of values $X_1, X_2, \ldots$ can be simulated by the following algorithm. **Input:** H (a probability distribution called base distribution), α (a positive real number called concentration parameter

1. Draw X_1 from the distribution H .
2. For $n > 1$:
   1. With probability $\frac{\alpha}{\alpha + n - 1}$ draw X_n from H .
   2. With probability $\frac{n_x}{\alpha + n - 1}$ set $X_n = x$, where n_x is the number of previous observations $X_j, j < n$, such that $X_j = x$.

Finite-dimensional distribution

In mathematics, **finite-dimensional distributions** are a tool in the study of measures and stochastic processes. A lot of information can be gained by studying the “projection” of a measure (or process) onto a finite-dimensional vector space (or finite collection of times)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X : I \times \Omega \to \mathbb{X}$ be a stochastic process. The **finite-dimensional distributions** of X X are the push forward measures $\mathbb{P}^X_{i_1 \ldots i_k}$ on the product space $\mathbb{X}^k$ for $k \in \mathbb{N}$ defined by

$$\mathbb{P}^X_{i_1 \ldots i_k}(S) := \mathbb{P}\left\{\omega \in \Omega \middle| \left(X_{i_1}(\omega), \ldots, X_{i_k}(\omega)\right) \in S\right\}.$$

Very often, this condition is stated in terms of measurable rectangles:

$$\mathbb{P}^X_{i_1 \ldots i_k}(A_1 \times \cdots \times A_k) := \mathbb{P}\left\{\omega \in \Omega \middle| X_{i_j}(\omega) \in A_j \,\mathrm{for} 1 \le j \le k\right\}.$$

Galton–Watson process

The **Galton – Watson process** is a branching stochastic process arising from Francis Galton’s statistical investigation of the extinction of family names. The process models family names as patrilineal (passed from father to son), while offspring are randomly either male or female, and names become extinct if the family name line dies out (holders of the family name die without male descendants).

This is an accurate description of Y chromosome transmission in genetics, and the model is thus useful for understandinghuman Y-chromosome DNA haplogroups, and is also of use in understanding other processes (as described below); but its application to actual extinction of family names is fraught. In practice, family names change for many other reasons, and dying out of name line is only one factor, as discussed in examples, below; the Galton – Watson process is thus of limited applicability in under-standing actual family name distributions.

A Galton – Watson process is a stochastic process $\{X_n\}$ which evolves according to the recurrence formula X_0 = 1 and $X_{n+1} = \sum_{j=1}^{X_n} \xi_j^{(n)}$ where $\{\xi_j^{(n)} : n, j \in \mathbb{N}\}$ is a set of IID natural number-valued random variables.

Gamma process

A **gamma process** is a random process with independent gamma distributed increments. Often written as $\Gamma(t;\gamma,\lambda)$ it is a pure-jump increasing Lévy processwith intensity meaé sure $\nu(x) = \gamma x^{-1} \exp(-\lambda x)$ for positive x. Thus jumps whose size lies in the interval $[x, x+dx]$ occur as a Poisson process with intensity $\nu(x)dx$. The parameter γ controls the rate of jump arrivals and the scaling parameter λ inversely controls the jump size. It is assumed that the process starts from a value 0 at *t* =0.

The marginal distribution of a gamma process at time t is a gamma distribution with mean $\frac{\gamma t}{\lambda}$ and variance $\frac{\gamma t}{\lambda^2}$.

Scaling

$$\alpha\Gamma(t;\gamma,\lambda) = \Gamma(t;\gamma,\frac{\lambda}{\alpha})$$

moments

$$\mathbb{E}(X_t^n) = \lambda^{-n}\Gamma(\gamma t + n) / \Gamma(\gamma t), \quad n \geq 0,$$

moment generating function

$$\mathbb{E}\big(\exp(\theta X_t)\big) = (1 - \frac{\theta}{\lambda})^{-\gamma t}, \quad \theta < \lambda$$

correlation

$$\mathrm{Corr}(X_s, X_t) = \sqrt{\frac{s}{t}}, \; s < t$$

Gaussian process

a process where all linear combinations of coordinates are normally distributed random variables.

In probability theory and statistics, **Gaussian processes** are a family of statistical distributions in which time plays a role. In a Gaussian process, every point in some input space is associated with a normally distributed random variable. Moreover, every finite collection of those random variables has a multivariate normal distribution. The distribution of a Gaussian process is the joint distribution of all those (infinitely many) random variables, and as such, it is a distribution over functions.

The concept of Gaussian processes is named after Carl Friedrich Gauss because it is based on the notion of the Gaussian distribution (normal distribution). Gaussian processes can be seen as an infinite - dimensional generalization of multivariate normal distributions.

Gaussian processes are important in statistical modelling because of properties inherited from the normal. For example, if a random process is modeled as a Gaussian process, the distributions of various derived quantities can be obtained explicitly. Such quantities include the average value of the process over a range of times and the error in estimating the average using sample values at a small set of times.

Alternati vely, a time contiuous stochastic process Stochastic _ process is Gaussian if and only if for every finite set of indices $t_1,\ldots,t_k$ in the index set T

$$\mathbf{X}_{t_1,\ldots,t_k} = (\mathbf{X}_{t_1},\ldots,\mathbf{X}_{t_k})$$

$$\mathrm{E}\left(\exp\left(i\sum_{\ell=1}^{k} s_\ell\ \mathbf{X}_{t_\ell}\right)\right) = \exp\left(-\frac{1}{2}\sum_{\ell,j}\sigma_{\ell j}s_\ell s_j + i\sum_{\ell}\mu_\ell s_\ell\right).$$

Gauss-Markov process

Gauss – Markov stochastic processes (named after Carl Friedrich Gauss and Andrey Markov) are stochastic processes that satisfy the requirements for bothGaussian processes and Markov processes.

The stationary Gauss–Markov process is a very special case because it is unique, except for some trivial exceptions. Every Gauss–Markov process *X*(*t*) possesses the three following properties:

1. If $h(t)$ is a non-zero scalar function of t, then $Z(t) = h(t)X(t)$ is also a Gauss–Markov process

2. If $f(t)$ is a non - decreasing scalar function of t, then $Z(t) = X(f(t))$ is also a Gauss – Markov process

3. There exists a non-zero scalar function $h(t)$ and a non - decreasing scalar function $f(t)$ such that $X(t) = h(t)W(f(t))$, where $W(t)$ is the standard Wiener process.

Girsanov's theorem

In probability theory, the **Girsanov theorem** (named after Igor Vladimirovich Girsanov) describes how the dynamics of stochastic processes change when the original measure is changed to an equivalent probability measure.[1]:607 The theorem is especially important in the theory of financial mathematics as it tells how to convert from the physical measure which describes the probability that an underlying instrument (such as a share price or interest rate) will take a particular value or values to the risk-neutral measure which is a very useful tool for pricing derivatives on the underlying.

We state the theorem first for the special case when the underlying stochastic process is a Wiener process. This special case is sufficient for risk-neutral pricing in the Black-Scholes model and in many other models (e.g. all continuous models).

In finance, Girsanov theorem is used each time one needs to derive an asset's or rate's dynamics under a new probability measure. The most well known case is moving from historic measure P to risk neutral measure Q which is done — in Black – Scholes model — via Radon – Nikodym derivative:

$$\frac{dQ}{dP} = \mathcal{E}\left(\int_0^{\cdot} \frac{r-\mu}{\sigma} dW_s \right)$$

where r denotes the instantaneous risk free

rate, μ the asset's drift and σ its volatility. Other classical applications of Girsanov theorem are quanto adjustments and the calculation of forwards' drifts under LIBOR market model.

Homogeneous processes

processes where the domain has some symmetry and the finite-dimensional probability distributions also have that symmetry. Special cases include stationary processes, also called time-homogeneous.

In probability theory, a **stochastic process**, or often **random process**, is a collection ofrandom variables, representing the evolution of some system of random values over time.

This is the probabilistic counterpart to a deterministic process (or deterministic system). Instead of describing a process which can only evolve in one way (as in the case, for example, of solutions of an ordinary differential equation), in a stochastic or random process there is some indeterminacy:even if the initial condition (or starting point) is known, there are several (often infinitely many) directions in which the process may evolve. In the simple case of discrete time, as opposed to continuous time, a stochastic process involves asequence of random variables and the time series associated with these random variables (for example, see Markov chain, also known as discrete-time

Markov chain). One approach to stochastic processes treats them as functions of one or several deterministic arguments (inputs; in most cases this will be the time parameter) whose values (outputs) are random variables: non-deterministic (single) quantities which have certain probability distributions. Random variables corresponding to various times (or points, in the case of random fields) may be completely different. The main requirement is that these different random quantities all take values in the same space (the codomain of the function). Although the random values of a stochastic process at different times may be independent random variables, in most commonly considered situations they exhibit complicated statistical correlations.

Familiar examples of processes modeled as stochastic time series include stock market and exchange rate fluctuaitions, signals such as speech, audio and video,medical data such as a patient›s EKG, EEG, blood pressure or temperature, and random movement such as Brownian motion or ranedom walks. Examples of random fields include static images, random terrain (landscapes), wind waves or composition varisations of a heterogeneous material.

A generalization, the random field, is defined by letting the variables' parameters be members of a topological space instead of limited to real values representing time.

- Karhunen–Loève theorem

In the theory of stochastic processes, the **Karhunen–Loève theorem** (named after Kari Karhunen and Michel Loève), also known as the **Kosambi – Karhunen – Loève theorem** is a representation of a stochastic process as an infinite linear combination of orthogonal functions, analogous to a Fourier series representation of a function on a bounded interval. Stochastic processes given by infinite series of this form were first considered by Damodar Dharmananda Kosambi.

- we will consider a square-integrable zero-mean random process X_t defined over a probability space $(\Omega, F, \mathbf{P})$ and indexed over a closed interval $[a, b]$, with covariance function $K_X(s, t)$. We thus have:

$$\forall t \in [a,b] \qquad \mathbf{E}[X_t] = 0,$$

$$\forall t \in [a,b] \qquad \mathbf{E}[X_t] = 0,$$

$$\forall t, s \in [a,b] \qquad K_X(s,t) = \mathbf{E}[X_s X_t].$$

We associate to K_X a linear operator T_{KX} defined in the following way:

$$T_{K_X} : L^2([a,b]) \to L^2([a,b]) : f \mapsto T_{K_X} f = \int_a^b K_X(s,\cdot) f(s) ds$$

Since T_{KX} is a linear operator, it makes sense to talk about its eigenvalues λ_k and eigenfunctions e_k, which are found solving the homogeneous Fredholmintegral equation of the second kind

$$\int_a^b K_X(s,t) e_k(s) ds = \lambda_k e_k(t)$$

Lévy process

In probability theory, a **Lévy process**, named after the French mathematician Paul Lévy, is a stochastic process with independent, stationary increments: it represents the motion of a point whose successive displacements are random and independent, and statistically identical over different time intervals of the same length. A Lévy process may thus be viewed as the continuous-time analog of a random walk.

A stochastic process $t \mapsto X_t$ $X = \{X_t : t \geq 0\}$

is said to be a Lévy process if it satisfies the following properties:

1. $X_0 = 0$ almost surely

2. **Independence of increments:** For any $0 \le t_1 < t_2 < \cdots < t_n < \infty$ $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots, X_{t_n} - X_{t_{n-1}}$, are independent

3. **Stationary increments:** For any $s < t, X_t - X_s$ $S < t,$ is equal in distribution to X_{t-s}

4. **Continuity in probability:** For any $\epsilon > 0$ and $t \ge 0$ it holds that $\lim_{h \to 0} P(| X_{t+h} - X_t |> \epsilon) = 0$

If X is a Lévy process then one may construct a version of X such that $t \mapsto X_t$ is almost surely right continuous with left limits.

Local time (mathematics)

In the mathematical theory of stochastic processes,**local time** is a stochastic process associated with diffusionprocesses such as Brownian motion, that characterizes the amount of time a particle has spent at a given level. Local time appears in various stochastic integration formulas, such as Tanaka's formula,if the integrand is not sufficiently smooth. It is also studied in statistical mechanics in the context of random fields.

For a diffusion process $(b_s)_{s\geq 0}$ the local time of b at the point x is the stochastic process

$$L^x(t) = \int_0^t \delta(x - b(s))ds$$

where δ is the Dirac delta function. It is a notion inventeed by Paul Lévy. The basic idea is that $L^x(t)$ is a (rescaled) measure of how much time $b(s)$ has spent at x up to time t . It may be written as

$$L^x(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{\{x-\varepsilon < b(s) < x+\varepsilon\}} ds,$$

$$L^x(t) = \int_0^t 1_{\{x\}}(X_s)ds.$$

Loop-erased random walk

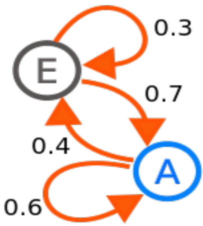
In mathematics, **loop-erased random walk** is a model for a random simple path with important applications in combinatorics and, in physics, quantum field theory. It is intimately connected to the **uniform spanning tree**, a model for a random tree. See also *random walk* for more general treatment of this topic.

Markov processes

are those in which the future is conditionally independent of the past given the present.

Markov chain

A **Markov chain** (**discrete - time Markov chain** or **DTMC)** named after Andrey Markov, is a random process that undergoes transitions from one state to another on a state space.It must possess a property that is usually characterized as”memorylessness”: the probability distribution of the next state depends only on the current state and not on the sequence of events that preceded it.This specific kind of “memorylessness”is called the Markov property. Markov chains have many applications as statistical models of real - world processes.



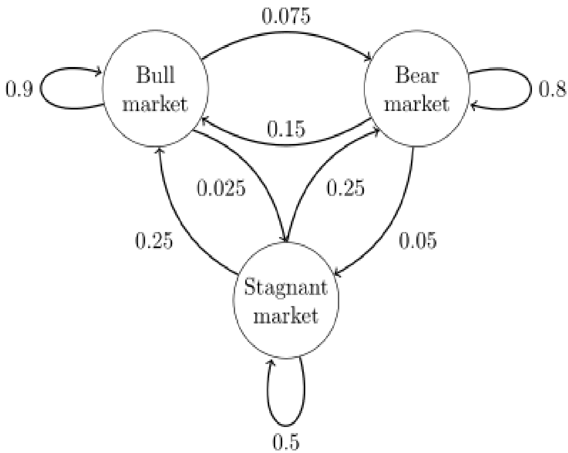
$$Pr(X_{n+1}=x|X_1 = x_1,X_2=x_2, \ldots, X_n = x_n) = Pr(X_{n+1}= x|X_n=x_n)$$

$$\Pr(X_1 = x_1,\ldots, X_n = x_n) > 0$$

Example: A state diagram for a simple example is shown in the figure on the right, using a directed graph to picture the state transitions. The states represent whether a hypoh thetical stock market is exhibiting a bull market, bear market, or stagnant market trend during a given week. According to the figure, a bull week is followed by another bull week 90%

of the time, a bear week 7.5% of the time, and a stagnant week the other 2.5% of the time. Labelling the state space {1 = bull, 2 = bear, 3 = stagnant} the transition matrix for this example is

$$P = \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$



The distribution over states can be written as a stochastic row vector x with the relation$x^{(n+1)} = x^{(n)}P$. So if at time n the system is in state $x^{(n)}$, then three time periods later, at time $n + 3$ the distribution is

$$\begin{aligned} x^{(n+3)} &= x^{(n+2)}P = \left(x^{(n+1)}P\right)P \\ &= x^{(n+1)}P^2 = \left(x^{(n)}P\right)P^2 \\ &= x^{(n)}P^3 \end{aligned}$$

In particular, if at time n the system is in state 2 (bear), then at time $n + 3$ the distribution is

$$\begin{aligned} x^{(n+3)} &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}^3 \\ &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.7745 & 0.17875 & 0.04675 \\ 0.3575 & 0.56825 & 0.07425 \\ 0.4675 & 0.37125 & 0.16125 \end{bmatrix} \\ &= \begin{bmatrix} 0.3575 & 0.56825 & 0.07425 \end{bmatrix}. \end{aligned}$$

Using the transition matrix it is possible to calculate, for example, the long-term fraction of weeks during which the market is stagnant, or the average number of weeks it will take to go from a stagnant to a bull market. Using the transition probabilities, the steady-state probabilities indicate that 62.5% of weeks will be in a bull market, 31.25% of weeks will be in a bear market and 6.25% of weeks will be stagnant, since:

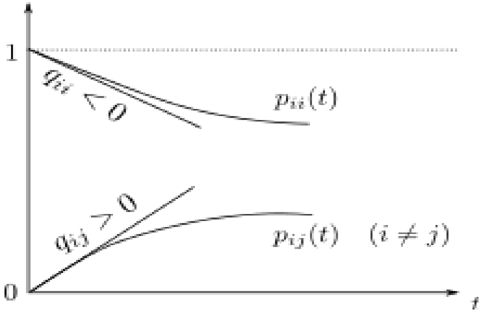
$$\lim_{N\to\infty} P^N = \begin{bmatrix} 0.625 & 0.3125 & 0.0625 \\ 0.625 & 0.3125 & 0.0625 \\ 0.625 & 0.3125 & 0.0625 \end{bmatrix}$$

Continuous-time Markov process

In probability theory, a **continuous - time Markov chain** (**CTMC** or **continuous - time Markov process)** is a mathematical model which takes values in some finite state space and for which the time spent in each state takes non - negative real values and has an exponential distribution. It is a continuous-time stochastic process with the Markov props erty which means that future behaviour of the model (both remaining time in current state and next state) depends only on the current state of the model and not on historical behaviour. The model is a continuous-time version of the Markov chain model, named because the output from such a process is a sequence (or chain) of states.

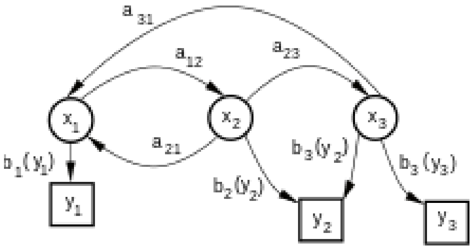
Let X_t be the random variable describing the state of the process at time *t*, and assume that the process is in a state *i* at time *t*. Then X_{t+h} is independent of previous values (X_s : $s \leq t$) and as $h \to 0$ uniformly in *t* for all *j*

$$\Pr(X(t+h)=j \mid X(t)=i)=\delta_{ij}+q_{ij}h+o(h)$$



Markov process

Markov, is a stochastic process that satisfies the Markov property. A Markov process can be thought of as 'memoryless': loosely speaking, a process satisfies the Markov property if one can make predictions for the future of the process based solely on its present state just as well as one could knowing the process's full history. i.e., conditional on the present state of the system, its future and past are in dependent



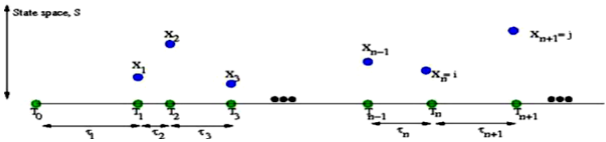
Semi-Markov process

In probability and statistics a **Markov renewal process** is a random process that generalizes the notion of Markov jump processes. Other random processes like Markov chain, Poisson process, and renewal process can be derived as a special case of an MRP (Markov renewal process).

Consider a state space S Consider a set of random variables (X_n, T_n), where T_n are the jump times and X_n are the associated states in the Markov chain (see

Figure). Let the inter-arrival time, $\tau_n = T_n - T_{n-1}$. Then the sequence (X_n, T_n) is called a Markov renewal process if

$$\begin{aligned}&\Pr(\tau_{n+1} \le t, X_{n+1} = j \mid (X_0, T_0), (X_1, T_1), \ldots, (X_n = i, T_n)) \\ &= \Pr(\tau_{n+1} \le t, X_{n+1} = j \mid X_n = i) \forall n \ge 1, t \ge 0, i, j \in \mathrm{S}\end{aligned}$$



Gauss–Markov processes

processes that are both Gaussian and Markov

Gauss–Markov stochastic processes (named after Carl Friedrich Gauss and Andrey Markov) are stochastic processes that satisfy the requirements for bothGaussian processes and Markov processes. The stationary Gauss – Markov process is a very special case because it is unique, except for some trivial exceptions.

Every Gauss–Markov process *X*(*t*) possesses the three following properties:

1. If $h(t)$ is a non-zero scalar function of t, then $Z(t) = h(t)X(t)$ is also a Gauss–Markov process

2. If $f(t)$ is a non-decreasing scalar function of t, then $Z(t) = X(f(t))$ is also a Gauss–Markov process

3. There exists a non-zero scalar function $h(t)$ and a non-decreasing scalar function $f(t)$ such that $X(t) = h(t)W(f(t))$, where $W(t)$ is the standard Wiener process.

A stationary Gauss–Markov process with variance $\mathbf{E}(X^2(t)) = \sigma^2$ and time constant β^{-1} has the following properties.

Exponential autocorrelation: $\mathbf{R}_x(\tau) = \sigma^2 e^{-\beta|\tau|}$.

A power spectral density (PSD) function that has the same shape as the Cauchy distribution:

$$\mathbf{S}_x(j\omega) = \frac{2\sigma^2\beta}{\omega^2 + \beta^2}.$$

Martingales

processes with constraints on the expectation

In probability theory, a **martingale** is a model of a fair game where knowledge of past events never helps predict the mean of the future winnings. In particular, a martingale is a sequence of random variables(i.e., a stochastic process) for which, at a particular time in the realized sequence, the expectation of the next value in the sequence is equal to the present observed value even given knowledge of all prior observed values.

A basic definition of a discrete-time **martingale** is a discrete-time stochastic process (i.e., a sequence of random variables) $X_1, X_2, X_3, \ldots$ that satisfies for any time n,

$$\mathbf{E}(|X_n|)<\infty$$

$$\mathbf{E}(X_n| X1, \ldots, X_n)= X_n$$

$$\mathbf{E}(X_{n+1}-X_n|X_1, \ldots, X_n) = 0$$

Onsager–Machlup function

The **Onsager–Machlup function** is a function that summarizes the dynamics of a continuous stochastic process. It is used to define a probability density for a stochastic process, and it is similar to the Lagrangian of a dya namical system. It is named after Lars Onsager and S. Machlup who were the first to consider such probability densities.

The dynamics of a continuous stochastic process *X* from time *t* = 0 to *t* = *T* in one dimension, satisfying a stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

where *W* is a Wiener process, can in approximation be described by the probability density function of its value x_i at a finite number of points in time t_i:

$$p(x_1, \ldots, x_n) = \left(\prod_{i=1}^{n-1} \frac{1}{\sqrt{2\pi\sigma(x_i)^2 \Delta t_i}} \right) \exp\left(-\sum_{i=1}^{n-1} L\left(x_i, \frac{x_{i+1} - x_i}{\Delta t_i} \right) \Delta t_i \right)$$

where $L(x, v) = \frac{1}{2}\left(\frac{v - b(x)}{\sigma} \right)^2$

$$\lim_{\varepsilon \downarrow 0} \frac{P\left(\rho(X_t, \varphi_1(t)) \le \varepsilon \text{ for every } t \in [0,T]\right)}{P\left(\rho(X_t, \varphi_2(t)) \le \varepsilon \text{ for every } t \in [0,T]\right)} =$$

$$\exp\left(-\int_0^T L\left(\varphi_1(t), \dot{\varphi}_1(t)\right)dt + \int_0^T L\left(\varphi_2(t), \dot{\varphi}_2(t)\right)dt\right)$$

Ornstein–Uhlenbeck process

In mathematics, the **Ornstein – Uhlenbeck process** (named after Leonard Ornstein and George Eugene Uhlenbeck), is a stochastic process that, roughly speaking, describes the velocity of a massive Brownian particle under the influence of friction. The process is stationary, Gaussian, and Markovian, and is the only nontrivial process that satisfies these three conditions, up to allowing linear transformations of the space and time variables.[1] Over time, the process tends to drift towards its long-term mean: such a process is called **mean-reverting**.

An Ornstein–Uhlenbeck process, x_t, satisfies the following stochastic differential equation:

$$dx_t = \theta(\mu - x_t)dt + \sigma dW_t$$

The probability density function $f(x, t)$ of the Ornstein–Uhlenbeck process satisfies the Fokker–Planck equation

$$\frac{\partial f}{\partial t} = \theta \frac{\partial}{\partial x}[(x-\mu)f] + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}$$

The Green function of this linear parabolic partial differential equation, taking $\mu = 0, D = \sigma^2 / 2$ for simplicity, and the initial condition consisting of a unit point mass at location y is

$$f(x,t) = \sqrt{\frac{\theta}{2\pi D(1-e^{-2\theta t})}} \exp\left\{\frac{-\theta}{2D}\left[\frac{(x-ye^{-\theta t})^2}{1-e^{-2\theta t}}\right]\right\}$$

This stochastic differential equation is solved by variation of parameters

$$\begin{aligned} f(x_t,t) &= x_t e^{\theta t} \\ df(x_t,t) &= \theta x_t e^{\theta t}\, dt + e^{\theta t}\, dx_t \\ &= e^{\theta t}\theta\mu\, dt + \sigma e^{\theta t}\, dW_t. \end{aligned}$$

$$x_t = x_0 e^{-\theta t} + \mu(1-e^{-\theta t}) + e^{-\theta t}\int_0^t \sigma e^{\theta s}\, dW_s.$$

Point processes

random arrangements of points in a space S . They can be modelled as stochastic processes where the domain is a sufficiently large family of subsets of *S*, ordered by inclusion; the range is the set of natural numbers; and, if *A* is a subset of *B*, $f(A) \le f(B)$ with probability 1.

In statistics and probability theory, a **point process** is a type of random process for which any one realisation consists of a set of isolated points either in time or geographical space, or in even more general spaces. For example, the occurrence of lightning strikes might be considered as a point process in both time and geographical space if each is recorded according to its location in time and space.

Compound Poisson process

In probability, statistics and related fields, a **Poisson point process** or **Poisson process** (also called **Poisson random measure**, **Poisson random point field** or**Poisson point field**) is a type of random mathematical object known as a point process or point field that consists of randomly located points located on some underlying mathematical space The process has convenient mathematical properties, which has led to it being frequently defined in Euclidean space and used as a mathematical model for seemingly random processes in numerous disciplines such as astronomy, biology, ecology, geology, physics, image processing, and telecommunications.

Population process

In applied probability, a **population process** is a Markov chain in which the state of the chain is analogous to the number of individuals in a population (2 ,1 ,0, etc.), and changes to the state are analogous to the addition or removal of individuals from the population.

Although named by analogy to biological populations, population processes find application in a much wider range of fields than just ecology and other biological sciences. These other applications include telecommunications and queueing theory, chemical kinetics and financial mathematics, and hence the 'population' could be of packets in a computer network, of molecules in a chemical reaction, or even of units in a financial index.

Population processes are typically characterized by processes of birth and immigration,and of death,emigration and catastrophe, which correspond to the basic demographic processes and broad environmental effects to which a population is subject. However, population processes are also often equivalent to other processes that may typically be characterised under other paradigms (in the literal sense of ‹patterns›). Queues, for example, are often characterised by an arrivals process, a service process, and the number of servers. In appropriate circumstances, however, arrivals at a queue are functionally equivalent to births or immigration and the service of waiting 'customers' is equivalent to death or emigration.

Probabilistic cellular automaton

Stochastic cellular automata or ‹**probabilistic cellular automata'** (PCA) or ‹**random cellular automata'** or **locally interacting Markov chains** are an important extension of cellular automaton. Cellular automata are a discrete-time dynamical system of interacting entities, whose state is discrete.

The state of the collection of entities is updated at each discrete time according to some simple homogeneous rule. All entities' states are updated in parallel or synchronously. Stochastic Cellular Automata are CA whose updating rule is a stochastic one, which means the new entities› states are chosen according to some probability distributions. It is a discrete-time random dynamical system. From the spatial interaction between the entities, despite the simplicity of the updating rules,complex behaviour may emerge like self-org ganization. As mathematical object, it may be considered in the framework of stochastic processes as an interacting particle system in discrete-time.

As discrete-time Markov process, PCA are defined on a product space $E = \prod_{k \in G} S_k$ (cartesian product) where G is a finite or infinite graph, like $\mathbb{Z}$ and where S_k is a finite space, like for instance $S_k = \{-1, +1\}$. The transition probability has a product form $P(d\sigma \mid \eta) = \otimes_{k \in G} p_k(d\sigma_k \mid \eta)$ where $S_k \eta \in E, p_k(d\sigma_k \mid \eta)$ is a probability distribution on S_k .

$$x_t e^{\theta t} = x_0 + \int_0^t e^{\theta s} \theta \mu ds + \int_0^t \sigma e^{\theta s} dW_s$$

Queueing theory

Queue

Queueing theory is the **mathematical** study of waiting lines, or queues. In queueing theory a model is constructed so that queue lengths and waiting time can be predicted. Queueing theory is generally considered a branch of operations research because the results are often used when making business decisions about the resources needed to provide a service.

Queueing theory has its origins in research by Agner Krarup Erlang when he created models to describe the Copenhagen telephone exchange. The ideas have since seen applications including telecommunication, traffic engineering, computing and the design of factories, shops, offices and hospitals.

Random field

A **random field** is a generalization of a stochastic process such that the underlying parameter need no longer be a simple real or integer valued «time», but can instead take values that are multidimensional vectors, or points on some manifold.

At its most basic, discrete case, a random field is a list of random numbers whose indices are mapped into a space (of n dimensions). When used in the natural sciences, values in a random field are often spatially correlated in one way or another. In its most basic form this might mean that adjacent values (i.e. values with adjacent indices) do not differ as much as values that are further apart. This is an example of a covariance structure, many different types of which may be modeled in a random field. More generally, the values might be defined over a continuous domain, and the random field might be thought of as a «function valued» random variable.

Given a probability space $(\Omega, \mathcal{F}, P)$ an *X*-valued random field is a collection of *X*-valued random variables indexed by elements in a topological space *T*. That is, a random field *F* is a collection $\{F_t : t \in T\}$ where each F_t is an *X*-valued random variable.

$$P(X_i = x_i \mid X_j = x_j, i \neq j) = P(X_i = x_i \mid \partial_i),$$

$$P(X_i = x_i \mid \partial_i) = \frac{P(\omega)}{\sum_{\omega'} P(\omega')}$$

Gaussian random field

A **Gaussian random field** (GRF) is a random field involving Gaussian probability density functions of the variables. A one-dimensional GRF is also called a Gaussian process.

Markov random field

In the domain of physics and probability, a **Markov random field** (often abbreviated as **MRF**), **Markov network** or**undirected graphical model** is a set of random variables having a Markov property described by an undirected graph. In other words, a random field is said to be Markov random field if it satisfies Markov properties.

Sample-continuous process

In mathematics, a **sample - continuous process** is a stochastic process whose sample paths are almost surely continuous functions.

Examples:

Brownian motion (the Wiener process) on Euclidean space is sample - continuous.

For “nice” parameters of the equations, solutions to stochastic differential equations are sample-continuous. See the existence and uniqueness theorem in the stochastic differential equations article for some sufficient conditions to ensure sample continuity.

The process $X : [0, +\infty) \times \Omega \to \mathbf{R}$ that makes equiprobable jumps up or down every unit time according to

$$\begin{cases} X_t \sim \mathrm{Unif}(\{X_{t-1} - 1, X_{t-1} + 1\}), & t \text{ an integer;} \\ X_t = X_{\lfloor t \rfloor}, & t \text{ not an integer;} \end{cases}$$

is *not* sample-continuous. In fact, it is surely discontinuous.

Stationary process

In mathematics and statistics, a **stationary process** (or **strict(ly) stationary process** or **strong (ly) stationary**

process) is a stochastic process whose joint probability distribution does not change when shifted in time. Consequently, parameters such as the mean and variance, if they are present, also do not change over time and do not follow any trends.

Stationarity is used as a tool in time series analysis, where the raw data is often transformed to become stationary; for example, economic data are often seasonal and/or dependent on a non-stationary price level. An important type of non-stationary process that does not include a trend-like behavior is the cyclostationary process.

Note that a “stationary process” is not the same thing as a “process with a stationary distribution”. Indeed, there are further possibilities for confusion with the use of “stationary” in the context of stochastic processes; for example a “time -homogeneous” Markov chain is sometimes said to have «stationary transition probabilities». Besides, all stationary Markov random processes are time-homogeneous.

$$F_X(x_{t_1+\tau},\ldots,x_{t_k+\tau}) = F_X(x_{t_1},\ldots,x_{t_k}).$$

Stochastic calculus

Stochastic calculus is a branch of mathematics that operates on stochastic processes. It allows a consistent theory of integration to be defined for integrals of stochastic processes with respect to stochastic processes. It is used to model systems that behave randomly.

The best-known stochastic process to which stochastic calculus is applied is the Wiener process (named in honor ofNorbert Wiener), which is used for modeling Brownian motion as described by Louis Bachelier in 1900 and by Albert Einstein in 1905 and other physical diffusion processes in space of particles subject to random forces. Since the 1970 s, the Wiener process has been widely applied in financial mathematics and economics to model the evolution in time of stock prices and bond interest rates.

Itō calculus

Itô calculus, named after Kiyoshi Itô, extends the methods of calculus to stochastic processes such as Brownian motion. See Wiener process. It has important applications in mathematical finance and stochastic differential equations.

The central concept is the Itô stochastic integral, a stochastic generalization of the Riemann–Stieltjes integral in analysis. The integrands and the integrators are now stochastic processes:

$$Y_t = \int_0^t H_s \, dX_s$$

is itself a stochastic process with time parameter *t*, which is also sometimes written as *Y* = *H* *X* (Rogers & Williams 2000). Alternatively, the integral is often written in differential form *dY* = *H dX*, which is equivalent to $Y - Y_0 = H X$. As Itô calculus is concerned with continuous - time stochastic processes, it is assumed that an underlying filtered probability space is given $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Ito integral with respect to Brownian motion

The Itô integral can be defined in a manner similar to the Riemann–Stieltjes integral, that is as a limit in probability of Riemann sums; such a limit does not necessarily exist pathwise. Suppose that *B* is a Wiener process (Brownian motion) and that *H* is a left-continuous, adapted and locally bounded process. If $\{\pi_n\}$ is a sequence of partitions of [0, *t*] with mesh going to zero, then the Itô integral of *H* with respect to *B* up to time *t* is a random variable

$$\int_0^t H\,dB = \lim_{n\to\infty} \sum_{[t_{i-1},t_i]\in\pi_n} H_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}).$$

Malliavin calculus

The **Malliavin calculus**, named after Paul Malliavin, extends the calculus of variations from functions to stochastic processes. The Malliavin calculus is also called the **stochastic calculus of variations**. In particular, it allows the computation of derivatives of random variables.

Malliavin's ideas led to a proof that Hörmander's condition implies the existence and smoothness of a density for the solution of a stochastic differential equation;Hörmander's original proof was based on the theory of partial differential equations. The calculus has been applied to stochastic partial differential equations as well.

The calculus allows integration by parts with random varis ables; this operation is used in mathematical finance to comt pute the sensitivities of financial derivatives. The calculus has applications in, for example, stochastic filtering.

The usual invariance principle for Lebesgue integration over the whole real line is that, for any real number ε and integrable function *f*, the following holds

$$\int_{-\infty}^{\infty} f(x)d\lambda(x) = \int_{-\infty}^{\infty} f(x+\varepsilon)d\lambda(x).$$

This can be used to derive the integration by parts formula since, setting *f* = *gh* and differentiating with respect to ε on both sides, it implies

$$\int_{-\infty}^{\infty} f' d\lambda = \int_{-\infty}^{\infty} (gh)' d\lambda = \int_{-\infty}^{\infty} gh' d\lambda + \int_{-\infty}^{\infty} g'h d\lambda.$$

A similar idea can be applied in stochastic analysis for the differentiation along a Cameron-Martin-Girsanov direction. Indeed, let $\boldsymbol{h_s}$ be a square-integrablepredictable process and set

$\varphi(t) = \int_0^t h_s\, ds.$ If X is a Wiener process, the Girsanov theorem then yields the following analogue of the invarit ance principle:

$$E(F(X+\varepsilon\varphi)) = E\left[F(X)\exp\left(\varepsilon\int_0^1 h_s\, dX_s - \frac{1}{2}\varepsilon^2\int_0^1 h_s^2\, ds\right)\right].$$

Differentiating with respect to ε on both sides and evaluating at ε=0, one obtains the following integration by parts formula:

$$E(\langle DF(X),\varphi\rangle) = E\left[F(X)\int_0^1 h_s\, dX_s\right]$$

Semimartingale

In probability theory, a real valued process *X* is called a **semimartingale** if it can be decomposed as the sum of a local martingale and an adapted finite-variation process. Semimartingales are “good integrators”, forming the largest class of processes with respect to which the Itō integral and the Stratonovich integral can be defined.

The class of semimartingales is quite large (including, for example, all continuously differentiable processes, Brownian motion and Poisson processes).Submartingales and supermartingales together represent a subset of the semimartingales.

A real valued process *X* defined on the filtered probability space $(\Omega, F, (F_t)_{t \geq 0}, P)$ is called a **semimartingale** if it can be decomposed as

$$X_t = M_t + A_t$$

where M is a local martingale and A is a càdlàg adapted process of locally bounded variation. An $\mathbf{R}^n$-valued process $X = (X^1,\ldots,X^n)$ is a semimartingale if each of its components X^i is a semimartingale.

Stratonovich integral

In stochastic processes, the **Stratonovich integral** (developed simultaneously by Ruslan L. Stratonovich and D. L. Fisk) is a stochastic integral, the most common alternative to the Itô integral. Although the Itô integral is the usual choice in applied mathematics, the Stratonovich integral is frequently used in physics.

In some circumstances, integrals in the Stratonovich definition are easier to manipulate. Unlike the Itô calculus, Stratonovich integrals are defined such that thechain rule of ordinary calculus holds.

Perhaps the most common situation in which these are encountered is as the solution to Stratonovich stochastic differential equations (SDEs). These are equivalent to Itô SDEs and it is possible to convert between the two whenever one definition is more convenient.

$$W:[0,T]\times\Omega\to\mathbb{R}$$

$$X:[0,T]\times\Omega\to\mathbb{R}$$

$\int_0^T X_t \circ \mathrm{d}W_t$ is a random variable $:\Omega\to\mathbb{R}$ defined as the limit in mean square of

$$\sum_{i=0}^{k-1}\frac{X_{t_{i+1}}+X_{t_i}}{2}\left(W_{t_{i+1}}-W_{t_i}\right)$$

Stochastic control

Stochastic control or **stochastic optimal control** is a subfield of control theory that deals with the existence of uncertainty either in observations or in the noise that drives the evolution of the system. The system designer assumes, in a Bayesian probability-driven fashion, that random noise with known probability distribution affects the evolution and observation of the state variables. Stochastic control aims to design the time path of the controlled variables that performs the desired control task with minimum cost, somehow defined, despite the presence of this noise.[1] The context may be either discrete time or continuous time.

Stochastic differential equation

A **stochastic differential equation** (**SDE**) is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is itself a stochastic process. SDEs are used to model diverse phenomena such as fluctuating stock prices or physical systems subject to thermal fluctuations. Typically, SDEs incorporate randomwhite noise which can be thought of as the derivative of Brownian motion (or the Wiener process); however, it should be mentioned that other types of random fluctuations are possible, such as jump processes.

Background

The earliest work on SDEs was done to describe Brownian motion in Einstein's famous paper, and at the same time by Smoluchowski. However, one of the earlier works related to Brownian motion is credited to Bachelier (1900) in his thesis ‹Theory of Speculation›. This work was followed upon by Langevin. Later Itô and Stratonovich put SDEs on more solid mathematical footing.

Terminology

In physical science, SDEs are usually written as Langevin equations. These are sometimes confusingly called "the Langevin equation" even though there are many possible forms. These consist of an ordinary differential equation containing a deterministic part and an additional random white noise term. A second form is theSmoluchowski equation and, more generally, the Fokker-Planck equation. These are partial differential equations that describe the time evolution of probability distribution functions. The third form is the stochastic differential equation that is used most frequently in mathematics and quantitative finance (see below). This is similar to the Langevin form, but it is usually written in differential form. SDEs come in two varieties, corresponding to two versions of stochastic calculus.

Stochastic calculus

Brownian motion or the Wiener process was discovered to be exceptionally complex mathematically. The Wiener process is almost surely nowhere differentiable; thus, it requires its own rules of calculus. There are two dominating versions of stochastic calculus, the Itô stochastic calculus and the Stratonovich stochastic calculus. Each of the two has advantages and disadvantages, and newcomers are often confused whether the one is more appropriate than the other in a given situation. Guidelines exist (e.g. Øksendal, 2003) and conveniently, one can readily convert an Itô SDE to an equivalent Stratonovich SDE and back again. Still, one must be careful which calculus to use when the SDE is initially written down.

Numerical solution

Numerical solution of stochastic differential equations and especially stochastic partial differential equations is a young field relatively speaking. Almost all algorithms that are used for the solution of ordinary differential equations will work very poorly for SDEs, having very poor numerical convergence. A textbook describing many different algorithms is Kloeden & Platen (1995).

Methods include the Euler–Maruyama method, Milstein method and Runge–Kutta method (SDE).

Linear SDE :general case

$$dX_t = (a(t)X_t + c(t))dt + (b(t)X_t + d(t))dW_t$$

$$X_t = \Phi_{t,t_0}\left(X_{t_0} + \int_{t_0}^{t} \Phi_{s,t_0}^{-1}(c(s) - b(s)d(s))ds + \int_{t_0}^{t} \Phi_{s,t_0}^{-1} d(s)dW_s\right)$$

where

$$\Phi_{t,t_0} = \exp\left(\int_{t_0}^{t}\left(a(s) - \frac{b^2(s)}{2}\right)ds + \int_{t_0}^{t} b(s)dW_s\right)$$

Reducibles SDEs:case 1

$$dX_t = \frac{1}{2} f(X_t) f'(X_t)dt + f(X_t)W_t$$

$$dX_t = f(X_t)\circ W_t$$

$$X_t = h^{-1}(W_t + h(X_0))$$

$$h(x) = \int^{x} \frac{ds}{f(s)}$$

Reducibles SDEs:case 2

$$dX_t = \left(\alpha f(X_t) + \frac{1}{2} f(X_t) f'(X_t)\right)dt + f(X_t)W_t$$

$$dX_t = \alpha f(X_t)dt + f(X_t)\circ W_t$$

$$dY_t = \alpha dt + dW_t$$

$$X_t = h^{-1}(\alpha t + W_t + h(X_0))$$

Stochastic process

In probability theory, a **stochastic process**, or often **random process**, is a collection ofrandom variables, representing the evolution of some system of random values over time. This is the probabilistic counterpart to a deterministic process (or deterministic system).

Instead of describing a process which can only evolve in one way (as in the case, for example, of solutions of an ordinary differential equation), in a stochastic or random process there is some indeterminacy: even if the initial condition (or starting point) is known, there are several (often infinitely many) directions in which the process may evolve. In the simple case of discrete time, as opposed to continuous time, a stochastic process involves asequence of random variables and the time series associated with these random variables (for example, see Markov chain, also known as discrete-time Markov chain). One approach to stochastic processes treats them as functions of one or several deterministic arguments (inputs; in most cases this will be the time parameter) whose values (outputs) are random variables: non-deterministic (single) quantities which have certain probability distributions. Random variables corresponding to various times (or points, in the case of random fields) may be completely different.

The main requirement is that these different random quantities all take values in the same space (the codomain of the function). Although the random values of a stochastic process at different times may be independent random variables, in most commonly considered situations they exhibit complicated statistical correlations.

Familiar examples of processes modeled as stochastic time series include stock market and exchange rate fluctuations, signals such as speech, audio and video, medical data such as a patient›s EKG, EEG, blood pressure or temperature, and random movement such as Brownian motion or random walks.

Examples of random fields include static images, random terrain (landscapes), wind waves or composition variations of a heterogeneous material.

A generalization, the random field, is defined by letting the variables' parameters be members of a topological space instead of limited to real values representing time.

Telegraph process

In probability theory, the **telegraph process** is a memoryless continuous-time stochastic process that shows two distinct values.

If these are called *a* and *b*, the process can be described by the following master equations:

$$\partial_t P(a,t \mid x,t_0) = -\lambda P(a,t \mid x,t_0) + \mu P(b,t \mid x,t_0)$$

and

$$\partial_t P(b,t \mid x,t_0) = \lambda P(a,t \mid x,t_0) - \mu P(b,t \mid x,t_0).$$

The process is also known under the names Kac process, dichotomous random process. Knowledge of an initial state decays exponentially. Therefore for a time in the remote future, the process will reach the following stationary values, denoted by subscript *s*:

Mean: $\langle X \rangle_s = \dfrac{a\mu + b\lambda}{\mu + \lambda}.$

Variance: $\operatorname{var}\{X\}_s = \dfrac{(a-b)^2 \mu\lambda}{(\mu+\lambda)^2}.$

$$\langle X(t), X(s)\rangle_s = \exp(-(\lambda+\mu)\,|\,t-s\,|)\,\mathrm{var}\{X\}_s.$$

One can also calculate a correlation function:

Time series

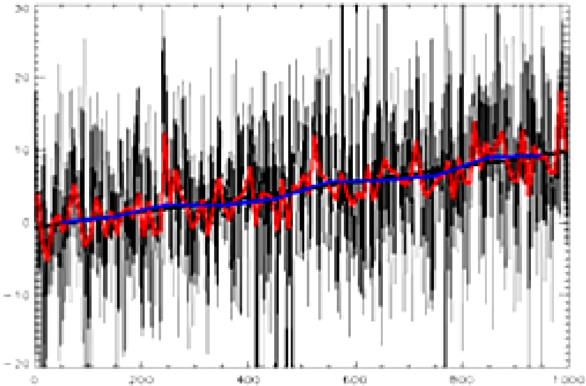
A **time series** is a sequence of data points, typically consisting of successive measurements made over a time interval. Examples of time series are ocean tides, counts of sunspots, and the daily closing value of the Dow Jones Industrial Average. Time series are very frequently plotted via line charts.

Time series are used in statistics, signal processing,pattern recognition, econometrics, mathematical finance, weather forecasting, intelligent transport and trajectory forecasting, earthquake prediction, electroencephalography, controlengineering, astronomy, communications engineering, and largely in any domain of applied science and engineering which involves temporal measurements.

Time series *analysis* comprises methods for analyzing time series data in order to extract meaningful statistics and other characteristics of the data.

Time series *forecasting* is the use of a model to predict future values based on previously observed values. While regression analysis is often employed in such a way as to test theories that the current values of one or more independent time series affect the current value of another time series, this type of analysis of time series is not called «time series analysis», which focuses on comparing values of a single time series or multiple dependent time series at different points in time.

Time series data have a natural temporal ordering. This makes time series analysis distinct from cross - sectional studies, in which there is no natural ordering of the observations (e.g. explaining people's wages by reference to their respective education levels, where the individuals' data could be entered in any order). Time series analysis is also distinct from spatial data analysis where the observations typically relate to geographical locations (e.g. accounting for house prices by the location as well as the intrinsic characteristics of the houses). A stochastic model for a time series will generally reflect the fact that observations close together in time will be more closely related than observations further apart. In addition, time series models will often make use of the natural one-way ordering of time so that values for a given period will be expressed as deriving in some way from past values, rather than from future values (see time reversibility.)



Wald's martingale

In probability theory **Wald's martingale**, named after Abraham Wald and more commonly known as the geometric Brownian motion, is a stochastic process of the form

$$\left\{\exp\left(\lambda W_t - \frac{1}{2}\lambda^2 t\right), t \geq 0\right\}$$

for any real value λ where W_t is a Wiener process. The process is a martingale.

Wiener process

In mathematics, the **Wiener process** is a continuous-time stochastic process named in honor of Norbert Wiener. It is often called standard **Brownian motion**, after Robert Brown. It is one of the best known Lévy processes (càdlàg stoschastic processes with stationary independent increments) and occurs frequently in pure and applied mathematics, economics, quantitative finance, and physics.

The Wiener process plays an important role both in pure and applied mathematics. In pure mathematics, the Wiener process gave rise to the study of continuous time martingales. It is a key process in terms of which more complicated stochastic processes can be described. As such, it plays a vital role in stochastic calculus, diffusion processes and even potential theory. It is the driving process of Schramm–Loewner evolution. In applied mathematics, the Wiener process is used to represent the integral of a white noise Gaussian process, and so is useful as a model of noise in electronics engineering (see Brownian noise), instrument errors in filtering theoryand unknown forces in control theory.

The Wiener process has applications throughout the mathematical sciences. In physics it is used to study Brownian motion, the diffusion of minute particles suspended in fluid, and other types of diffusion via theFokker–Planck and Langevin equations. It also forms the basis for the rigorous path integral formulation ofquantum mechanics (by the Feynman–Kac formula, a solution to the Schrödinger equation can be represented in terms of the Wiener process) and the study of eternal inflation in physical cosmology. It is also prominent in the mathematical theory of finance, in particular the Black–Scholes option pricing model.

The Wiener process W_t is characterised by the following properties

1. $W_0 = 0$ a.s.
2. W has independent increments: $W_{t+u} - W_t$ is independent of $\sigma(W_s : s \leq t)$ for $u \geq 0$
3. W has Gaussian increments: $W_{t+u} - W_t$ is normally distributed with mean 0 and

variance u, $W_{t+u} - W_t \sim N(0, u)$

4. W has continuous paths: With probability 1, W_t is continuous in t.

The independent increments means that if $0 \le s_1 < t_1 \le s_2 < t_2$ then $W_{t1} - W_{s1}$ and $W_{t2} - W_{s2}$ are independent random variables, and the similar condition holds for n increments.

Basic properties:

$$f_{W_t}(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

$$E[W_t] = 0.$$

$$\mathrm{Var}(W_t) = E\left[W_t^2\right] - E^2[W_t] = E\left[W_t^2\right] - 0 = E\left[W_t^2\right] = t.$$

$$\mathrm{cov}(W_s, W_t) = \min(s, t),$$

$$\mathrm{corr}(W_s, W_t) = \frac{\mathrm{cov}(W_s, W_t)}{\sigma_{W_s}\sigma_{W_t}} = \frac{\min(s,t)}{\sqrt{st}} = \sqrt{\frac{\min(s,t)}{\max(s,t)}}.$$

brief reference letter

Stochastic processes	

Discrete time	• Bernoulli process
	• Branching process
	• Chinese restaurant process
	• Galton–Watson process
	• Independent and identically distributed random variables
	• Markov chain
	• Moran process
	• Random walk
	    • Loop-erased
	    • Self-avoiding

Continuous time	• Bessel process • Birth–death process • Brownian motion • Bridge • Excursion • Fractional • Geometric • Meander • Cauchy process • Contact process • Continuous-time random walk • Cox process • Diffusion process • Empirical process • Feller process • Fleming–Viot process • Gamma process

	• Hunt process • Interacting particle systems • Itô diffusion • Itô process • Jump diffusion • Jump process • Lévy process • Local time • Markov additive process • McKean–Vlasov process • Ornstein–Uhlenbeck process • Poisson process • Compound • Non-homogeneous • Point process • Schramm–Loewner evolution • Semimartingale

	• Sigma-martingale • Stable process • Superprocess • Telegraph process • Variance gamma process • Wiener process • Wiener sausage

Both	• Branching process • Gaussian process • Hidden Markov model (HMM) • Markov process • Martingale • Differences • Local • Sub- • Super-

	• Random dynamical system • Regenerative process • Renewal process • White noise

Fields and other	• Dirichlet process • Gaussian random field • Gibbs measure • Hopfield model • Ising model • Potts model • Boolean network • Markov random field • Percolation • Pitman–Yor process • Point process • Cox • Poisson • Random field • Random graph

Time series models	• Autoregressive conditional heteroskedasticity (ARCH) model • Autoregressive integrated moving average (ARIMA) model • Autoregressive (AR) model • Autoregressive–moving-average (ARMA) model • Generalized autoregressive conditional heteroskedasticity (GARCH) model • Moving-average (MA) model

Financial models	• Black–Derman–Toy • Black–Karasinski • Black–Scholes • Chen • Constant elasticity of variance (CEV) • Cox–Ingersoll–Ross (CIR) • Garman–Kohlhagen • Heath–Jarrow–Morton (HJM)

	• Heston • Ho–Lee • Hull–White • LIBOR market • Rendleman–Bartter • SABR volatility • Vašíček • Wilkie

Actuarial models	• Bühlmann • Cramér–Lundberg • Risk process • Sparre–Anderson

Queueing models	• Bulk • Fluid • Generalized queueing network • M/G/1 • M/M/1 • M/M/c

Properties	• Càdlàg paths • Continuous • Continuous paths • Ergodic • Exchangeable • Feller-continuous • Gauss–Markov • Markov • Mixing • Piecewise deterministic • Predictable • Progressively measurable • Self-similar • Stationary • Time-reversible

Limit theorems	• Central limit theorem • Donsker's theorem • Doob's martingale convergence theorems • Ergodic theorem • Fisher–Tippett–Gnedenko theorem

	• Large deviation principle • Law of large numbers (weak/ strong) • Law of the iterated logarithm • Maximal ergodic theorem • Sanov's theorem

Inequalities	• Burkholder–Davis–Gundy • Doob's martingale • Kunita–Watanabe

Tools	• Cameron–Martin formula • Convergence of random variables • Doléans-Dade exponential • Doob decomposition theorem • Doob–Meyer decomposition theorem • Doob's optional stopping theorem • Dynkin's formula • Feynman–Kac formula • Filtration

	• Girsanov theorem • Infinitesimal generator • Itô integral • Itô’s lemma • Kolmogorov continuity theorem • Kolmogorov extension theorem • Lévy–Prokhorov metric • Malliavin calculus • Martingale representation theorem • Optional stopping theorem • Prokhorov’s theorem • Quadratic variation • Reflection principle • Skorokhod integral • Skorokhod’s representation theorem • Skorokhod space • Snell envelope • Stochastic differential equation • Tanaka

	• Stopping time • Stratonovich integral • Uniform integrability • Usual hypotheses • Wiener space • Classical • Abstract

Disciplines	• Actuarial mathematics • Econometrics • Ergodic theory • Extreme value theory (EVT) • Large deviations theory • Mathematical finance • Mathematical statistics • Probability theory • Queueing theory • Renewal theory • Ruin theory • Statistics • Stochastic analysis • Time series analysis • Machine learning