Shortest paths on manifolds
Algorithms and applications

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Discretizing Anisotropic PDEs on Cartesian grids

Distance maps and Shortest Paths

Pontryagin’s principle

Riemannian metrics and Lattice Basis Reduction

A word on anisotropic diffusion

Finsler metrics and the Stern-Brocot tree

A word on cones of convex functions

Conclusion
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Grid discretizations of anisotropic PDEs

Prescribed anisotropy

- **Anisotropic diffusion**, with respect to a positive definite tensor field $D$. Structure: Maximum principle.
- Eikonal equations, with respect to an anisotropic Riemannian or Finsler metric $F$. Structure: Causality.

Spontaneous anisotropy

- Optimization problems subject to the constraint of convexity. Structure: convexity.
- Optimal transportation. Structure: Degenerate ellipticity.

On the domain’s interior

\[
\partial_t u(x) = \text{div}(D \nabla u(x)), \quad F_x^*(\nabla u(x)) = 1, \\
\min\{\mathcal{E}(u); \ u \text{ convex}\}, \quad \det(\nabla^2 u(x)) = f(x).
\]
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Current methods and results

- Most numerical schemes restricted to “tame” anisotropy, or based on (impractically) large stencils, or violating the PDE structural properties.

- Non existence of schemes based on bounded stencils.

Objective

- Adaptive numerical schemes, relying on sparse stencils, of limited extension, without restrictions on anisotropy.

- Quantitative results on minimal stencil cardinality and size.

Tools and techniques

- Lattice basis reduction. Notion(s) of “best” base of a discrete subgroup of $\mathbb{R}^d$.

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Introduction

Distance maps and Shortest Paths

Figure: A distance map, its level lines, and the associated shortest paths.
Asymmetric norms and Finsler metrics

**Definition (Asymmetric norm)**

A map $F : \mathbb{R}^d \to \mathbb{R}_+$ such that

- (Definiteness) $F(u) = 0$ implies $u = 0$.
- (Homogeneity) $F(\lambda u) = \lambda F(u)$ for $\lambda \geq 0$.
- (Triangle inequality) $F(u + v) \leq F(u) + F(v)$.

**Definition (Finsler metric on a domain $\Omega$)**

A continuous map $\mathcal{F} : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}_+$, $(z, u) \to \mathcal{F}_z(u)$, such that $\mathcal{F}_z$ is an asymmetric norm for all $z \in \overline{\Omega}$. 
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Path Length and asymmetric Distance

Connected domain \( \Omega \subset \mathbb{R}^d \) equipped with a Finsler Metric \( \mathcal{F} \).

Definition (Length of a path \( \gamma \in C^1([0, 1], \Omega) \))

\[
\text{length}(\gamma) := \int_0^1 \mathcal{F}_{\gamma(t)}(\gamma'(t)) \, dt
\]

Definition (Asymmetric Distance on \( \Omega \))

\[
D(x, y) := \inf\{\text{length}(\gamma); \gamma(0) = x, \gamma(1) = y\}.
\]

Addressed Problem.

Input: \( \mathcal{F}, z \).

Output: \( D(\cdot, z) \), paths of minimal length to \( z \).
Path Length and asymmetric Distance

Connected domain $\Omega \subset \mathbb{R}^d$ equipped with a Finsler Metric $\mathcal{F}$.

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Definition (Asymmetric Distance on $\bar{\Omega}$)

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Leaving an expo of centre Pompidou
Tubular structure extraction

Figure: Localization of the centerline of blood vessels in a medical image. Credit: L. D. Cohen and F. Benmansour
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Escape time from $x \in \Omega$

$$u(x) := \inf_{y \in \partial \Omega} D(x,y) = \inf_{\substack{\gamma(0) = x \\ \gamma(1) \in \partial \Omega}} \int_{0}^{1} \mathcal{F}_{\gamma}(\gamma'(t)) \, dt$$

Pontryagin's principle

If $x \in V \subset \Omega$, then to escape $\Omega$ one must cross $\partial V$

$$u(x) = \min_{y \in \partial V} D(x,y) + u(y).$$
Escape time from $x \in \Omega$

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$$u(x) = \min_{y \in \partial V} D(x, y) + u(y).$$
Choose a disk neighborhood $V = B(x, h)$, $h > 0$ small. Write $y = x + hv$, $v \in S$ (unit sphere). Assume enough smoothness.

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$$= \min_{v \in S} D(x, x + hv) + u(x + hv).$$

$$\approx \min_{v \in S} \mathcal{F}_x(hv) + u(x) + \langle \nabla u(x), hv \rangle.$$

Subtract $u(x)$ and divide by $h$.

$$\min_{v \in S} \mathcal{F}_x(v) + \langle \nabla u(x), v \rangle = 0$$

The eikonal equation

$$\mathcal{F}_x^*(-\nabla u(x)) = 1,$$

with

$$\mathcal{F}_x^*(g) = \max_{v \in S} \frac{\langle g, v \rangle}{\mathcal{F}(v)}.$$
Choose a disk neighborhood \( V = B(x, h), \ h \geq 0 \) small. Write \( y = x + hv, \ v \in S \) (unit sphere).
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Let $X$ and $\partial X$ be finite point sets discretizing $\Omega$, $\partial \Omega$.

**Definition (Hopf-Lax update operator)**

For $u : X \cup \partial X \to \mathbb{R}$, $x \in X$ with polygonal stencil $V(x)$.

$$\Lambda(u, x) := \min_{y \in \partial V(x)} F_x(y - x) + u(y),$$

where $u$ is piecewise-linearly interpolated on the faces of $\partial V(x)$.
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**Discrete fixed point problem**

$$
\begin{cases}
    u(x) = \Lambda(u, x) & \text{for all } x \in X, \\
    u(x) = 0 & \text{for all } x \in \partial X.
\end{cases}
$$
Definition (\(F\)-Acute cone \(K\))

Let \(K \subset \mathbb{R}^d\) be a convex cone, and \(F\) an asymmetric norm. We say that \(K\) is \(F\)-acute iff \(F(u + v) \geq F(u)\) for all \(u, v \in K\).

- A cone \(K\) is \(\| \cdot \|\)-acute iff \(\langle u, v \rangle \geq 0\) for all \(u, v \in K\).
- Let \(G\) be a smooth asymmetric norm. The quadrant \(K := \mathbb{R}_+^d\) is \(G\)-acute iff \(\forall i, \partial_i G \geq 0\) on \(K\).

Acuteness condition

Holds if the cone \(K\) spanned by any facet of \(V(x)\) is \(\mathcal{F}_x\)-acute.

Acuteness \(\Rightarrow\) Causality \(\Rightarrow\) Single pass resolution

If the minimum defining \(\Lambda(u, x)\) is attained in some minimal facet \([y_0, \cdots, y_k]\) of \(V(x)\), then \(\forall i \in [0, k], \Lambda(u, x) > u(y_i)\).
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Definition (*F*-Acute cone *K*)

Let $K \subset \mathbb{R}^d$ be a convex cone, and $F$ an asymmetric norm. We say that $K$ is *F*-acute iff $F(u + v) \geq F(u)$ for all $u, v \in K$.

- A cone $K$ is $\| \cdot \|$-acute iff $\langle u, v \rangle \geq 0$ for all $u, v \in K$.
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**Acuteness condition**

Holds if the cone $K$ spanned by any facet of $V(x)$ is $\mathcal{F}_x$-acute.

**Acuteness $\Rightarrow$ Causality $\Rightarrow$ Single pass resolution**

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**Conclusion**
**Definition (F-Acute cone K)**

Let $K \subset \mathbb{R}^d$ be a convex cone, and $F$ an asymmetric norm. We say that $K$ is $F$-acute iff $F(u + v) \geq F(u)$ for all $u, v \in K$.

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**Acuteness condition**

Holds if the cone $K$ spanned by any facet of $V(x)$ is $\mathcal{F}_x$-acute.

**Acuteness $\Rightarrow$ Causality $\Rightarrow$ Single pass resolution**

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Acuteness $\Rightarrow$ Causality. Heuristic of proof.

Let $G(\alpha) := \mathcal{F}_x \left( \sum_{i=1}^{k} \alpha_i v_i \right)$, with $v_i = y_i - x$. Let $U = (u(y_1), \cdots, u(y_k)) \in \mathbb{R}^k$, $1 = (1, \cdots, 1)$.

$$\Lambda := \min \{ G(\alpha) + \langle U, \alpha \rangle; \ \alpha \in \mathbb{R}^k_+, \langle \alpha, 1 \rangle = 1 \}$$

At the minimizer $\alpha$, for some Lagrange multiplier $\lambda \in \mathbb{R}$

$$\nabla G(\alpha) + U = \lambda 1. \quad (1)$$

Take the scalar product of (1) with $\alpha$, use Euler’s identity

$$\Lambda = \langle \alpha, \nabla G(\alpha) \rangle + \langle U, \alpha \rangle = \lambda \langle 1, \alpha \rangle = \lambda.$$

Consider (1) componentwise

$$\partial_i G(\alpha) + U_i = \lambda = \Lambda, \quad \text{and } \partial_i G(\alpha) \geq 0 \text{ by acuteness.}$$
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Acuteness ⇒ Causality. Heuristic of proof.

Let $G(\alpha) := \mathcal{F}_x \left( \sum_{i=1}^k \alpha_i v_i \right)$, with $v_i = y_i - x$. Let $U = (u(y_1), \cdots, u(y_k)) \in \mathbb{R}^k$, $1 = (1, \cdots, 1)$.

$$\Lambda := \min \{ G(\alpha) + \langle U, \alpha \rangle; \ \alpha \in \mathbb{R}_+^k, \langle \alpha, 1 \rangle = 1 \}$$

At the minimizer $\alpha$, for some Lagrange multiplier $\lambda \in \mathbb{R}$

$$\nabla G(\alpha) + U = \lambda 1. \quad (1)$$

Take the scalar product of (1) with $\alpha$, use Euler’s identity

$$\Lambda = \langle \alpha, \nabla G(\alpha) \rangle + \langle U, \alpha \rangle = \lambda \langle 1, \alpha \rangle = \lambda.$$

Consider (1) componentwise

$$\partial_i G(\alpha) + U_i = \lambda = \Lambda, \quad \text{and} \quad \partial_i G(\alpha) \geq 0 \text{ by acuteness.}$$
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The euclidean case

Acuteness implies causality (heuristic)

- Let \( T \in V(x) \) be the triangle containing the front normal.
- Causality ⇔ front last reaches vertex \( x \) of \( T \).
- Unconditional causality ⇔ the angles of \( T \) at \( x \) are acute.
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- Let $T \in V(x)$ be the triangle containing the front normal.
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Conclusion
Let \( \| e \|_M := \sqrt{\langle e, Me \rangle} \), for \( e \in \mathbb{R}^d \), \( M \in S^+_d \).

**Definition (Voronoi cell and facets)**

For each matrix \( M \in S^+_d \), introduce the Voronoi cell and facet

\[
\text{Vor}(M) := \{ g \in \mathbb{R}^d; \forall e \in \mathbb{Z}^d, \| g \|_M \leq \| g - e \|_M \},
\]

\[
\text{Vor}(M; e) := \{ g \in \text{Vor}(M); \| g \|_M = \| g - e \|_M \}.
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Let $\|e\|_M := \sqrt{\langle e, Me \rangle}$, for $e \in \mathbb{R}^d$, $M \in S_d^+$. 

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\end{align*}
\]
Proposition (Connected Voronoi vertices form acute angles)

If \( \text{Vor}(M; e) \cap \text{Vor}(M; f) \neq \emptyset \), then \( \langle e, Mf \rangle \geq 0 \).

Indeed, let \( p \in \text{Vor}(M; e) \cap \text{Vor}(M; f) \). Then

\[
\|p\|_M = \|p - e\|_M = \|p - f\|_M \leq \|p - (e + f)\|_M.
\]

\[
0 \leq \|p - (e + f)\|_M^2 - \|p - e\|_M^2 - \|p - f\|_M^2 + \|p\|_M^2 = 2 \langle e, Mf \rangle.
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Causal stencils for Riemannian metrics, on a cartesian grid

\( \mathcal{F}_x = \| \cdot \|_{M(x)} \), and \( X = \Omega \cap \mathbb{Z}^d \). Define \( V(x) \) as the collection of \( M(x) \)-Voronoi vectors, with the topology dual to \( \text{Vor}(M(x)) \).

Theorem (Optimality of Voronoi based stencils, 2D)

Voronoi based stencils are the smallest, in the sense of convex hull inclusion, satisfying the acuteness condition.

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Causal stencils for Riemannian metrics, on a cartesian grid $\mathcal{F}_x = \|\cdot\|_{M(x)}$, and $X = \Omega \cap \mathbb{Z}^d$. Define $V(x)$ as the collection of $M(x)$-Voronoi vectors, with the topology dual to $\text{Vor}(M(x))$.

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Voronoi based stencils are the smallest, in the sense of convex hull inclusion, satisfying the acuteness condition.

Size of the stencils

For each $\kappa \geq 1$, $\theta \in [0, \pi]$, introduce the symmetric matrix

$$M_\kappa(\theta) := e_\theta \otimes e_\theta + \kappa^2 e_\perp \otimes e_\perp$$

Let $V_\kappa(\theta)$ be the $M_\kappa(\theta)$-Voronoi vectors, and

$$R_\kappa(\theta) := \max_{e \in V_\kappa(\theta)} \| e \|, \quad S_\kappa(\theta) := \max_{e \in V_\kappa(\theta)} \| e \| M_\kappa(\theta)$$

Theorem (Euclidean and intrinsic stencil radius, as $\kappa \to \infty$)

$$\| R_\kappa \|_{L^p} \approx \kappa^{\frac{1}{2}} \| S_\kappa \|_{L^p}. \quad \| S_\kappa \|_{L^p} \approx \begin{cases} \kappa^{-\frac{1}{2}} \frac{1}{p} & \text{if } p > 2, \\
\left(\ln \kappa\right)^{\frac{1}{2}} & \text{if } p = 2, \\
1 & \text{if } p < 2. \end{cases}$$
Taking advantage of Anisotropy

Anisotropic fast marching (left) allows to take smaller steps in iterative extraction of retinal vessel trees.

Da Chen, Laurent Cohen, J.-M. M, Vessel Extraction Using Anisotropic Minimal Paths and Path Score, ICIP 2014
Three dimensional Riemannian Shortest paths

Same construction based on Voronoi’s vectors.
Three dimensional Riemannian Shortest paths

Same construction based on Voronoi’s vectors.
Three dimensional Riemannian Shortest paths

Same construction based on Voronoi’s vectors.
Petitot’s model: curvature penalized length

$\gamma : [0, 1] \rightarrow \mathbb{R}$, $s$: curvilinear abcissa, $\kappa$: curvature.

$$\int_{\gamma} \sqrt{1 + \kappa^2} ds = \int_{0}^{1} \sqrt{||\gamma'||^2 + |\theta'|^2} dt, \quad \text{with } \det(\gamma', e_\theta) = 0.$$ 

$\theta : [0, 1] \rightarrow \mathbb{R}$, $e_\theta := (\cos \theta, \sin \theta)$.

Riemannian taming $\kappa^2 = \Lambda + 1 \gg 1$

$$\int_{0}^{1} \sqrt{||\gamma'||^2 + |\theta'|^2 + \Lambda \det(\gamma', e_\theta)^2} dt,$$

$$\begin{pmatrix} M_\kappa(\theta) & 0 \\ 0 & 1 \end{pmatrix}$$
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Conclusion
Anisotropic diffusion $\partial_t u = \text{div}(D \nabla u)$

- Weickert’s Edge enhancing / Plane enhancing PDEs

Similarly use $M(x)$-Delaunay stencils, $M(x) := D(x)^{-1}$.

Equivalence with a Finite-Element discretization on Shewchuk’s Anisotropic Delaunay triangulation. (2D)
Anisotropic diffusion $\partial_t u = \text{div}(D \nabla u)$

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AD–LBR ; T=10

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Conclusion
The Stern Brocot tree

Obtain the \((n+1)\)-th line by inserting \(\frac{a+a'}{b+b'}\) between consecutive elements \(\frac{a}{b}\) and \(\frac{a'}{b'}\) of the \(n\)-th line.

- Each rational number appears exactly once, in its irreducible form.
Off topic: Fun facts on the Stern-Brocot tree

The Stern-Brocot tree can be represented as a complete infinite binary tree. Some images from Wikipedia.

Minkowski’s question mark function, \( \mathcal{M} : [0, 1] \rightarrow [0, 1] \)

\( \mathcal{M}(x) \) is the continuous function mapping the Stern-Brocot labels in \([0, 1]\) to the dyadic labels. Properties:

- \( \mathcal{M}'(x) = 0 \) for almost every \( x \). (“Slippery Devil’s staircase”)
- \( \mathcal{M} \) is Holder continuous, with exponent \( \frac{\ln 2}{2 \ln \Phi}, \Phi := \frac{1+\sqrt{5}}{2} \).
- \( \mathcal{M}(x) \) is rational for every quadratic irrational.
Off topic: Fun facts on the Stern-Brocot tree

\[
\begin{array}{cccc}
  a & A & a+a' & a' \\
  b & B & b+b' & b' \\
  & & & \\
  & & & \\
  & & & \\
  a & a+A & A & A+a' & a' \\
  b & b+B & B & B+b' & b' \\
  & & & & \\
  & & & & \\
  & & & & \\
  x & x+X & X & 1 & x'
\end{array}
\]

Figure: Dyadic rationals can be organized in a similar (complete infinite binary) tree. Some images from Wikipedia.

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Definition (A tree of triangles)

Children of $T = [0, u, v]$: $T' = [0, u, u + v]$, $T'' = [0, u + v, v]$.

Root: $[(0, 0), (1, 0), (0, 1)]$.

Correspondence with the Stern-Brocot tree: to each node $\frac{a+a'}{b+b'}$ associate

$$ T = [0, u, v], \quad \text{where} \quad u = (a, b), \quad v = (a', b'). $$
Definition (A tree of triangles)

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Correspondence with the Stern-Brocot tree: to each node $\frac{a+a'}{b+b'}$ associate

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Stencil and tree structure

$V(F)$: mesh obtained by recursively refining the 4 element mesh $T_0$ (bottom left), until all triangles are $F$-acute.

\[ V(F) \]

Applications of Finsler shortest paths

- Ascent is harder than descent.
- Navigation at unit speed + drift due to currents.
- Segmentation with black on right, white on left. (Zach, Chan, Niethammer, 09)
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(a) Geodesic active contour  (b) Finsler active contour
Euler elastica: squared curvature penalized length
\( \gamma : [0, 1] \rightarrow \mathbb{R} \), \( s \): curvilinear abcissa, \( \kappa \): curvature

\[
\int_{\gamma} (1 + \kappa^2)ds = \int_0^1 \|\gamma'| + \frac{\|\theta'\|^2}{\|\gamma'\|}, \text{ with } \langle \gamma', e_\theta \rangle = \|\gamma'\|.
\]

\( \theta : [0, 1] \rightarrow \mathbb{R} \), \( e_\theta = \cos(\theta, \sin \theta) \).

Finslerian taming \( \lambda \gg 1 \)

\[
\int_0^1 \sqrt{\lambda^2 \|\gamma'|^2 + 2\lambda |\theta'|^2 - (\lambda - 1) \langle e_\theta, \gamma' \rangle}
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Conclusion
$\Omega$ convex. $X = \Omega \cap \mathbb{Z}^2$. $N := \#(X)$. $\mathcal{T}$: triangulation of $X$.

$\text{Cone}(\mathcal{T}) := \{u \in \mathbb{P}_1(\mathcal{T}), \text{ convex}\}$. $K := \{u|_X, u \text{ convex}\}$.

**Figure**: Principal agent problem: Minimize $\int_\Omega (\|\nabla u(x) - x\|^2 + u) \, dx$, subject to $u \geq 0, u \text{ convex}$. Center: $\text{det}(\nabla^2 u)$. Right: $u \in \text{Cone}(\mathcal{T})$.

- $\text{Cone}(\mathcal{T}) \sim N$ linear constraints. $K \sim N^2$ linear const.
- Choné, Lemeur: non-density with regular triangulations.
- $K$ is partitioned by the $\text{Cone}(\mathcal{T})$, $\mathcal{T}$ triangulation of $X$. 

Anisotropy, non-locality
Distance maps and Shortest Paths
Pontryagin’s principle
Riemannian metrics and Lattice Basis Reduction
Diffusion
Finsler metrics and the Stern-Brocot tree
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\[ \Omega \text{ convex. } \quad X = \Omega \cap \mathbb{Z}^2. \quad N := \#(X). \quad \mathcal{T}: \text{ triangulation of } X. \]

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**Figure**: Principal agent problem: Minimize \( \int_{\Omega} (\|\nabla u(x) - x\|^2 + u) \, dx \), subject to \( u \geq 0, \text{ u convex} \). Center: \( \det(\nabla^2 u) \). Right: \( u \in \text{Cone}(\mathcal{T}) \).

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Ω convex. \( X = \Omega \cap \mathbb{Z}^2 \). \( N := \#(X) \). \( \mathcal{T} \): triangulation of \( X \).

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**Figure**: Principal agent problem: Minimize $\int_{\Omega}(\|\nabla u(x) - x\|^2 + u)dx$, subject to $u \geq 0$, $u$ convex. Center: $\det(\nabla^2 u)$. Right: $u \in \text{Cone}(\mathcal{T})$.

- $\text{Cone}(\mathcal{T}) \sim N$ linear constraints. $K \sim N^2$ linear const.
- Choné, Lemeur: non-density with regular triangulations.
- $K$ is partitioned by the $\text{Cone}(\mathcal{T})$, $\mathcal{T}$ triangulation of $X$. 

Plus courts chemins
Jean-Marie Mirebeau

Anisotropy, non-locality
Distance maps and Shortest Paths
Pontryagin’s principle
Riemannian metrics and Lattice Basis Reduction
Diffusion
Finsler metrics and the Stern-Brocot tree
Convex functions
Conclusion
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