Complex Lectures

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1 The Complex Plane

1.1 Complex Plane Intro

Complex Numbers

Definition 1. $\mathbb{C} = \{(x, y) : x, y \in \mathbb{R}\}$ *Let* $z = (x, y), w = (u, v)$. *We have the following field structure:*

$$z + w = (x + u, y + v)$$

$$z \cdot w = (xu - yv, sv + yu)$$

So we have

$$(x + iy)(u + iv) = (xu - yv) + i(xv + yu)$$

Theorem 1. $\mathbb{C}$ *is a field.*

Proof. Let $z, w \in \mathbb{C}$ with $w \neq 0$. We will show that there exists unique $s \in \mathbb{C}$ such that $z = w \cdot s$. We have

$$\begin{aligned} \frac{z}{w} &= \frac{x + iy}{u + iv} \cdot \frac{(u - iv)}{(u - iv)} \\ &= \frac{(xu + yv) + i(yu - xv)}{u^2 + v^2} \\ &= \frac{xu + yv}{u^2 + v^2} + i\frac{yu - xv}{u^2 + v^2} \\ &= s \end{aligned}$$

Can check this works. But we need to show it is unique. Suppose that

$$z = ws_1 = ws_2$$

This implies that

$$0 = w(s_1 - s_2)$$

Let $\alpha + i\beta = s_1 - s_2$. Then we have

$$\begin{aligned} 0 &= w(\alpha + i\beta) \\ &= w(\alpha^2 + \beta^2) \\ &= (u + iv)(\alpha^2 + \beta^2) \\ &= (\alpha^2 + \beta^2)u + i(\alpha^2 + \beta^2)v \end{aligned}$$

Thus $\alpha^2 = \beta^2 = 0$ and $\alpha = \beta = 0$. □

NOTATION:

$$\text{Re}(z) = x$$

$$\text{Im}(z) = y$$

$$\overline{z} = x - iy$$

FACTS:

- $\overline{z + w} = \overline{z} + \overline{w}$

- $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$
- $\overline{z} + z = 2\,\mathrm{Re}(z)$
- $z - \overline{z} = 2i\,\mathrm{Im}(z)$
- $\mathrm{Im}(z) = \frac{z - \overline{z}}{iz}$

Definition 2. *The Modulus of $z = x + yi$ is*

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z \cdot \overline{z}}$$

FACTS ABOUT MODULUS:

- $\overline{z} \cdot z = |z|^2$
- $-|z| \leq \mathrm{Re}\, z \leq |z|$
- $-|z| \leq \mathrm{Im}\, z \leq |z|$
- $|\overline{z}| = |z|$

Proposition 1. *Suppose $z, w \in \mathbb{C}$, then*

1. $|z \cdot w| = |z| \cdot |w|$
2. $|z + w| \leq |z| + |w|$
3. $|\prod z_j| = \prod |z_j|$
4. $|\sum z_j| \leq \sum |z_j|$

Proof. (1) We square the quantity

$$\begin{aligned} |zw|^2 &= (zw)(\overline{zw}) \\ &= z\overline{z}w\overline{w} \\ &= |z|^2|w|^2 \end{aligned}$$

□

Definition 3. *The distance formula is*

$$d(z, w) = |z - w|$$

FACTS:

1. $|z - w| \geq 0$
2. $|w - z| = |z - w|$
3. $z_1 - z_3| \leq |z_1 - z_2| + |z_2 - z_3|$

POLAR REPRESENTATION OF COMPLEX NUMBERS

Let $r = |z|$ we have

$$z = x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}$$

Multiplication is given by

$$re^{i\theta_1} \cdot se^{i\theta_2} = rse^{i(\theta_1 + \theta_2)}$$

Complex and Holomorphic Functions

Definition 4. *The term* $\lim_{z\to\infty} z_n = z$ *means that given* $\epsilon > 0$ *there exists* $N = N_\epsilon$ *such that for all* $n \geq N$

$$|z_n - z| < \epsilon$$

Definition 5. *Let* f *be a real or complex valued function defined on a set* S. *Then*

1. $\lim_{z\to z_0} f(z) = L$, $L \in \mathbb{C}$ *means that* z_0 *is a limit point of* S *and for all* $\epsilon > 0$ *there exists a* $\delta = \delta_\epsilon$ *such that if* $0 < |z - z_0| < \delta$ *then* $|f(z) - L| < \epsilon$.

2. f *is continuous at* z_0 *if* $z_0 \in S$ *and when* z_0 *is a limit point of* S *we have*

$$\lim_{z\to z_0} f(z) = f(z_0)$$

3. f *is continuous on* S *i.e* $f \in C(s)$ *if* f *is continuous at every point of* S.

1.1.1 NOTATION

$\mathrm{Re}(f) : \mathbb{C} \to \mathbb{R}$ is the function $z \mapsto \mathrm{Re}(f(z))$ $\mathrm{Im}(f) : \mathbb{C} \to \mathbb{R}$ is the function $z \mapsto \mathrm{Im}(f(z))$

1.1.2 EXAMPLE

$$f(z) = z^3 = (x + iy)^3 = x^3 + 3x^y i - 3xy^2 - iy^3 = (x^3 - 3xy^2) + i(3x^2y + y^3)$$

So we have

$$\mathrm{Re}\, f = x^3 - 3xy^2$$

1.1.3 EXERCISE

Proof the following things

- f is continuous at z_0 if and only if $\mathrm{Re}\, f$, $\mathrm{Im}\, f$, $\overline{f}$ and $|f|$ are also continuous at z_0. (use $\epsilon/2$)
- Show by example that $|f|$ continuous at z_0 does not imply f is continuous.

1.1.4 NOTATION

$$\mathbb{D}(z_0, \delta) = \{z \in \mathbb{C} : |z - z_0| < \delta\}$$
$$\partial\mathbb{D}(z_0, \delta) = \{z \in \mathbb{C} : |z - z_0| = \delta\}$$
$$\overline{\mathbb{D}(z_0, \delta)} = \partial\mathbb{D}(z_0, \delta) \cup \mathbb{D}(z_0, \delta)$$

Definition 6. *A set* $D \subseteq \mathbb{C}$ *is called open if for all* $z \in D$ *there is a* $\delta > 0$ *such that* $\mathbb{D}(z, \delta) \subseteq D$.

Definition 7. *Let* $f : \mathbb{D}(z_0, R) \to \mathbb{C}$. *The* ***derivative*** *of* f *at* z_0 *is*

$$f'(z) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

(Alternatively we use the $z + h$ *form* $h \in \mathbb{C}$.*)*

$$f(z_0 + h) = f(z_0) + hf'(z) + h\epsilon(h)$$

Where $\epsilon(h)$ *is defined for small enough* h *and* $\lim_{h\to 0} \epsilon(h) = 0$.

Note: Limit must exist and be independent of the direction of h

1.2 Derivatives

Facts about Derivative

1. If f' exists then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist.

   Proof.

$$\frac{\partial f}{\partial x}(z_0) = \lim_{t\to 0} \frac{f(x_0+t)-f(z_0)}{t} = f'(z_0)$$

$$\begin{aligned}\frac{\partial f}{\partial y}(z_0) &= \lim_{t\to 0} \frac{f(x_0+it)-f(z_0)}{t} \\ &= \lim_{t\to 0} \frac{i(f(x_0+it)-f(z_0))}{it} \\ &= if'(z_0)\end{aligned}$$

□

2. Existence of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ do not imply $f'(z_0)$ exists.

   Proof. $f(z) = f(x+iy) = x$ has partials exist but does not have f'.

□

3. If $f'(z_0)$ exists then f is continuous at z_0.

Definition 8. *Let D be an open set and $f : D \to \mathbb{C}$. The function f is said to be* ***holomorphic in** D* *if $f'(z)$ exists for all $z \in D$. Alternate terms are* ***analytic, regular, monogenic, syndectic****.*

Notation: $H(D)$ means the set of holomorphic functions on D.

Theorem 2. *Let $f, g \in H(D)$, then*

1. $f + g \in H(D)$
2. $f - g \in H(D)$
3. $f \cdot g \in H(D)$
4. $\frac{f}{g} \in H(D)$
5. *And in all of the above the derivatives are what we would expect.*

Theorem 3 (Chain Rule)**.** *Let $f \in H(D)$, $g \in H(D_1)$ with $f(D) \subseteq D_1$. Then $g \circ f \in H(D)$ with*

$$(g \circ f)'(z) = g'(f(z))f'(z)$$

Proof. Fix $z_0 \in D$. Then for small enough h:

$$f(z_0 + h) = f(z_0) + hf'(z_0) + h\epsilon(h)$$

Then

$$g(w_0 + h) = g(w_0) + hg'(w_0) + h\epsilon_1(h)$$

We now have

$$\begin{aligned} g(f(z_0 + h)) &= g\left(f(z_0) + [hf'(z_0) + h\epsilon(h)]\right) \\ &= g(f(z_0)) + h[f'(z_0) + \epsilon(h)]g'(f(z_0)) + h(f'(z_0) + \epsilon(h))\epsilon_1(hf'(z_0) + \epsilon(h)) \end{aligned}$$

$$\begin{aligned} \lim_{h\to 0} \frac{g \circ f(z_0 + h) - g \circ f(z_0)}{h} &= \lim_{h\to 0}(f'(z_0) + \epsilon(h))g'(f(z_0)) + (f'(z_0) + \epsilon(h))\epsilon_1(h(f'(z_0) + \epsilon(h)) \\ &= g'(f(z_0))f'(z_0) \end{aligned}$$

□

Definition 9. *If $f \in H(\mathbb{C})$ then we say f is* ***entire****.*

EXAMPLES

- $f(z) = z_0$, $f'(z) = 0$
- $f(z) = z$, $f'(z) = 1$

Definition 10. *A* ***polynomial*** *is a function of the form $p(z) = \sum_{j=0}^{n} a_j z^j$.*

$$p'(z) = \sum_{j=1}^{n} j a_j z^{j-1}$$

1.3 The Riemann Sphere

Definition 11. $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. *We add a topology to $\mathbb{C}^*$ by defining basic neighborhoods of ∞ to be sets of the form*

$$\mathbb{D}(\infty, R) = \{z \in \mathbb{C} : |z| > R\}$$

So $\{z_n\} \to \infty$ means that for all $R > 0$ there is an $N = N_R$ such that if $n \geq N$ then $|z_n| > R$.

If $f : \mathbb{D}(\infty, R) \setminus \{\infty\} \to \mathbb{C}$

$$\lim_{z\to\infty} f(z) = L$$

if for all $\epsilon > 0$ there exists R_ϵ such that if $|z| > R$ then

$$|f(z) - L| < \epsilon$$

Definition 12. *For $z_0 \in \mathbb{C}$*

$$\lim_{z\to z_0} f(z) = \infty$$

means that given an $R > 0$ there is a $\delta > 0$ such that

$$0 < |z - z_0| < \delta$$

then $|f(z)| > R$.

Definition 13.

$$\lim_{z\to\infty} f(z) = \infty$$

if for all $R > 0$ there exists an $R_2(R)$ such that if $|z| > R_2$ then $|f(z)| > R$

Definition 14. *$\mathbb{C}^*$ is called the **one point compactification** of $\mathbb{C}$.*

Example

$$Q(z) = \frac{1}{z-1}$$

for $z \in \mathbb{C} \setminus \{1\}$. $Q(\infty) = 0$, $Q(1) = \infty$, then Q is continuous from $\mathbb{C}^* \to \mathbb{C}^*$.

REMARK:
There is a big difference between Real ∞ and complex ∞. For example, $f(x) = 1 - x^3$

$$\lim_{x\to\infty^+} f(x) = -\infty$$

$$\lim_{x\to-\infty} f(x) = +\infty$$

$$\lim_{z\to\infty} (1 - z^3) = \infty$$

1.3.1 Stereographic Projection

Let S^2 be the uni sphere in $\mathbb{R}^3$. Let $T : S^2 \to \mathbb{C}^*$ as follows

$$T(0,0,1) = \infty$$

$$x_3 > 0 \Rightarrow |T(x)| > 1$$

$$x_3 < 0 \Rightarrow |T(x)| < 1$$

$$x_3 = 0 \Rightarrow |T(x)| = 1$$

$$T(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}$$

Let $P = T^{-1}$.

$$P(re^{i\theta}) = \left(\frac{2r\cos\theta}{1+r^2}, \frac{2r\sin\theta}{1+r^2}, \frac{r^2-1}{r^2+1}\right)$$

Theorem 4. *T is a homeomorphism.*

Proof. It is visually apparent. The formulas would work though. □

This gives rise to a new metric on $\mathbb{C}^*$.

Definition 15. *For $z, w \in \mathbb{C}^*$ let*

$$(z, w) = |T^{-1}(z) - T^{-1}(w)|_{\mathbb{R}^3}$$

*this is called the **spherical metric**.*

FACT
(C^*, χ) is a compact metric space.
FORMULAS

$$\chi(z,w) = \frac{2|z-w|_{\mathbb{C}}}{((1+|z|^2)(1+|w|^2))^{\frac{1}{2}}}$$

$$\chi(z,\infty) = \frac{2}{(1+|z|^2)^{\frac{1}{2}}}$$

1.4 Mobius Transformations

Definition 16. *A **domain** is a connected open set.*

Theorem 5 (The Uniqueness Theorem)**.** *Suppose $D \subseteq \mathbb{C}$ is a domain, $f \in H(D))$ and $f'(z) \equiv 0$ for all $z \in D$. Then f is constant.*

Proof. Use fact that a connected set in the plane is C^1-pathwise connected. Let $z_0, z_1 \in D$ be given and let $\gamma : I \to D$ be a C^1 path from z_0, z_1. Let $h(t) = f(\gamma(t))$, by the chain rule

$$h'(t) = f'(\gamma(t))\gamma'(t)$$

Lets write

$$h(t) = h_1(t) + ih_2(t)$$

So we have

$$0 = h_1' + ih_2'$$

So

$$h_1' \equiv h_2' \equiv 0$$

The MVT says that

$$h_1(1) - h_1(0) = h_1'(t^*), \qquad \text{for some } t^* \imath (0,1)$$

So,

$$h_1(1) = h_1(0)$$
$$h_2(1) = h_2(0)$$
$$h(1) = h(0)$$
$$f(z_1) = f(z_0)$$

□

Definition 17. *A rational function R is one of the form $R = \frac{P}{Q}$ with $P, Q \in \mathbb{C}[z]$ and $R \in H(\mathbb{C} \setminus \{\text{zeros of } Q\})$*

1.4.1 Remark

- We will later see that at the zeros of Q, say z_0 is one of them,

$$\lim_{z\to z_0} \frac{P(z)}{Q(z)}$$

exists in $\mathbb{C}^*$.

- Also $R(\infty) = \lim_{z\to\infty} \frac{P(z)}{Q(z)}$ also exists in $\mathbb{C}^*$.

Definition 18. *IF $\mathbb{R} : \mathbb{C}^* \to \mathbb{C}^*$ is continuous we say that R is* ***meromorphic*** *in $\mathbb{C}^*$.*

Definition 19. *A nonconstant function of the form $T(z) = \frac{az+b}{cz+d}$ for $a, d, c, d \in \mathbb{C}$ with at least one of c, d nonzero defined initially on $\mathbb{C} \setminus \{-\frac{d}{c}\}$ is called a* ***Möbius Transformation****. Alternatively called a* ***Linear Fractional Transformation****.*

Definition 20. *Let $\mathcal{M} = \{T(z) : T$ is a möbius transformation$\}$.*

Facts

- A function T of this form is nonconstant if and only if $ad - bc \neq 0$.

  Proof. Note that $T' = \frac{ad-bc}{(cz+d)^2}$, hence if $ad - bc = 0$ then $T' \equiv 0$ on $\mathbb{C} \setminus \{\frac{-d}{c}\}$ and thus T is constant via the uniqueness theorem.

  Suppose that T is constant. This implies $T' \equiv 0$ and thus $ad - bc = 0$. □

- For $T \in \mathcal{M}$ if $c \neq 0$ define $T(\infty) = \frac{a}{c}$ and $T(\frac{-d}{c}) = \infty$. If $c = 0$ then let $T(\infty) = \infty$. Then T is a homeomorphism of $\mathbb{C}^* \to \mathbb{C}^*$.

  Proof. Suppose $c \neq 0$. We show that T is one-to-one. Suppose $T(z_1) = T(z_2)$ and $z_1, z_2 \neq \frac{-d}{c}, \infty$. Then

$$\frac{az_1 + b}{cz_1 + d} = \frac{az_2 + b}{cz_2 + b}$$

  So

$$ad(z_2 - z_1) = bc(z_2 - z_1)$$

  If $z_1 \neq z_2$ then $ad = bc$, if T is nonconstant we have a contradiction. If z_1, z_2 are $\frac{-d}{c}, \infty$ then it is also straightforward (CHECK).

  Now we prove surjectivity. Let $w_0 \in \mathbb{C}^*$, set

$$w_0 = \frac{az_0 + b}{cz_0 + d}$$

  then we obtain

$$z_0 = \frac{b - dw_0}{cw_0 - a}$$

  which works so long as $w_0 \neq \frac{a}{c}$. However, $T(\infty) = \frac{a}{c}$ so we can map to w_0. □

Theorem 6. $(\mathcal{M}, \cdot)$ *is a group. It is isomorphic to*

$$SL(2, \mathbb{C})/\{I, -I\} \cong PSL(2, \mathbb{C})$$

Proof. Define $\varphi : SL(2, \mathbb{C}) \to \mathcal{M}$ by

$$\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = T(z)$$

Where $T(z) = \frac{az+b}{cz+d}$ Show φ is a homomorphism and $\ker \varphi = \{I, -I\}$. ☐

1.5 $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$

Definition 21. *Suppose* $f : \mathbb{D}(z_0, R) \to \mathbb{C}$, $f = u + iv$. *Assume that* $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *exist at* z_0. *Let*

$$\frac{\partial f}{\partial z} = \frac{1}{2}(f_x - if_y)$$
$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}(f_x + if_y)$$

These operators are called the formal complex derivatives of f *at* z_0.

Examples

- $f(z) = \overline{z} = x - iy$
$$\frac{\partial f}{\partial x} = 1, \frac{\partial f}{\partial y} = -i$$
$$\frac{\partial f}{\partial z} = 0, \frac{\partial f}{\partial \overline{z}} = 1$$
- $f(z) = z = x + iy$
$$\frac{\partial f}{\partial z} = \frac{1}{2}(1 - i(i)) = 1$$
$$\frac{\partial f}{\partial \overline{z}} = 0$$

Remarks

- We saw earlier that if $f'(z_0)$ exists then $f' = f_x$ and $f_y = if'$ and so
$$\frac{\partial f}{\partial z} = \frac{1}{2}(f' - i(if')) = f'$$
$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}(f' + i(if')) = 0$$
- Suppose that $f \in H(D)$ then $\frac{\partial f}{\partial z} = f'(z)$, and $\frac{\partial f}{\partial \overline{z}} = 0$ in D. Additionally,
$$\begin{aligned} f_{\overline{z}} &= \frac{1}{2}[(u_x + iv_x) + i(u_y + iv_y) \\ &= \frac{1}{2}[(u_x - v_y) + i(v_x + u_y)] \\ &= 0 \end{aligned}$$

So if f' exists then

$$u_x = v_y$$

$$v_x - u_y$$

- $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ are $\mathbb{C}$-linear operators. They satisfy product and quotient rules.

$$f(z) = \sum_{j,k\geq 0} a_{jk} z^j z^k$$

- Exercise: Show that $\text{Re}(f_x) = u_x$ and $\text{Re}(f_y) = u_y$, but $\text{Re}(f_z) \neq u_z$ and $\text{Re}(f_{\overline{z}}) \neq u_{\overline{z}}$.

Lemma 1. *Suppose $f : \mathbb{D}(z_0, R) \to \mathbb{C}$ for some $R > 0$ and that f_x and f_y exist everywhere in $\mathbb{D}(z_0, R)$ and are continuous at z_0. Then for $z \in \mathbb{D}(z_0, R)$*

$$f'(z) = f(z_0) + f_x(z_0)(x - x_0) + f_y(z_0)(y - y_0) + \epsilon(z)$$

$$= f(z_0) + f_z(z_0)(z - z_0) + f_{\overline{z}}(z_0)(z - \overline{z}_0) + \epsilon(z)$$

Where $\epsilon(z)$ is a function defined by these equations and $\lim_{z\to z_0} \frac{\epsilon(z)}{z-z_0} = 0$

1.5

Lemma 2. *Suppose $f : \mathbb{D}(z_0, R) \to \mathbb{C}$ for some $R > 0$ and that f_x and f_y exist everywhere in $\mathbb{D}(z_0, R)$ and are continuous at z_0. Then for $z \in \mathbb{D}(z_0, R)$*

$$f'(z) = f(z_0) + f_x(z_0)(x - x_0) + f_y(z_0)(y - y_0) + \epsilon(z)$$

$$= f(z_0) + f_z(z_0)(z - z_0) + f_{\overline{z}}(z_0)(z - \overline{z}_0) + \epsilon(z)$$

Where $\epsilon(z)$ is a function defined by these equations and $\lim_{z\to z_0} \frac{\epsilon(z)}{z-z_0} = 0$

Proof. First assume $z_0 = 0$ and that f is real valued. By the Mean Value Theorem there exist $z_2, z_3 \in \mathbb{C}$ such that

$$z_2 \in [0, \text{Re}(z)]$$

$$z_3 \in \{\text{Re}(z) + i[0, \text{Im}(z)]$$

So that

$$\begin{aligned} f(z) - f(0) = f(z) - f(z_1) + f(z_1) - f(0) \\ + f_y(z_3) \cdot y + f_z(z_2) \cdot x \end{aligned}$$

The above holds for any $z \in \mathbb{D}(0, R) \setminus \{0\}$. So we can write

$$f(z) = f(0) + f_y(0)y + f_x(0)x + y(f_y(z_3) - f_y(0)) + x(f_x(z_2) - f_x(0))$$

Define

$$\epsilon(z) = y(f_y(z_3) - f_y(0)) + x(f_x(z_2) - f_x(0))$$

Notice that

$$|\epsilon(z)| \leq |z| \left(|f_y(z_3) - f_y(0|) + |f_x(z_2) - f_x(0)|\right)$$

Since f_x and f_y are continuous we have $\lim_{z\to 0} \epsilon/z = 0$.

If f is $\mathbb{C}$-valued then we apply the above argument to $f = u + iv$ separately.
Next we verify the second formula.

$$\begin{aligned} f_z(0) + f_{\overline{z}}(0)\overline{z} &= \frac{1}{2}(f_x(0) - if_y(0))(x+iy) + \frac{1}{2}(f_x(0) + if_y(0))(x-iy) \\ &= f_x(0)x + f_y(0)y \end{aligned}$$

For general z_0, apply lemma to $g(z) = f(z + z_0)$. □

1.6 Cauchy-Riemann Equations

Theorem 7. *Suppose that $f : \mathbb{D}(z_o, R) \to \mathbb{C}$ for some $R > 0$ and $f = u + iv$, then*

1. *If $f'(z_0)$ exists then*

$$u_x(z_0) = v_y(z_0)$$
$$u_y(z_0) = -v_x(z_0)$$

2. *If u_x, u_y, v_x, v_y exist and are continuous at z_0 and the Cauchy-Riemann equations hold, then $f'(z_0)$ exists.*

Proof. (1) If $f'(z)$ exists we saw earlier that $f'(z) = f_x(z) = u_x(z) + iv_x = \frac{1}{i}f_y = v_y - iu_y$, equating the real and imaginary parts gives the CR equations.

(2) By the lemma in the previous section we know that

$$f(z) = f(z_0) + f_z(z_0)(z - z_0) + f_{\overline{z}}(z_0)(\overline{z} - \overline{z}_0) + \epsilon(z)$$

We saw earlier that the CR equations hold if and only if $f_{\overline{z}}(z) = 0$, moving things around we obtain

$$\frac{f(z) - f(z_0)}{z - z_0} = f_z(z_0) + \frac{\epsilon(z)}{z - z_0}$$

Hence

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f_z(z_0)$$

□

Corollary 1. *If $f = u + iv \in C^1(D)$ for a domain D and u, v satisfy the CR equations in D, then $f \in H(D)$.*

Examples

- $f(z) = f(z + iy) = x^2 + 2y^2 + 2ixy$

$$u_x = 2x, \qquad v_x = 2y$$
$$u_y = 4y, \qquad v_y = 2x$$

Since the CR equations only hold on the real line. So

$$f'(z) \text{ exists} \iff z \in \mathbb{R}$$

- $f(z) = x^2 - y^2 + i2xy = z^2$
$$u_x = 2x, \quad u_y = -2y$$
$$v_x = 2y, \quad v_y = 2x$$
CR hold and so f is entire.
- $f(z) = x^2 + y^2 = |z|^2$
$$u_x = 2x, \quad v_x = 0$$
$$u_y = 2y, \quad v_y = 0$$

1.7 Series of Complex Functions

Definition 22. *Let $\{f_n\}$ be a sequence of functions defined on E. We say the sequence* ***converges pointwise*** *on E to f if*
$$\lim_{n\to\infty} f_n(z) = f(z)$$
i.e. for every point $z_0 \in E$ we have the sequence
$$\{f_n(z_0)\} \to f(z_0)$$
The sequence of functions ***converges uniformly*** *on E if given an $\epsilon > 0$ there exists an N_0 such that for all $n \geq N_0$ we have*

Definition 23. *A series of functions $\sum f_n$ is said to converge pointwise (uniformly) if the sequence of partial sums converges pointwise (uniformly).*

Examples

- $\sum^\infty z^n$, for $z \neq 1$
$$s_n(z) = \sum_{k=0}^{n} z^k - \frac{1 - z^{n+1}}{1 - z}$$
If $z = 1$, then $s_n(z) = n + 1$, and so it diverges.
If $|z| > 1$ the sum diverges.
If $|z| = 1$ the sum diverges because $|z^k| = 1$.
If $|z| < 1$, as $\lim_{n\ to\infty} z^{n+1} = 0$. So s_n converges to $\frac{1}{1-z}$.
- The above discussed pointwise convergence. To test for uniform convergence we have
$$\left| s_n(z) - \frac{1}{1-z} \right| = \left| \frac{z^{n+1}}{1-z} \right| = \frac{|z|^{n+1}}{|1-z|}$$
By letting $z \to 1$ from inside $\mathbb{D}(0,1)$, we see the convergence is not uniform since for all n
$$\sup_{\mathbb{D}} |s_n(z) - \frac{1}{1-z}| = \infty$$
Fix $R \in (0,1)$.
$$\sup_{|z|\leq R} |s_n(z) - \frac{1}{1-z}| = \frac{|R|^{n+1}}{|1-R|} < \infty$$
for all n and as $n \to \infty$, this goes to 0. So $\sum z^n$ converges pointwise on $\mathbb{D}(0,1)$ but not uniformly on $\overline{\mathbb{D}(0,R)}$, $R \in (0,1)$.

Series of Complex Functions

Facts about series of constants

1. If a series $\sum_{n=k}^{\infty}$ converges $a_k \in \mathbb{C}$ then $\lim_{k\to\infty} a_k = 0$.
2. If $a_k > 0$, $\sum_{k=n}^{\infty} a_k < \infty$ then

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} a_k = 0$$

Proof.

$$\begin{aligned}\sum_{k=n}^{\infty} a_k &= \lim_{m\to\infty} \sum_{k=n}^{m} a_k \\ &= \lim_{m\to\infty} s_m - s_{n-1} \\ &= \lim_{m\to\infty} s_m - \lim +m \to \infty s_{n-1} \\ &= S - s_{n-1}\end{aligned}$$

So if we let $n \to \infty$ we have the result.

□

3. If $\sum |a_n| < \infty$ converges then $\sum a_n$ converges.

Proof. We say that

$$\begin{aligned}|s_m - s_n| &= |\sum_{k=n+1}^{m} a_k| \\ &\leq \sum_{k=n+1}^{m} |a_k| \\ &\leq \sum_{=k+1}^{\infty} |a_k|\end{aligned}$$

So if n is large enough, by 2 we can make this as small as we like. Hence $\{s_n\}$ is Cauchy in $\mathbb{C}$ and converges.

□

4. (Weierstrass M-test) If $\sup_{z\in E} |f_n(z)| \leq M_n$ and $\sum M_n$ converges, then $\sum f_n(z)$ converges uniformly and absolutely on E.

- Can you solve the CR at z_0 without the partials being continuous at z_0?

yes

- Can you solve the CR equations in a domain without f being continuous on D?

Lemma 3. *(Weyl) If f has distributional derivates, solving $\frac{\partial f}{\partial \bar{z}} = 0$ on a domain D, then f is holomorphic in D.*

Theorem 8 (Weierstrass M-test)**.** *If* $\sup_{z\in E}|f_n(z)| \leq M_n$ *and* $\sum_{n=1}^{\infty} M_n$ *converges, then* $\sum f_n(z)$ *converges uniformly and absolutely on* E*.*

Proof. For $m \geq n \geq n_1$ and all $z \in E$

$$\begin{aligned}
|S_n(z) - S_m(z)| &= \left|\sum_{k=n+1}^{m} f_k(z)\right| \\
&\leq \sum_{k=n+1}^{m} |f_k(z)| \\
&\leq \sum_{k=n+1}^{m} M_k \\
&\leq \sum_{k=n+1}^{\infty} M_k
\end{aligned}$$

So if n is large enough, this difference is less than ϵ. So $\{S_n(z)\}$ is Cauchy, so it converges to $S(z)$.

Also, letting $m \to \infty$,

$$|S_n(z) - S_m(z)| \leq \sum_{k=n+1}^{\infty} M_k$$

Which given $\epsilon > 0$ we may make smaller than ϵ if n is large enough uniformly on E. So we conclude that the convergence is uniform. The same analysis will show the convergence is absolute. ☐

1.8 Power Series

Definition 24. *A* ***power series centerd at*** $z_0 = 0$ *is a series of the form*

$$\sum_{n=0}^{\infty} a_n z^n$$

where $a_n \in \mathbb{C}$*.*

EXAMPLES

- $\sum z^n$
- $\sum \frac{z^n}{n!}$
- $\sum n! z^n$

Theorem 9. *Given a power series centered at 0 with coefficients* $\{a_n\} \in \mathbb{C}$*, there exists an* $0 \leq R \leq \infty$ *such that the series converges absolutely if* $|z| < R$*. The convergence is uniform on* $\overline{\mathbb{D}(0,R)}$*. The convergence is uniform on* $\overline{\mathbb{D}(0,R_1)}$ *if* $R_1 < R$*. The series diverges if* $|z| > R$ *and*

$$\lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = \frac{1}{R}$$

(Called Hademard's Formula.) Such an R *is called a* ***Radius of Convergence****.*

Proof. Let R be defined as in Hadamard Formula. We will show that the theorem is true for this R. If $R = 0$, then the series diverges for all z (this will be clear from the next case). Assume that R is finite. Let $R_1 < R$ and let R_2 be $R_1 < R_2 < R$. Hence

$$\frac{1}{R} < \frac{1}{R_2} < \frac{1}{R_1}$$

There exists an n_0 such that if $n > n_0$ then

$$|a_n|^{\frac{1}{n}} < \frac{1}{R_2}$$

So,

$$|a_n| R_1^n \leq \frac{1}{R_2^n} R_1^n = \left(\frac{R_1}{R_2}\right)^n = \lambda^n$$

With $\lambda = \frac{R_1}{R_2}$, hence

$$\sum \lambda^n < \infty$$

So by Weirstrauss M-test $\sum a_n z^n$ converges uniformly on $\overline{\mathbb{D}(0, R_1)}$.

If $R = \infty$, $\limsup |a_n|^{\frac{1}{n}} = 0$, so fix $R_1 < \infty$ and $0 < \lambda < 1$. These is an n_0 such that

$$|a_n|^{\frac{1}{n}} < \frac{\lambda}{R}$$

when $n \geq n_0$, so

$$|a_n| R_1^n \leq \lambda^n$$

Repeat the M-test argument from above. So, the convergence is uniform in $\overline{\mathbb{D}(0, R_1)}$.

We now look at the divergence part of the theorem. If $R = \infty$ there is nothing to prove. Suppose R is finite. Let z with $|z| > R$ be given. Hence

$$\frac{1}{|z|} < \frac{1}{R}$$

Thus there are infinitely many n such that

$$|a_n|^{\frac{1}{n}} > \frac{1}{|z|}$$

Hence $|a_n||z|^n > 1$ for infinitely many n. So the series diverges. If $R = 0$ the same argument works. □

Definition 25.

$$\limsup_{n\to\infty} b_n = \lim_{n\to\infty} (\sup\{b_n, b_{n+1}, \dots\})$$

We say that $\limsup b_n = B$ *when given* $\epsilon > 0$ *there exists an* N_0 *such that* $n \geq n_0$ *implies* $b_n < B + \epsilon$ *and for infinitely many* b_n, $b_m \geq B - \epsilon$. *For* $B = \pm\infty$ *make appropriate modifications. Also have*

$$\liminf_{n\to\infty} b_n = \lim_{n\to\infty} (\inf\{b_n, \dots\})$$

EXAMPLES

- $\sum z^n, a_n \equiv q$ So we have
$$\limsup |a_n|^{\frac{1}{n}} = 1, \qquad R = 1$$
- $\sum \frac{z^n}{n!}$
$$\limsup |\frac{1}{n!}|^{\frac{1}{n}} = \limsup \left| \frac{1}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} \right|^{\frac{1}{n}} = 0$$
- $\sum n! z^n, R = 0$

What happens if $|z| = R$?

1. $\sum z^n = f(z)$
2. $\sum \frac{z^n}{n} = g9z)$
3. $\sum \frac{z^n}{n^2} = h(z)$

All of the above have $R = 1$.

1. We know that $f(z)$ diverges.
2. g is complicated. If $z = 1$ then g diverges. If $z = -1$ then g converges. If $z = i$ then g converges. In fact on all of $\partial\mathbb{D}(0,1) \setminus \{1\}$ the series converges.
3. By Weirstrauss M-test we know that $h(z)$ conerges on $\overline{\mathbb{D}(0,1)}$

Theorem 10. *Suppose $\sum_0^\infty a_n z^n$ has a radius of convergence R, then*

1. *$\sum_1^\infty n a_n z^{n-1}$ has radius of convergence R.*
2. *If $R > 0$ and we let $f(z) = \sum_0^\infty a_n z^n$ for $z \in \mathbb{D}(0,R)$ then $f \in H(\mathbb{D}(0,R))$ and $f' = \sum_1^\infty n a_n z^{n-1}$.*

1.8.1 Examples

- Consider $f(x) = \sum_1^\infty 2^{-n} \cos(3^n x) \in C(\mathbb{R})$. It is know that f' exists nowhere on R, but differentiation of the series gives $\sum \left(\frac{3}{2}\right)^n \sin(3^n x)$ If $x = 0$ then this is 0, but that is not the case for $f'(x)$.
- Let $f_n = \frac{x}{1+n^2 x}$ then $\{f_n\} \to 0$ locally uniformly on $\mathbb{R}$. Also, $f_n'(x) = \frac{1}{(1+n^2x)^2}$, but $f_n'(0) = 1$ for all $n \geq 0$. So
$$\lim_{n\to\infty} f_n'(0) = 1$$

Theorem 11. *Suppose $\sum_0^\infty a_n z^n$ has a radius of convergence R, then*

1. *$\sum_1^\infty n a_n z^{n-1}$ has radius of convergence R.*
2. *If $R > 0$ and we let $f(z) = \sum_0^\infty a_n z^n$ for $z \in \mathbb{D}(0,R)$ then $f \in H(\mathbb{D}(0,R))$ and $f' = \sum_1^\infty n a_n z^{n-1}$.*

Proof. (1)

$$\limsup_{n\to\infty} |na_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty} n^{\frac{1}{n}} \cdot \limsup |a_n|^{\frac{1}{n}}$$
$$= 1 \cdot \frac{1}{R}$$

But $|a_n|^{\frac{1}{n}} \leq |na_n|^{\frac{1}{n}}$ and

$$\frac{1}{R} \leq \limsup |na_n|^{\frac{1}{n}}$$

So the radius of convergence is R.

(2) Let

$$g(z) = \sum^{\infty} na_n z^{n-1}, \quad z \in \mathbb{D}(0,R)$$

Then

$$S_n = \sum_{k=0}^{n} a_k z^k$$

$$R_n = \sum_{k=n+1}^{\infty} a_k z^k$$

So $f(z) = S_n(z) + R_n(z)$. Fix some $z_0 \in \mathbb{D}(0,R)$ and take $\rho > |z_0|$, but $\rho < R$. For $z \neq z_0$ we have

$$\frac{f(z)-f(z_0)}{z-z_0} - g(z_0) = \left[\frac{S_n(z) - S_n(z_0}{z-z_0} - S_n(z_0)\right] + [S_n(z_0) - g(z_0)] + \left[\frac{R_n(z)-R_n(z_0)}{z-z_0}\right] \quad (1)$$

Where we have

$$\frac{R_n(z) - f_n(z_0)}{z-z_0} = \sum_{k=n+1}^{\infty} a_k \frac{z^k - x_0^k}{z-z_0}$$

Assume $z \in \overline{\mathbb{D}(0,\rho)}$, since $z_0 \in \overline{\mathbb{D}(0,\rho)}$ we have

$$\left|\frac{z^k - z_0^k}{z-z_0}\right| = |z^{k-1} + z_0^{k-2} + \cdots + z_0^{k-1}| \leq k\rho^{k-1}$$

So we have

$$\left|\frac{R_n(z)-R_n(z_0)}{z-z_0}\right| \leq \sum_{k=n+1}^{\infty} k|a_k|\rho^{k-1}$$

This sum converges by part (1) of the theorem and thus there exists N_0 such that for $n > N_0$ we have

$$\sum_{k=n+1}^{\infty} k|a_k|\rho^{k-1} \leq \epsilon/3$$

Next, by previous theorems we have $S_n' \to g$ pointwise in $\overline{\mathbb{D}(0\rho)}$, thus there exists an N_1 such that for all $n \geq N_1$ we have

$$|S_n'(z_0) - g(z_0)| < \epsilon/3$$

Lastly, since S_n is a polynomial, it is differentiable on all of $\mathbb{C}$, so take $\delta > 0$ such that

$$\left|\frac{S_n(z) - S_n(z_0)}{z - z_0} - S_n'(z_0)\right| < \epsilon/3$$

So if we put these inequalities into (1) we have the result. □

Corollary 2. *Let $f(z) = \sum_0^\infty a_n z^n$, when the series converges uniformly on $\mathbb{D}(0, R)$ then f and all of its derivatives are holomorphic in $\mathbb{D}(0, R)$. Also,*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n z^{n-k} = k!a_k + \frac{(k+1)!}{1!}a_{k+1}z + \frac{(k+2)!}{2!}z^2 + \cdots$$

Proof. By induction. □

1.8.2 Moving the Center from Zero

Now consider the series $f = \sum a_n(z - z_0)^n$. Letting $g(w) = f(w + z_0)$ we have g is a zero-centered powerseries and

- The series for f converges on $\mathbb{D}(z_0, R)$ when $R = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$ and is holomorphic there.
- For $z \in \mathbb{D}(z_0, R)$, $f'(z) = \sum_1^\infty n a_n (z - z_0)^{n-1}$.

1.8.3 Examples

1. $f(z) = \sum_0^\infty z^n = \frac{1}{1-z}$ the ROC is 1

$$f'(z) = \sum_1^\infty n z^{n-1} = \frac{1}{(1-z)^2}$$

   Hence

$$\frac{z}{(1-z)^2} = \sum_1^\infty n z^n \qquad \text{(Koebe Function)}$$

$$\frac{1}{1-z^2} = \sum z^{2n}$$

$$\frac{1}{1=z} = \sum (-1)^n z^n$$

2. Lets get a series representative for $\frac{1}{1-2z}$ around $z_0 = 0$? How about

$$\frac{1}{1-2z} = \sum_0^\infty (2z)^n, \qquad \text{valid for } |2z| < 1$$

3. (a) Let's expand $\frac{1}{2-z}$ in powers of $(z+1)$. Let $w = z + 1$

$$\frac{1}{2-z} = \frac{1}{3-w} = \frac{1}{3}\cdot\frac{1}{1-\frac{w}{3}}$$
$$= \frac{1}{3}\sum_0^\infty \left(\frac{w}{3}\right)^n$$
$$= \sum_0^\infty 3^{-n-1}(z+1)^n$$

   So valid if $|w| < 3$ and for $|z+1| < 3$.

(b) $\frac{z+2}{(z-2)^2}$ in powers of $(z+1)$.

$$\begin{aligned}\frac{z+2}{(z-2)^2} = \frac{w+1}{(3-w)^2} &= \frac{w}{(3-w)^2} + \frac{1}{(3-w)^2} \\ &= \frac{1}{3}\frac{w}{(1-\frac{w}{3})^2} + \frac{1}{3}\sum_0^\infty (n+1)w^n 3^{-n-1} \\ &= \frac{1}{3}\sum_1^\infty n\left(\frac{w}{3}\right)^n + \frac{1}{3}\sum_0^\infty (n+1)w^n 3^{-n-1} \\ &= \sum_0^\infty \left(\frac{1}{3}n3^{-n} + (n+1)3^{-n-1}\right)(z+1)^n\end{aligned}$$

Valid for $|w| < 3$, hence $z \in \mathbb{D}(-1,3)$.

1.8.4 Facts

- $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$
- $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$
- $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$

1.8.5 Examples

(a) Find ROC of

$$\sum_1^\infty (2^{-\frac{n}{2}} + 4^{-\frac{n}{2}})z^{2n}$$

Note that

$$(2^{-\frac{n}{2}} + 4^{-\frac{n}{2}})^{\frac{1}{2n}} = [2^{-\frac{n}{2}}(1+2^{-\frac{n}{2}})]^{\frac{1}{2n}} = 2^{-\frac{1}{4}}[1+2^{-\frac{n}{2}}]^{\frac{1}{2n}}$$

Since

$$1 \le (1+2^{-\frac{n}{2}})^{\frac{1}{2n}} \le 2^{\frac{1}{2n}} \to 1$$

Hence

$$\lim |a_n|^{\frac{1}{n}} = \frac{1}{2^{\frac{1}{4}}}$$

And $R = \sqrt[4]{2}$

(b) $\sum_1^\infty 2^n z^{n^2}$

$$\limsup |2^n|^{\frac{1}{n^2}} = \lim 2^{\frac{1}{n}} = 1$$

So $R = 1$

(c) $\sum_1^\infty \frac{z^{2^n}}{n!}$

$$\begin{aligned}\limsup \left|\frac{1}{n!}\right|^{\frac{1}{2^n}} &= \lim \left(\left(\frac{e}{n}\right)^n (2\pi n)^{-\frac{1}{2}}\right)^{\frac{1}{2^n}} \\ &= \lim \left(\frac{e}{n}\right)\frac{n}{2^n} \lim(2\pi n)^{\frac{-1}{2^{n+1}}} = 1 \\ \lim \frac{n}{n^2}\log\left(\frac{e}{n}\right) &= \lim \frac{n}{2^n}\left(\log(e) - \log(n)\right) = 0\end{aligned}$$

Hence $ROC = 1$

1.9 The Exponential e^z

Definition 26. *Let*

$$E(z) = \sum_0^\infty \frac{z^n}{n!}$$

For $x \in \mathbb{R}$ let

$$A(x) = E(x)$$

$$B(x) = E(ix)$$

Theorem 12 (Theorem 10)**.** *:*

1. $E(z_1 + z_2) = E(z_1)E(z_2)$
2. *A maps $\mathbb{R}$ onto $(0, \infty)$ moreover, $e = A(1)$, then $A(x) = e^x$.*
3. *$|B(x)| = 1$ for $x \in \mathbb{R}$. Moreover, there exists a number $\pi \in (0, \infty)$ so that B maps $[0, \pi]$ 1-1, onto the upper half of the unit circle, $(\partial\mathbb{D})^+$, with $B(0) = 1$, $B(\pi) = -1$*

Proof. :

(1) Fix $w \in C$, let $f(z) = E(z + w)(E(-z)$. We then have $f'(z) = E'(z + w)E(-z) - E(z + w)E'(-z)$, but $E = E'$, hence $f'(z) = 0$. Thus, f is constant and $f(0) = f(z) = E(w)$. Let $z = -z_2$ and $w = z_1 + z_1$. We then have

$$E(z_1 + z_2) = E(z_1)E(z_2)$$

(2) $A(x) = 1 + x + x^2/2! + \cdots$. Consider $A(x)$ on the interval $[0, \infty)$. We have

$$A(0) = 1$$

All coefficients are positive, thus A maps $[0, \infty) \to \mathbb{R}^+$. Additionally, $A(x) > x$ so A is strictly increasing. So, $A : [0, \infty) \to [1, \infty)$. As A is continuous, we conclude by the intermediate value theorem that A is onto $[1, \infty)$. On $(-\infty, 0)$ we have $A(0) = A(x)A(-x)$, thus A maps $(-\infty, 0)$ 1-1 and onto the interval $(0, 1]$.

Set $e = A(1)$. For $x = n \in \mathbb{Z}^+$ let

$$a^x = \underbrace{a \cdot a \cdots a}_{x \text{ times}}$$

For $x = \frac{1}{n}$ let a^x be the unique positive number such that $b^n = a$. For $x = \frac{p}{q}, p, q \in \mathbb{Z}^+$ let

$$a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p$$

For $x \in \mathbb{R}^+$ we let $a^x = \sup\{a^y : y \in \mathbb{Q}, y < x\}$.

Note that $E(n) = (E(1))^n = e^n$, $n \in \mathbb{Z}^+$.

COMPUTER PROBLEMS SO I DIDN'T FINISH THE PROOF, BUT IT WAS TOO FUNDAMENTAL TO BE USEFUL ANYWAYS!

□

1.9.1 Remarks

1. $B(-x) = \overline{B(x)}$. So B maps $[-\pi, \pi]$ onto $|z| = 1$
2. $B(2\pi) = B(\pi)^2 = (-1)^2 = 1$ So $B(x + 2\pi) = B(x)$.
3. For each $z \in \partial D$ there is a unique $\theta_0 \in [-\pi, \pi]$ such that $B(\theta_0) = z$.

1.9.2 Facts

- $e^{z_1+z_2} = e^{z_1}e^{z_2}$
- $e^{2\pi i} = 1$
- $e^{z+2\pi i} = 1$
- $e^{z+e\pi ik} = e^z$
- $|e^z| = |e^x||e^{iy}| = |e^x|$
- For fixed y, $x \mapsto e^{z+iy}$ is a 1-1 map onto a ray in $\mathbb{C} \setminus \{0\}$. For fixed x we have $y \mapsto e^{x+iy}$ is a 2π periodic cover of the circle of radius e^x and center 0.
- $e^{2\pi ik} = 1$
- $e^{\pi i} = -1$
- If $z = e^{i\theta_0}$ with $\theta_0 \in (0, \pi)$ then $z^2 = e^{i\varphi}$ with $\varphi \in (0, 2\pi)$. Hence $e^{\frac{\pi}{2}i} = i$. So we can take square roots.

1.9.3 Taking Square Roots Examples

$$z = \frac{1+i}{\sqrt{2}}$$
$$x^2 = \frac{2i}{2} = i$$
$$e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}}$$

1.9.4 Recap

Definition 27 (Properties of Exponential)**.**

$$E(z) = \sum \frac{z^n}{n!}$$
$$A(x) = E(x)$$
$$B(x) = E^{ix}$$
$$E(z_1 + z_2) = E(z_1)E(z_2)$$
$$e = A(1)$$
$$|B(x)| = 1$$
$$B : [0, 2\pi] \xrightarrow{\cong} \{z : |z| = 1\}$$

1.9.5 Polar Representation

Theorem 13. *Every $z \in \mathbb{C} \setminus \{0\}$ can be written in the form $z = re^{i\theta}$ with $r > 0, \theta \in \mathbb{R}$, with $r = |z|$, each θ has the form $\theta_0 + 2\pi k$, $k \in \mathbb{Z}$.*

Proof. Consider $\frac{z}{|z|} \in S^1$. Since B is surjective, there exists θ_0 such that

$$z = |z|e^{i\theta_0}$$

$\square$

Definition 28. *The argument of z is $\arg z = \{\theta : z = |z|e^{i\theta}\}$. This is often viewed as a multi-valued function.*

1.9.6 Example

$$2 + 2i = 2\sqrt{2}e^{i(\pi/4)}$$
$$\arg(2 + 2i) = \frac{\pi}{4}$$

1.10 Trigonometric Functions

Definition 29.

$$\cosh z = \frac{e^z + e^{-z}}{2} = \sum \frac{z^{2n}}{(2n)!}$$
$$\sinh z = \frac{e^z - e^{-z}}{2} = \sum \frac{z^{2n+1}}{(2n+1)!}$$
$$\cos z = \cosh(iz) = \sum \frac{(-1)^n z^{2n}}{(2n)!}$$
$$\sin z = \frac{1}{i}\sinh(iz) = \sum \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$
$$\tan z = \frac{\sin z}{\cos z}, \qquad \tanh z = \frac{\sinh z}{\cosh z}$$

1.10.1 Facts

$$\mathrm{Re}(e^{i\theta}) = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta$$
$$\mathrm{Im}(e^{i\theta}) = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta$$
$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$z = x + iy \Rightarrow \tan\theta = \frac{y}{x}$$

1.10.2 Addition Formulas of $\sin z$ and $\cos z$

- $$\sin(z_1 + z_2) = \sin z_1 \cos z_1 + \cos z_1 \sin z_2$$

- $$e^z = e^x \cos y + ie^x \sin y$$

- $$\sin(x + iy) = \sin x \cosh y_i \cos x \sinh y$$

1.11 Complex n^{th} Roots

Problem: For $n \in \mathbb{Z}^+$ want to find all z such that $z^n = 1$.
Solution:

$$\{e^{i\frac{\theta k}{n}} : 0 \leq k < n\}$$

Theorem 14. *Each $z = re^{i\theta} \in \mathbb{C} \setminus \{0\}$ has exactly n n^{th} roots, given by*

$$z_j = r^{\frac{1}{n}} e^{i(\frac{\theta}{n} + \frac{2\pi j)}{n}}, \qquad j = 0, \ldots, n-1$$

1.11.1 Example

$$(2i)^{\frac{1}{4}}$$

$$2i = 2e^{i(\pi/2)}$$

Let $\alpha = 2^{1/4}$

$$(2i)^{\frac{1}{4}} = \left\{\alpha e^{i(\pi/8)}, \alpha e^{i(\pi/8+\pi/4)}, \alpha e^{i(\pi/8)+\pi/2}, \alpha e^{i(\pi/8+3\pi/4)}\right\}$$

1.12 Logarithms and Powers

1.12.1 Natural log of z

Definition 30. *For $z \in \mathbb{C} \setminus \{0\}$, $\log z := \left\{\exp^{-1}(z)\right\} = \{w \in \mathbb{C} : e^w = z\}$.*
For $w = u + iv$ we have $w = |w|e^{i\theta}$ and so

$$\log(z) = \{\log(|w|) + i\theta + 2\pi ik, k \in \mathbb{Z}\}$$

1.12.2 Example

- Find all values of $\log(2 + 3i)$. Well, $2 + 3i = \sqrt{13}e^{i\tan^{-1}(3/2)}$. We then have

$$\log(2 + 3i) = \log\sqrt{13} + i\theta + 2\pi ik, \qquad k \in \mathbb{Z}$$

1.12.3 Complex Powers

Definition 31. $z^w := e^{w \log z}$. *(Note this is a multivalued function.)*

1.12.4 Example

- $$i^i = i\log i = i(0+\frac{\pi}{2}i+2\pi ik) = -\frac{\pi}{2}-2\pi k$$

- Find all values of $(1+i)^{2+i} = e^{(2+i)\log(1+i)}$. First we have

$$1+i = \sqrt{2}2^{\frac{\pi i}{4}}$$

$$\log(1+i) = \log(\sqrt{2})+\frac{\pi i}{4}+2\pi ik$$

$$(2+i)\log(1+i) = (2+i)\frac{\log 2}{2}+\frac{\pi}{4}(2i-1)+2\pi k(2i-1) = (\log 2-\frac{\pi}{4}+2\pi k)+i(\frac{\pi}{2}+4\pi k)$$

$$(1+i)^{2+i} = 2e^{-\frac{\pi}{4}+2\pi k}e^{i(\frac{\pi}{2}+4\pi k+\frac{\log 2}{2})} = 2e^{-\frac{\pi}{4}+2\pi k}e^{i(\frac{\pi}{2}+\frac{\log 2}{2})}$$

Notice that every value has the same θ and thus belongs to the same ray.

- $|z^w| = |e^{w\log z}| = e^{\mathrm{Re}(w\log z)}$

- What about with real numbers?

$$(1+i)^2 = e^{2\log(1+i)} = e^{2(\log\sqrt{2}+\frac{\pi i}{4}+2\pi ik)} = e^{\log 2+\frac{\pi i}{2}+2\pi ik} = 2e^{i\frac{\pi]}{2}} = 2i$$

1.13 Inverse Trig and Hypertrig Functions

Definition 32.

$$\arcsin z := \sin^{-1}(z) = \{w\in\mathbb{C} : \sin w = z\} = \frac{1}{i}\log(iz+\sqrt{1-z^2})$$

$$\arccos z = \frac{\pi}{2}+i\log\left(iz+\sqrt{1-z^2}\right)$$

Proof. So, we want to solve

$$\sin w = \frac{e^{iw}-e^{-iw}}{2i} = z$$

$$e^{iw}-e^{-iw} = 2iz \Rightarrow e^{2iw}-1 = 2ie^{iw}z$$

Solving the quadratic

$$\zeta^2-(2iz)\zeta-1 = 0$$

we obtain

$$\zeta = iz\pm\sqrt{1-z^2}$$

□

2 Cauchy's Theorem

2.1 Line Integrals

Definition 33. *Let $h\in C([a,b],\mathbb{C})$, $[a,b]\subseteq\mathbb{R}$*

$$\int_a^b h(t)dt = \int_a^b \mathrm{Re}(h)dt+\int_a^b \mathrm{Im}(h)dt$$

2.1.1 Facts

1. Linearity: $\forall, c_1, c_2, h_1, h_2$ it holds that

$$\int_a^b (c_1 h_1 + c_2 h_2)dt = c_1 \int_a^b h_1 dt + c_2 \int_a^b h_2 dt$$

2. $\left|\int h(t)dt\right| \leq \int_a^b |h(t)|dt$

Definition 34. *A curve $\gamma \in C([a,b],\mathbb{C})$ is called a **curve**, $[a,b]$ is called the **parametric interval of** γ. The image of γ is called the **trace of** γ.*

2.1.2 Example

- Let $\gamma(t) = z_0 + re^{it}$ with $t \in [0, 2\pi)$. This has the same trace as the different curve $\gamma_1(t) = z_0 + re^{it}, t \in [0, 2\pi]$.
- An oriented line segment connecting z_0 and z_1 is given by

$$\gamma(t) = z_0(1-t) + z_1 t$$

Definition 35. *A curve $\gamma : [a,b] \to \mathbb{C}$ is said to be **piecewise** C^1 or often **paths** if there exist $a \leq s_1, \dots, s_n \leq b$ such that $\gamma\big|_{[s_j, s_{j+1}]} \in C^1([s_j, s_{j+1}]$.*

$$\gamma' = (\operatorname{Re}\gamma)' + i(\operatorname{Im}\gamma)'$$

Definition 36. *Let $\gamma : [a,b] \to \mathbb{C}$ be a path, and $f \in C(\gamma[a,b]m\mathbb{C})$. Define*

- $\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt$
- $\int_\gamma f(z)|dz| := \int_a^b f(\gamma(t))|\gamma'(t)|dt$
- $\int_\gamma f(z)dx := \int_a^b f(\gamma(t))\operatorname{Re}(\gamma'(t))dt$
- $\int_\gamma f(z)dy := \int_a^b f(\gamma(t))\operatorname{Im}(\gamma'(t))dt$

Definition 37. *The **length** of γ is*

$$\int_\gamma |dz| = \int_a^b |\gamma'(t)|dt$$

2.1.3 Examples

- $\gamma(t) = re^{it}$, $t \in [\theta_1, \theta_2]$. We then have

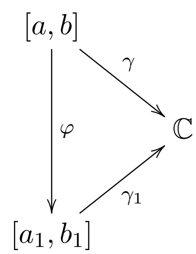
$$l(\gamma) = \int_{\theta_1}^{\theta_2} |\gamma'(t)|dt = r(\theta_2 - \theta_1)$$

$$|\gamma'(t) = |ire^{it}| = r$$

- $$\int_\gamma \overline{z}^2 dz = \int_{\theta_1}^{\theta_2} \overline{\gamma(t)}^2 \gamma'(t) dt$$
$$= \int_{\theta_1}^{\theta^2} r^2 e^{-2it} i r e^{it}$$
$$= i r^3 \int_{\theta_1}^{\theta^2} e^{-it} dt$$
$$= r^3 [e^{-i\theta_2} - e^{-i\theta_1}]$$

- $$\int_\gamma \overline{z}^2 dy = \int_{\theta_1}^{\theta^2} r^2 e^{-2it} r \cos t dt = r^3 \int_{\theta_1}^{\theta^2} e^{-2it} \left(\frac{e^{it} + e^{-it}}{2} \right) dt = \cdots$$

Definition 38. *$\gamma : [a,b] \to \mathbb{C}$, then another curve $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ is called a* ***reparametrization*** *if there exists a $\varphi : [a,b] \to [a_1, b_1]$ strictly increasing C^1-mapping so that the diagram commutes.*



Definition 39. *Suppose $\gamma : [a,b] \to \mathbb{C}$ is a curve, then $\gamma^\circ(t) = \gamma(b + a - t)$ is called* ***the curve opposite*** *γ, denoted by γ°.*

2.1.4 Facts

1. $$\left| \int_\gamma f(z) dz \right| \leq \int_\gamma |f(z)||dz| \leq \sup_\gamma |f(z) \cdot l(\gamma)|$$

2. If γ_1 is a reparametrization of γ_2 then
$$\int_{\gamma_1} f(z) dz = \int_\gamma 2 f(z) dz$$
(Also holds for integral over $dx, dy, |dz|$)

Proof.
$$\int_{\gamma_1} f(z) dz = \int_{a_1}^{b_1} f(\gamma_1(t)\gamma_1'(t) dt$$
$$= \int_a^b f(\gamma_1(\varphi(u))\gamma_1'(\varphi(u))\varphi'(u) du$$

Then substitute in $t = \varphi(u)$.

□

3. $\int_\gamma f(z)dz = -\int_{\gamma^\circ} f(z)dz$
4. $\int_\gamma f(z)|dz| = \int_{\gamma^\circ} f(z)|dz|$

Theorem 15 (Cauchy's Theorem)**.** *Suppose that D is a contractible open set and γ is a closed curve in D and F is holomorphic in D then*

$$\int_\gamma F(z)dz = 0$$

We need to gear up to prove this...

Theorem 16 (Theorem 1)**.** *Suppose that D is open in $\mathbb{C}$ and $f \in H(D)$, also suppose there exists $F \in H(D)$ such that $F'(z) = f(z)$ for all $z \in D$. Then for all paths $\gamma : [a, b] \to D$ with $\gamma(a) = z_1$ and $\gamma(b) = z_2$ then*

$$\int_\gamma f(z)dz = F(z_2) - F(z_1)$$

Proof. We have

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t_1)) \cdot \gamma'(t_1)dt$$

Let $h(t) = F(\gamma(t))$ then

$$h'(t) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t)$$

Thus,

$$\int_\gamma f(z)dz = \int_a^b h'(t)dt = h(b) - h(a) = F(z_2) - F(z_1)$$

□

Corollary 3 (Corollary 1)**.** *If γ is closed then $\int_\gamma f(z)dz = 0$.*

Corollary 4. *For all complex polynomials $P(z)$ and a closed path γ we have*

$$\int_\gamma P(z)dz = 0$$

2.2 Cauchy's Theorem

Corollary 5. *Suppose γ is a closed path in $\mathbb{C} \setminus \{z_0\}$ and $n \geq 2$ we have*

$$\int_\gamma \frac{1}{(z - z_0)^n} dz = 0$$

Proof. Let $F(z) = (z - z_0)^{1-n} - \frac{1}{1-n}$ then $F'(z) = \frac{1}{(z-z_0)^n}$ and $F \in H(\mathbb{C} \setminus \{z_0\})$, so apply Corollary 1.

□

2.2.1 Example

Let $f(z) = \frac{1}{z}$, $\gamma(t) = Re^{it}$, $0 \le t \le 2\pi$.

$$\int_\gamma \frac{dz}{z} = \in_0^{2\pi} \frac{1}{Re^{it}} Rie^{it} dt = i \int_0^{2\pi} dt = 2\pi i$$

Definition 40. *A* ***rectangle*** *in $\mathbb{C}$ is a set of the form $[z,b] \times i[c,d]$ with $a,b,c,d \in \mathbb{R}$. Note $\mathbb{R}$ is a closed set and ∂R is the boundary of this set. We also denote*

$$\partial R = \gamma(t) = \begin{cases} z_0(1-t) + tz_1, & t \in [0,1] \\ z_1(2-t) + (1-t)z_1, & t \in [1,2] \\ z_2(3-t) + (2-t)z_3, & t \in [2,3] \\ z_3(3-t) + (3-t)z_0, & t \in [3,4] \end{cases}$$

Theorem 17 (Theorem 2)**.** *(Cauchy's Theorem) Let R be a rectangle and suppose f is holomorphic in an open set which contains R. Then $\int_{\partial R} f(z)dz = 0$.*

Proof. Fix R and subdivide into four congruent rectangles, $R_1, \cdots, R_4$. Let $I(R) = \int_{\partial R} f(z)dz$. Observe that

$$I(R) = \sum_{j=1}^4 I(R_j)$$

Hence

$$|I(R)| \le \sum |I(R_j)|$$

This means that at least one of the R_j satisfies

$$|I(R_j)| \ge \frac{1}{4}|I(R)|$$

Call this rectangle R^1. Apply this argument to R^1 to get R^2. We then have

$$|I(R^2)| \ge \frac{1}{4^2}|I(R)|$$

Continuing this process we obtain a sequence of rectangles with

$$R^1 \supset R^2 \supset \cdots$$

and

$$|I(R^n)| \ge 4^{-n}|I(R)| \tag{2}$$

$$L(R^n) = 2^{-n}L(R) \tag{3}$$

This $\{R_n\}$ of nested compact sets must have nonempty intersection. Since $diam(R^n) \to 0$ there exists exactly one point z_0 so that $\{z_0\} = \bigcap R^n$. By assumption f is differentiable at z_0 so we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \lambda(z)(z - z_0)$$

Where $\lambda(z)$ is continuous in a nbhd of z_0 and $\lim_{z \to z_0} \lambda(z) = 0$ (4). Since $f(z_0) + f'(z_0)(z - z_0)$ is a polynomial and ∂R^n is a closed curve for all $n \ge 1$ its integral over ∂R^n is 0 for $n \ge 1$. So we have for n large enough

$$I(R^n) = \int_{\partial R^n} \lambda(z)(z - z_0)dz$$

Hence if $L(R^n)$ denotes the length of the perimeter of R^n we have

$$\begin{aligned} |I(R^n)| &\leq \int_{\partial R^n} |\lambda(z)||(z-z_0)||dz| \\ &\leq \sup_{z\in\partial R^n} |\lambda(z)| \cdot \sup_{z\in R^n} |z-z_0| L(R^n) \end{aligned}$$

Since $z \in R^n$ we have $\sup |z - z_0| < L(R^n)$. So by (3) we have

$$|I(R^n)| \leq (L(R^n))^2 \sup |\lambda(z)| = 2^{-2n} L(R^1)^2 \sup |\lambda(z)|$$

By (2) and the above line we have

$$|I(R)| \leq (L(R))^2 \sup_{\partial R^n} |\lambda(z)|$$

for all n. By (4) $\lambda \to 0$ as $n \to \infty$. So $I(R) = 0$. □

Theorem 18 (Green's Theorem)**.** *If* $f, g \in C^1(\mathbb{C}, \mathbb{C})$ *then*

$$\int_{\partial R} [f(z)dz + g(z)dy] = \int_R (g_x - f_y) dxdy$$

$$\begin{aligned} \int_{\partial R} h(z)dz &= \int_{\partial R} h(z)dx + ih(z)dy \\ &= \int\int_R (ih_x - h_y) dxdy \\ &= 2i \int\int_R (\frac{1}{2}h_x + \frac{1}{2}ih_y) dxdy \\ &= 2i \int\int_R h_{\overline{z}} dxdy \end{aligned}$$

If f is holomorphic in a set containing R so that $\frac{\partial f}{\partial \overline{z}} = 0$ so $I(R) = 0$.

2.3 More Cauchy

Theorem 19 (Theorem 3)**.** *(Improvement of Cauchy) Let* $R \subseteq D$ *and let* $f \in C^0(D)$ *and holomorphic in* D *except perhaps at a single point* $z_0 \in D$*. Then*

$$\int_{\partial R} f(z)dz = 0$$

Proof. :

1. If $z_0 \in D \setminus R$ proof of Cauchy still holds.

2. Assum z_0 is a corner of R. Subdivide R into four rectangles, but this time they are disproportional with an arbitrarily small rectangle, R_1 at the z_0 corner.

$$I(R) = \sum I(R_j)$$

Theorem 2 implies that $I(R_j) = 0$ for $j = 2, 3, 4$ So $I(R) = I(R_1)$. But f is bounded on R_1 and so

$$|I(R_1)| \leq M \cdot L(\partial R_1)$$

Since R_1 can be made arbitrarily small we have $l(R_1) = 0$ and so $I(R) = 0$

3. If $z_0 \in R$ then subdivide so on a corner and apply case 2.

$\square$

Theorem 20 (Theorem 4). *(Cauchy's Theorem for Disk) Suppose $f \in H(\mathbb{D}(z_0, R))$, then for every closed path, γ, in D*

$$\int_\gamma f dz = 0$$

Proof. Let $\Delta = \mathbb{D}(z_0, R)$. We will build an antiderivative for f. Let $F(z) = \int_\sigma f(\zeta)d\zeta$ where σ is the path that goes from z_0 to $\operatorname{Re} z$ and then from $\operatorname{Re} z$ to $\operatorname{Re}(z) + i \operatorname{Im} z$. Claim that if $F \in C^1(\Delta)$ and $F_x = -iF_y$ in Δ then $F' = \frac{\partial F}{\partial x} = f$. If this is valid, then $F \in H(\Delta)$, $F' = f$ and theorem 4 follows from Theorem 1.

With $z \in \Delta$ fixed, let h be small and positive. We have

$$\begin{aligned} F(z + ih) - F(z) &= \int_{L(z,z+ih)} f(\zeta)d\zeta \\ &= \int_0^h f(z + it)idt \end{aligned}$$

Since f is continuous we know that

$$\frac{F(z + ih) - F(z)}{h} = \frac{i}{h}\int_0^h f(z + it)idt \underset{h\to 0}{\longrightarrow} if(z)$$

Similarly as $h \to 0^-$ we have

$$\lim_{h\to 0^-} = \frac{F(z + ih) - F(z)}{h} = if(z)$$

So $\frac{\partial F}{\partial y}$ exists and is equal to $if(z)$. Similarly, we may show that $\frac{\partial F}{\partial [}x]$ exists and is $f(z)$. To do this we use a different path $\tau : z_0 \to \operatorname{Im}(z)i \to \operatorname{Re}() + \operatorname{Im}(z)i$ observe that we have a rectangle and thus by Cauchy we have

$$0 = \int_\sigma f(\zeta)d\zeta - \int_\tau f(\zeta)d\zeta$$

Since $f \in C(\Delta)$ then $F \in C^1(\Delta)$. Since $F_x = -iF_y$ and $\frac{\partial F}{\partial x} = f$ we have $F \in H(\Delta)$ and we are done by Theorem 1.

$\square$

Remark: The above prove used the fact that the rectangle connecting any two points in the disk is contained in the disk. So, we only need that property to hold for our domain.

Theorem 21 (Theorem 5). *Let $\Delta = \mathbb{D}(z_0, R)$ as above and let $f \in C(\Delta)$ and $f \in H(\Delta \setminus \{z_1\})$ for some $z_1 \in \Delta$. Then if γ is a closed path then*

$$\int_\gamma f(\zeta)d\zeta = 0$$

Proof. Same as above but invoke Theorem 3.

$\square$

2.4 Resumé of Connectivity

Definition 41. *A set $S \subseteq C^*$ is* ***connected*** *whenever $S = A \cup B$ with A, B relatively open and disjoint implies that A or B is empty.*

Proposition 2 (Proposition 2)**.** *Let $D \subseteq \mathbb{C}^*$ be open. D is connected if and only if it is arcwise connected.*

Theorem 22 (Jordan Curve Theorem)**.** *The complement of a Jordan Curve, $\mathbb{C}^* \setminus J = D_1 \cup D_2$ where D_1, D_2 are open, connected and disjoint.*

Definition 42. *A* ***component*** *of a set $S \subseteq \mathbb{C}^*$ is a connected subset which is contained in no strictly larger connected subset.*

Proposition 3 (Proposition 3)**.** *Let $S \subseteq \mathbb{C}^*$. (a) The components of S are disjoint. (b) S is equal to the union of its components.*

Proposition 4 (Proposition 4)**.** *The components of an open set are open.*

Proposition 5 (Proposition 5)**.** *A subset of $|R$ is connected iff it is an interval.*

Proposition 6 (Proposition 6)**.** *Let $S \subseteq \mathbb{C}^*$ and $f : S \to \mathbb{C}^*$, $f \in C(S)$, then $f(S)$ is connected.*

2.5 Winding Number

Definition 43. *Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. Let $a \in \mathbb{C} \setminus \operatorname{Im}(\gamma)$, then*

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - a}$$

is called the ***winding number of*** *γ* ***around*** *a.*

2.5.1 Example

1. Let $\gamma = z_0 + re^{it}$, $t \in [0, 2\pi k]$ then

$$n(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} dz = \frac{1}{2\pi i} \int_0^{2\pi k} \frac{1}{re^{it}} ire^{it} dt = \frac{1}{2\pi} \int_0^{2\pi k} dt = k$$

2. Note that if we reverse γ we get $n(\gamma^*, z_0) = -k$.

3. Let Δ be an open disk, $\gamma \subseteq \Delta$ and $a \in \mathbb{C} \setminus \Delta$. Then we have

$$n(\gamma, a) = 0$$

Because $\frac{1}{z-a} \in H(\Delta)$ so by Cauchy we have the integral is zero around any closed curve.

Theorem 23 (Theorem 6)**.** *Let γ be closed in $\mathbb{C}$ and let $D = \mathbb{C} \setminus \gamma$, then*

(a) $n(\gamma, a) \in \mathbb{Z}$

(b) If a, b belong to the same component of D then $n(\gamma, a) = n(\gamma, b)$.

(c) If a is in the unbounded component of D, then $n(\gamma, a) = 0$.

Proof. Suppose $\gamma : [\alpha, \beta] \to \mathbb{C} \setminus \{a\}$. We define $h(u) = \int_\alpha^u \frac{\gamma'(t)}{\gamma(t)-a} dt$. Then we have for $u \in [\alpha, \beta]$ that

1. $h(\beta = n(\gamma, a) - e\pi i$
2. $h'(u) = \frac{\gamma'(u)}{\gamma(u)-a}$ for $u \in [\alpha, \beta] \setminus F$ where F is finite.

We claim that $e^{-h(t)}[\gamma(t) - a]$ is constant on $[\alpha, \beta]$. Assume the claim, then

$$e^{-h(\alpha)}[\gamma(\alpha) - a] = e^{-h(\beta)}[\gamma(\beta) - a]$$

Since $\gamma(\alpha) = \gamma(\beta) \neq a$ so

$$e^{-h(\alpha)} = e^{-h(\beta)}$$

So

$$h(\beta) - h(\alpha) = 2\pi i k$$

for some $k \in \mathbb{Z}$. Also, $h(\alpha) = 0$

$$h(\beta) = n(\gamma, a)2\pi i = 2\pi i k$$

So $n(\gamma, a) = 2\pi i k$.

It remains to prove the claim. For $t \in [\alpha, \beta] \setminus F$ we have

$$\begin{aligned}
\frac{d}{dt}\left(e^{-h(t)}[\gamma(t) - a]\right) &= -e^{-h(t)}h'(t)[\gamma(t) - a] + e^{-h(t)}\gamma'(t) \\
&= -e^{-h(t)}\gamma'(t) + e^{-h(t)}\gamma'(t) \\
&= -e^{-h(t)}\gamma'(t) + e^{-h(t)}\gamma'(t) \\
&= 0
\end{aligned}$$

So the quantity $e^{-h(t)}[\gamma(t) - a]$ is constant on intervals in $[\alpha, \beta] \setminus F$. Since h is continuous, it is constant on $[\alpha, \beta]$. So we have (a).

Now we seek (b). We'll show that

$$a \mapsto n(\gamma, a)$$

is continuous on D. Since $n(\gamma, a)$ is integer valued, it must be constant on each component of D. So, take $a_0, a \in D$ and we have

$$\begin{aligned}
|n(\gamma, a) - n(\gamma, a_0)| &= \left|\frac{1}{2\pi i}\int_\gamma \frac{a - a_0}{(z - a)(z - a_0)} dz\right| \\
&\leq \frac{1}{2\pi} l(\gamma)|a - a_0| \sup_\gamma \frac{1}{|z - a|}|z - a_0|
\end{aligned}$$

Let $d = \inf_\gamma |z - a_0|$, then $d > 0$. If $|a - a_0| < \frac{1}{2}d$ then $|z - a| > d/2$. So

$$\sup \frac{1}{|z - a||z - a_0|} \leq \frac{2}{d^2}$$

So

$$|n(\gamma, a) - n(\gamma, a_0| \leq \frac{1}{2}|a - a_0|\frac{2}{d^2}l(\gamma)$$

So as the RHS goes to 0 we have the LHS goes to 0, thus $n(\gamma, blank)$ is continuous in D.

To prove (c) we note that $\gamma[\alpha,\beta]$ is bounded. So $\gamma[\alpha,\beta] \subseteq \mathbb{D}(0,R)$ for some $R > 0$. If $a_0 \in \mathbb{C} \setminus \mathbb{D}(0,R)$, $\frac{1}{z-a_0}$ is in $H(\mathbb{D}(0,R))$, using Cauchy's Theorem on the disk we find $n(\gamma, a_0) = 0$. As ta_0 for $1 \leq t \leq \infty$ is a path in $\mathbb{C} \setminus \mathbb{D}(0,R)$ from a_0 to ∞, we have a_0 is in the unbounded component, So $n(\gamma, a) = 0$ is in the unbounded component.

□

2.5.2 Remarks

1. For $K \subseteq \mathbb{C}$ compact let D_1 be the component of $\mathbb{C}^* \setminus K$ which contains the point at infinity, ∞. Then $D \setminus \{\infty\}$ is called the **unbounded component**.

2. Let $a = 0$ and $n(\gamma, 0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z}$. Heuristically this is equal to

$$= \frac{1}{2\pi i} \log z \Big|_{\gamma(\alpha)}^{\gamma(\beta)} = \frac{1}{2\pi i}\left[i(\arg(\gamma(\beta)) - \arg(\gamma(\alpha))\right] = k$$

3. We defined $n(\gamma, a)$ only for piecewise C^1 γ, but it can be defined for all curves. If $\gamma \in C([\alpha,\beta], \mathbb{C} \setminus \{a\})$ then γ is homotopic to some mapping $t \mapsto a + e^{it}$ with $t \in [0, 2\pi k]$. Then use the winding number for this path.

About the Exam

- Covers all material up to end of lecture today
- Should know statements of all theorems
- Proofs of all theorems up to Theorem 8 in Ch. 2
- Work Problems (For extra problems look in (Gamlin and Ahlfors)

2.6 Cauchy's Formula for a Disk

Theorem 24 (Theorem 7)**.** *Set $\Delta = \mathbb{D}(z_0, R)$, if $f \in H(\Delta)$ and γ is a closed path in Δ, then for $a \in \Delta \setminus \gamma$ we have*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = n(\gamma, a) f(a)$$

Proof. Set $g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a}, & z \in \Delta \setminus \{a\} \\ f'(a), & z = a \end{cases}$. So we have $g \in H(\Delta \setminus \{a\})$ and by the technical improvement to Cauchy's Theorem we have $g \in H(\Delta)$. Cauchy's Theorem says that

$$\int_\gamma g(z)dz = 0$$

So, as $a \notin Im(\gamma)$ we have

$$\begin{aligned}0 &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(z)}{z-a}\\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz - \frac{1}{2\pi i}\int_\gamma \frac{f(a)}{z-a}dz\\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz - f(a)n(\gamma,a)\end{aligned}$$

□

Definition 44. *Let $\overline{E}$ denote the **closure** of a set E. If $\Delta = \mathbb{D}(z_0, R)$ then $\partial\Delta$ is the curve $\gamma(t) = z_0 + Re^{it}$ for $0 \le t \le 2\pi$, but $\partial\Delta$ is also a point set.*

Theorem 25 (Theorem 8)**.** *Suppose D is an open set, Δ is a disk with $\overline{\Delta} \subseteq D$ and $f \in H(D)$, then for $a \in \Delta$ we have*

$$f(a) = \frac{1}{2\pi i}\int_{\partial\Delta} \frac{f(z)}{z-a}dz$$

Proof. We know that $n(\partial\Delta, a_0) = 1$. By Theorem 5b, $n(\partial\Delta, a) = 1$. Now there is a Δ_1 with

$$\overline{\Delta} \subseteq \Delta - 1 \subseteq \overline{\Delta} - 1 \subseteq D$$

Now use Theorem 7 on Δ_1 and we get the conclusion.

□

2.7 Power Series Representations

Lemma 4 (Lemma 1)**.** *Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a path and $\{f_n\}$ a sequence of continuous functions on $\gamma[\alpha, \beta]$ which converge uniformly on γ to f. Then*

$$\lim_{n\to\infty}\int_\gamma f_n dz = \int_\gamma f dz$$

Proof. Observe that if n is large enough, then for any $\epsilon > 0$

$$\left|\int_\gamma f_n dz - \int_\gamma f dz\right| \le \int_\gamma |f - f_n| dz < \epsilon l(\gamma)$$

□

Lemma 5 (Lemma 2)**.** *Let γ be as in Lemma 1. Suppose $\{g_n\}$ is a sequence of functions on γ and $\sum_0^\infty g)n$ converes uniformly on γ and f is continuous on γ then*

$$\int_\gamma f\left(\sum g_n\right)dz = \sum_0^\infty \int_\gamma f g_n dz$$

Proof. We have

$$f \cdot \sum_0^k g_n \to f \cdot \sum^\infty g_n$$

uniformly on γ since f is bounded on γ. By lemma 1 we have

$$\lim_{k\to\infty}\int_\gamma f\sum^k g_n dz = \int_\gamma f\sum^\infty g_n dz$$

But, for $k \in \mathbb{Z}^+$ the LHS is equal to

$$\sum^k \int_\gamma f g_n dz$$

□

Theorem 26 (Theorem 9)**.** *Let $f \in H(D)$, D open in $\mathbb{C}$. Then f has complex derivatives of all orders at every point in D, the functions $f^{(n)}$, $n \geq 1$ are all holomorphic in D.*

Theorem 27 (Theorem 10)**.** *Suppose $f \in H(D)$ and $\mathbb{D}(z_0, R) \subseteq D$ for some $R > 0$, $z_0 \in D$. Then for all $z \in \mathbb{D}(z_0, R)$ we have*

$$f(z) = \sum_0^\infty \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

In particular, the Taylor series at z_0 of f converges in $\mathbb{D}(z_0, R)$.

Proof. Let $\rho \in (0, R)$ and set

$$\Delta = \mathbb{D}(z_0, R)$$
$$\Delta_1 = (z_0, \rho)$$

By theorem 8 we know that $z \in \Delta_1$. Also

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Delta_1}\frac{f(\zeta)}{\zeta - z}d\zeta \tag{4}$$

Now we have

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0}\frac{1}{(1 - \frac{z - z_0}{\zeta - z_0})}$$

If $z \in \Delta_1$ then we have $\zeta \in \partial\Delta_1$,

$$\left|\frac{z - z_0}{\zeta - z_0}\right|$$

Convergence is uniform if $\zeta \in \partial\Delta_1$. By Lemma 2 we have

$$\int_{\partial\Delta_1} f(\zeta)\frac{d\zeta}{z - \zeta} = \int_{\partial\Delta_1} f(\zeta)\sum_0^\infty \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}d\zeta = \sum_{n=0}^\infty \left(\int_{\partial\Delta_1} f(\zeta)\frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}d\zeta \qquad (2)\right)$$

Let

$$c_n = \frac{1}{2\pi i}\int_{\partial\Delta_1}\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta, \qquad (3)$$

From (1) and (2) we have

$$f(z) = \sum^{\infty} c_n(z - z_0)^n, \qquad z \in \Delta_1, \qquad (4)$$

By Corollary to Theorem 9 in Chapter 1, f has complex derivatives of all orders in Δ_1. Since every $z \in \mathbb{D}$ belongs to some disk Δ_1 so that $\overline{\Delta}_1 \subseteq D$ we know that $f^{(n)} \in H(D)$ because differentiability is a local property.

Returning to (4) from chapter 1 we also know that $c_n = \frac{1}{n!} f^{(n)}(z_0)$. so for each $z \in \mathbb{D}(z_0, R)$ there is a $\rho \in (0, R)$ so that $z \in \mathbb{D}(z_0, \rho)$.

So, by (3) and (4) we have $f(z) = \sum_0^\infty \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$ for all $x \in \mathbb{D}(z_0, R)$.

□

Definition 45. *For $z \in \mathbb{C}$, $X \subseteq \mathbb{C}$ define $d(z, X) = \inf\{|z - x| : x \in X\}$.*

Theorem 28 (Theorem 11)**.** *Let $f \in H(D)$, D an open set, $z_0 \in D$ and $d = d(z_0, \partial D)$, where $d = \infty$ if $D = \mathbb{C}$. Then there is a poser series converging in $\mathbb{D}(z_0, d)$ whose sum is $f(z)$, namely the one with $c_n = \frac{f^{(n)}(z_0)}{n!}$. So the ROC is $\geq d$.*

Proof. Follows from previous Theorem.

□

2.8 Cauchy's Formula for Derivatives

Lemma 6 (Integration by Parts)**.** *Let $D \subseteq \mathbb{C}$ be open, γ a closed path in D, and $f, g \in H(D)$, then*

$$\int_\gamma fg' dz = -\int_\gamma f' g dz$$

Proof. Let $F = fg$, then Theorem 1 applied to F' says that

$$0 = \int_\gamma F' dz = \int_\gamma fg' + f'g dz$$

□

Corollary 6. *By induction,*

$$\int_\gamma fg^{(k)} dz = (-1)^k \int_\gamma f^{(k)} g dz$$

Theorem 29 (Theorem 12)**.** *Let $\Delta = \mathbb{D}(z_0, R)$, let $f \in H(\Delta)$ and γ a closed path in Δ. Choose $z \in \Delta \setminus \gamma$. Then for $k = 0, 1, 2 \ldots$*

$$n(\gamma, z) f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f(\zeta}{(\zeta - z)^{k+1}} d\zeta$$

Proof. We know that $f^{(k)} \in H(\Delta)$. By Theorem 7 we know that

$$n(\gamma, z_0) f^{(k)}(z) = \frac{1}{2\pi i} \int_\gamma \frac{f^{(k)}(\zeta)}{(\zeta - z_0)} d\zeta$$

for $z_0 \in \Delta$. Now

$$\frac{d^k}{d\zeta^k} \left(\frac{1}{\zeta - z_0} \right) = (-1)^k \frac{k!}{(\zeta - z)^{k+1}}$$

Using integration by parts lemma in $D = \Delta \setminus \{z_0\}$ we obtain

$$n(\gamma, z_0) f^{(k)}(z_) = (-1)^k \frac{1}{2\pi i} \int_\gamma f(\zeta)(-1)^k \frac{k!}{(\zeta - z_0)^{k+1}} d\zeta$$

□

Theorem 30 (Theorem 13)**.** *Let $f \in H(D)$, D open, suppose D contains the closed disk $\overline{\Delta}$. Then for $x \in \Delta$ and $n = 0, 1, 2 \ldots$ then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial\Delta} f(\zeta) \frac{d\zeta}{(\zeta - z)^{k+1}}$$

Proof. Use Theorem 8 instead of Theorem 7 in previous proof. □

2.9 Consequences of Cauchy's Formula

We've seen that if $f \in H(\Delta)$ with $\Delta = \mathbb{D}(z_0, R)$ and f has representation as a power series

$$f(z) = \sum_0^\infty a_n (z - z_0)^n, \qquad z \in \Delta$$

with $a_n = \frac{f^{(n)}(z_0)}{n!}$, with γ any closed path in $\Delta \setminus \{z_0\}$ with $n(\gamma, z_0) = 1$ then

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Theorem 31 (Theorem 14: Cauchy's estimates)**.** *Suppose $f(z) = \sum c_n (z - z_0)^n \in H(D)$,with $D = \mathbb{D}(z_0, R)$, then for $n = 0, 1, \ldots$ we have*

$$|c_n| = \frac{1}{n!} |f^{(n)}(z_0)| \leq \frac{1}{R^n} \cdot \sup_\Delta |f|$$

Proof. Let $\gamma = \partial\mathbb{D}(z_0, \rho)$ with $\mathbb{D}(z_0, \rho) = \Delta_1$ for some $\rho \in (0, R)$. We have

$$\begin{aligned}
|c_n| &= \frac{1}{n!} |f^{(n)}(z_0)| \\
&= \left| \frac{1}{2\pi i} \int_{\partial\Delta_1} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right| \\
&\leq \frac{1}{2\pi} l(\partial\Delta_1) \sup_{\partial\Delta_1} |z - z_0|^{-n-1} \sup_{\Delta_1} |f| \\
&= -\rho^{-n} \sup_{\Delta - 1} |f|
\end{aligned}$$

$$\leq \rho^{-n} \sup_D |f|$$

And if we let $\rho \nearrow R$ we obtain

$$|c_n| \leq \frac{1}{R^n} \sup_D |f|$$

Note that

$$|f^{(n)}(z_0)| \leq \frac{n!}{R^n} \sup_D |f|$$

□

Theorem 32 (Theorem 15, Liouville's Theorem)**.** *A bounded entire function is constant.*

Proof. Let $z_0 \in \mathbb{C}$, then for each $R > 0$ we have

$$|f'(z_0)| \leq \frac{\sup_{\mathbb{D}(z_0,R)} |f|}{R} \leq \frac{M}{R}$$

So if we let $R \to \infty$ then we have $|f'(z_0)| = 0$ everywhere, thus f is constant.

□

Theorem 33 (Theorem 16, Fundamental Theorem of Algebra)**.** *Each nonconstant polynomial $P(z) = \sum_0^n a_k z^k$ has at least one complex root.*

Proof. Suppose that P has no zeros. Then $f = \frac{1}{P}$ is also entire and for non-constant P we have

$$\lim_{z\to\infty} P(z) = \infty$$

So there exists an $R > 0$ such that $|P(z)| \geq 1$ outside of $\mathbb{D}(0, \mathbb{R})$. Hence $|f(z)| = \frac{1}{P} < 1$ outside of $\mathbb{D}(0, R)$. Inside of $\mathbb{D}(0, R)$ we have P is zero free, thus $|P|$ has a minimum value on $\overline{\mathbb{D}(0, R)}$, which is positive. So $f = \frac{1}{P}$ has a max on $\mathbb{D}(0, R)$ call it M. Then

$$\sup_{\mathbb{C}} |f| = max(1, M)$$

Since f is bounded and entire we then have f is constant and thus P is constant. □

Corollary 7. *Each polynomial of degree n has the form*

$$P(z) = c \prod_{j=1}^{n} (z - z_j)$$

2.10 Morera's Theorem

Theorem 34 (Theorem 17)**.** *Suppose D is an open set and $f \in C(D)$ and*

$$\int_{\partial R} f dz = 0 \qquad (1)$$

then $f \in H(D)$.

Proof. It suffices to show that $f \in H(\Delta)$ for all open $\Delta \subseteq D$. Let $\Delta = \mathbb{D}(z_0, R)$ define

$$F(z) = \int_\sigma f(\zeta) d\zeta$$

where σ is the path that goes along the real part then up the imaginary part (bottom half of rectangle between z_0 and z). Let τ be the upper part of the rectangle between z and z_0. Then hypothesis (1) says that

$$F(z) = \int_\tau f(\zeta) d\zeta$$

So by argument in proof of Theorem 4 we have $F \in H(\Delta)$ and $F' - f \in H(\Delta)$ since we may differentiate forever.

□

2.11 Local Uniform Convergence of Holomorphic Functions

2.11.1 Question:

If $f_n \to f$ uniformly does $f_n' \to f'$ uniformly?

2.11.2 Example

Definition 46. *Let $\{f_n\}$ be a sequence of function defined on a set $S \subseteq \mathbb{C}$. We say $f_n \to f$ **locally uniformly** (l.u.) if for each compact set $K \subseteq S$ we have $f_n \to f$ uniformly on K.*

$$\forall K \subseteq S, \epsilon > 0 \quad \exists N(\epsilon, K) : \forall n \geq N \quad \sup_K |f_n - f| < \epsilon$$

1. $f_n(x) = \frac{1}{n}\sin(nx)$ so $f_n(x) \to 0$ uniformly, but $f_n'(x) = \cos(nx)$ which converges to zero only if $x \in 2\pi\mathbb{Z}$
2. $f_n(x) = \sum_1^n 2^{-k}\cos(3^k x)$ we have $f_n \to f = \sum_1^\infty 2^{-k}\cos(3^k x)$ uniformly on $\mathbb{R}$ but f' exists nowhere.
3. Suppose radius of convergence of a powerseries $\sum_0^\infty c_n z^n = f$ is R. then $\sum_0^k c_n z^n \to f$ locally uniformly in $\mathbb{D}(0, R)$.

Fact 1. *If $f_n \in C(S)$, S open, $f_n \to f$ locally uniformly on S then f is continuous.*

Theorem 35 (Theorem 18)**.** *(Convergence Theorem) Let $\{f_n\}$ be a sequence of holomorphic functions on $D \subseteq \mathbb{C}$ open. Suppose $f_n \to f$ locally and uniformly on D, then*

(a) $f \in H(D)$

(b) $f_n^{(k)} \to f^{(k)}$ *locally and uniformly on D*

Proof. (a) Let $f \in C(D)$ by the fact above and let R be a rectangle in D. By Cauchy's Theorem we have

$$\int_{\partial R} f_n(z)dz = 0, \qquad n \in \mathbb{Z}^+$$

As $f_n \to f$ uniformly on R we have

$$\int_{\partial R} |f_n - f| dz \to 0, \qquad n \to \infty$$

Hence

$$\int_{\partial R} f(z)dz = 0$$

So by Morera's Theorem we have $f \in H(D)$.

(b) It suffices to prove the $k = 1$ case. Let$K \subseteq D$, then $K \subseteq \bigcup_j^n \overline{\Delta}_j$ where $\overline{\delta}_j = D$. If a sequence of functions converge uniformly on finitely many disks they also converge uniformly on the union of those disks. So we prove the assertion for one disk Δ such that $\overline{\Delta} \subseteq D$. Let Δ_1 be an open disk such that

$$\overline{\Delta} \subseteq \Delta_1 \subseteq \overline{\Delta}_1 \subseteq D$$

By Cauchy's formula, for $z \in \overline{D}$ we have

$$f_n'(z) - f'(z) = \frac{1}{2\pi i}\int_{\partial\Delta_1} \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^2} d\zeta$$

Let ρ be the radius of Δ_1 and

$$\delta = \inf\{|z - s| : z \in \overline{\Delta}, s \in \partial\Delta_1\} > 0$$

Observe that

$$|f_n'(z) - f(z)| \leq \frac{1}{2\pi} l(\partial\Delta_1) \cdot \sup_{\partial\Delta_1} |f_n - f| \cdot \delta^{-2}$$

As $\sup_{\partial\delta_1} |f_n - f| \to 0$ as $n \to \infty$ we have

$$f_n'(z) \to f(z)$$

uniformly on $\overline{\Delta}$. □

2.11.3 True or False

If $f_n \in H(D)$, $f_n \to f$ uniformly on D, does $f_n' \to f'$ uniformly on D?

Consider the series $\sum \frac{1}{n^2} z^n$ which has ROC of 1 (which is uniform). Taking $f_k = \sum_1^k \frac{1}{n^2}$ converges uniformly to something on $\mathbb{D} \cup \partial\mathbb{D}$

$$\boxed{\text{FALSE}}$$

2.12 Zeros of the Holomorphic Functions

Definition 47. *Recall that z_0 is a **limit point** of a set S if for all $\epsilon > 0$ there is a $z \in S$ so that $0 < |z - z_0| < \epsilon$.*

Theorem 36 (Theorem 19). *(Uniqueness Theorem) Let $f \in H(D)$ where D is a domain. Let $Z(f) = \{z \in D \mid f(z) = 0\}$. Then either*

1. *$Z(f)$ has no limit points in D, or*
2. *$f \equiv 0$ in D.*

Proof. Let $S = \{z \in D : z \text{ is a limit point of } Z(f)\}$. We want to show that $S = \emptyset$. We will use connectedness of D. First we show that S is closed. This is true since f is a continuous function. So if there exists $\{s_n\} \subseteq S, s_n \to s \in D$ then $f(s_n) \to f(s) = 0$.

Now we claim that S is also open. If $S = \emptyset$ we are done.

Suppose S is nonempty and take $z_0 \in S$. Since f is continuous we know that $f(z_0) = 0$. Let $R > 0$ be such that $\Delta = \mathbb{D}(z_0, R) \subseteq D$. In Δ, f has a representation $f(z) = \sum_0^\infty c_n(z - z_0)^n$. Suppose (to show a contradiction) that $c_n \neq 0$ for some n. Let m be the smallest term such that $c_m \neq 0$. Let

$$f(z) = c_m(z - z_0)^m + \cdots = (z - z_0)\underbrace{[c_m + c_{m+1}(z - z_0) + \cdots]}_{=g(z)}$$

Since g is a powerseries we know that the series for g converges in Δ so $g \in H(\Delta)$ and

$$\lim_{z \to z_0} g(z) = g(z_0) = c_m \neq 0$$

As g is continuous, there exists $\delta > 0$ so that $g(z) \neq 0$ on $\Delta(z_0, \delta)$. BUt $(z - z_0)^m$ is also zero free on $\mathbb{D}(z_0, \delta) \setminus \{z_0\}$ but

$$f = (z - z_0)^m g$$

is never zero on $\mathbb{D}(z_0, \delta) \setminus \{z_0\}$. Hence z_0 cannot be a limit point of S. Therefore $c_n = 0$ for all $n \geq 0$. And thus $f \equiv 0$ in Δ, so $\Delta \subseteq S$ and S is no open. Since S is now open and closed in D we have $S = \emptyset$ or $S = D$.

□

Corollary 8. *If $f \in H(D)$ and is nonconstant then for all $a \in \mathbb{C}$ we have $f^{-1}(a) \cap D$ is discrete and hence is at most countable.*

Corollary 9. *If $f, g \in H(D)$ and $f = g$ on a set with a limit point in D then $f = g$ on D.*

2.12.1 Remark

This is *very* different from the real theory. For example $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ then $f \in C^1(\mathbb{R})$, but $f \equiv 0$ on $\mathbb{R}^-$.

2.13 Zeros

Theorem 37 (Theorem 20)**.** *(Factoring out Zeros) Suppose $f \in H(D)$ with D open and connected. Also, $z_0 \in D$ where $f(z_0) = 0$ and $f \not\equiv 0$. Then there exists a unique positive integer m and a function $g \in H(D)$ such that $g(z_0) \neq 0$ and for all $z \in D$*

$$f(z) = (z - z_0)^m g(z)$$

The number m is called ***the order of the zero at*** *z_0*

Proof. Let $R > 0$ and $\Delta = \mathbb{D}(z_0, R)$ then

$$f(z) = \sum_0^\infty c_n (z - z_0)^n$$

since $f \not\equiv 0$, the zeros of f have no limit point, so sufficiently close to z_0 there is a value of f which is not zero. Thus, there exists n such that $c_n \neq 0$. Letting m be the first such non-zero term. Note that $m \geq 1$ since $f(z_0) = 0$. So we have

$$f(z) = (z - z_0)^m \sum_{n=m}^\infty c_n (z - z_0)^{n-m}, \quad \text{for } z \in \Delta$$

Let

$$g = \begin{cases} \sum_{n=m}^\infty c_n (z - z_0)^{n-m}, & z \in \Delta \\ (z - z_0)^{-m} f(z), & *z \notin \Delta \end{cases}$$

Notice that the definitions agree on the overlap, so g is well defined and holomorphic in D and

$$g(z_0) = c_m \neq 0$$

So g and m exist. If M is unique then so is g, so we seek to show this.

Suppose also that $f(z) = (z - z_0)^{m_1} g_1(z)$ with $m_1 \geq 1$, $g \in H(D)$ and $g_1(z_0) \neq 0$. Suppose that $m_1 > m$, then for $z \neq z_0$ we have

$$g(z) = (z - z_0)^{m_1 - m} g_1(z)$$

Letting $z \to z_0$ says that $g(z_0) = 0$. This is a contradiction. So $m_1 \leq m$. By symmetry $m_1 \geq m$. Thus $m_1 = m$ and m and g are unique.

□

2.13.1 Facts

1. c_m is the first non-vanishing Taylor coefficient in the power series.
2. $f^{(j)}(z_0) = 0$ for $j = 0, 1, \ldots, m - 1$
3. If f, g have zeros at z_0 of order m_1 and m_2 respectively then $f \cdot g$ has zero of order $m_1 + m_2$ at z_0 and $g + g$ has zeros of order $\geq \min(m_1, m_2)$.

2.13.2 Examples

1. $\sin z$ has a 0 of order 1 at 0
2. $z^k \sin z$ has a 0 of order $k + 1$ at 0
3. $\sin z - z$ has a 0 of order 3 at 0.
4. $\sin z + 1$ has a 0 of order 0 at 0

3 Zeros and Singularities

3.1 Isolated Singularites

Definition 48. *Suppose that D is an open set and that $z_0 \in D$ and $f \in H(D \setminus \{z_0\})$, then z_0 is called an **isolated singularity** of f.*

Definition 49. *Let z_0 be an isolated singularity of f, then z_0 is called*

*(a) A **removable singularity** if f can be defined at z_0 such that $f\ in H(D)$.*

*(b) A **pole** if it is not removable and there is an $m \in \mathbb{Z}^+$ such that $(z - z_0)^m f(z)$ does have a removable singularity at z_0.*

*(c) An **essential singularity** if neither (a) nor (b).*

Theorem 38 (Theorem 1)**.** *Suppose $f \in H(D \setminus \{z_0\})$ with D open. Then*

(a) z_0 is removable if and only if f is bounded in some neighborhood of z_0.

(b) z_0 is a pole if and only if $\lim_{z \to z_0} f(z) = \infty$

(c) (Casorati-Weierastrass) the pole is essential if and only if for any $\epsilon > 0$ with $\mathbb{D}(z_0, \epsilon) \subseteq \mathbb{D}$, it holds that $f(\mathbb{D}(z_0, \epsilon) \setminus \{z_0\})$ is dense in $\mathbb{C}^*$.

Proof. (a) ($\Rightarrow$) Suppose removable, so there exists a holomorphic extension to D, in particular f is continuous and $\mathbb{C}$-valued at z_0, so it must be bounded by the extreme value theorem.

($\Leftarrow$) Let

$$g(z) = \begin{cases} (z-z_0)^2 f(z), & z \in D \setminus \{z_0\} \\ 0, & z = z_0 \end{cases}$$

by our construction we have $g_1 H(D \setminus \{z_0\})$ and for $z \neq z_0$ we have

$$\frac{g(z) - g(z_0)}{z - z_0} = (z - z_0) f(z)$$

Since f is bounded in a neighborhood of z_0 we have the RHS $\to 0$ as $z \to z_0$. So, g is differentiable at z_0 with $g'(z_0) = 0$. So, $g \in H(D)$, let $\Delta = \mathbb{D}(z_0, R) \subseteq D$. Then $g(z) = \sum_0^\infty c_n (z - z_0)^n$ with $c_0 = c_1 = 0$. So, $g(z)$ by our last theorem is equal to

$$g(z) = (z - z_0)^2 [c_2 + c_3(z - z_0) + \cdots]$$

And by definition we have

$$g(z) = (z - z_0) f(z)$$

Thus, for all $z \in \Delta \setminus \{z_0\}$ we have

$$f(z) = c_2 + c_3(z - z_0) + c_4(z - z_0)^2 + \cdots$$

Setting $f(z_0) = c_2$ gives the desired extension.

(b) ($\Rightarrow$) Let m be the smallest integer such that

$$g(z) = (z - z_0)^m f(z)$$

has a removable singularity at z_0. By definition we know that $m \geq 1$. Also $g(z_0) \neq 0$. If otherwise we would have $g(z) = (z - z_0) g_1(z)$ and thus $g_1 \in H(D)$ and so $g_1(z) = (z - z_0)^{m-1} f(z)$ which contradicts the minimality of m. For $z \in D \setminus \{z_0\}$ we have

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

with $m \geq 1$ and $g(z_0) \in \mathbb{C} \setminus \{0\}$. Thus,

$$\lim_{z \to z_0} f(z) = \infty.$$

($\Leftarrow$) Since $\lim_{z \to z_0} f(z) = \infty$ there exists a disk $\Delta = \mathbb{D}(z_0, R) \subseteq D$ such that $|f(z)| > 1$ in $\Delta \setminus \{z_0\}$. So $\frac{1}{f} = g$ has a removable singularity. Now it is easy to finish.

(c) ($\Rightarrow$) Suppose, to show a contradiction, that there exists an $\epsilon > 0$ so that $\Delta(z_0, \epsilon) \subseteq D$ and $f(\Delta \setminus \{z_0\})$ is not dense in $\mathbb{C}^*$. So, $\mathbb{C} \setminus \overline{f(\Delta \setminus \{z_0\})} \neq \emptyset$. Let $w \in \mathbb{C} \setminus \overline{f(\Delta \setminus \{z_0\})}$ and $\delta > 0$ so that

$$\mathbb{D}(w_0, \delta) \cap \overline{f(\Delta \setminus \{z_0\})} = \emptyset$$

Hence $|f(w) - w_0| \geq \delta$ for all $w \in \Delta \setminus \{z_0\}$. Let

$$g(z) = \frac{1}{f(z) - w_0}$$

Then since $f \in H(D \setminus \{z_0\})$ we have $g \in H(\Delta \setminus \{z_0\}$. So by (a) we have

$$\lim_{z \to z_0} g(z)$$

exists in $\mathbb{C}$. Thus, $\lim_{z \to z_0} f(z)$ exists in $\mathbb{C}^*$. Call this limit b. If $b = \infty$ then f has a pole at z_0 and if b is finite then f has a removable singularity at z_0. Both contradictions, thus the image is dense.

($\Leftarrow$) We prove the contrapositive. If z_0 is not essential it is a pole or a removable singularity and it is easy to show non-density in these cases.

□

3.1.1 Examples

1. $\frac{\sin z}{z}$ is removable
2. $\frac{\cos z}{z} \to \infty$ as $z \to 0$ so a pole
3. $\sin(1/z)$ has no limit. So essential.
4. $e^{1/z}$ is also essential.

3.2 Poles

Theorem 39 (Theorem 2)**.** *Suppose $f \in H(D \setminus \{z_0\})$ where D is a domain, $z_0 \in D$ and f has a pole at z_0. Then f has a representation*

$$f(z) = h(z) + \sum_1^m \frac{c_k}{(z - z_0)^k}, \qquad z \in D \setminus \{z_0\}$$

Where $m \in \mathbb{Z}^+$, $c_1, \ldots, c_m \in \mathbb{C}$, $c_m \neq 0$ and $h \in H(D)$.

Proof. As before let m be the smallest integer such that $g(z) = (z - z_0)^m f(z)$ has a removable singularity at z_0. We saw before that $g(z_0) \neq 0$. Then $g(z) = \sum_0^\infty a_n (z - z_0)^n$ in some disk $\Delta \subseteq D$. And $a_0 = g(z_0) \neq 0$ for $z \in \Delta \setminus \{z_0\}$. We then have

$$\begin{aligned} f(z) &= \frac{g(z)}{(z - z_0)^m} \\ &= \frac{a_0}{(z - z_0)^m} + \frac{a_1}{(z - z_0)^{m-1}} + \cdots + \frac{a_m}{(z - z_0)} + \sum_{n=0}^\infty a_{m+n} (z - z_0)^n \end{aligned}$$

Let

$$h(z) = \begin{cases} \sum_{n=0}^\infty a_{m+n}(z - z_0)^n, & z \in \Delta \\ f(z) - \sum_{n=1}^m c_k (z - z_0)^{-k}, & z \in D \setminus \Delta \end{cases}$$

Where $c_j = a_{m-j}$, $j = 1, 2 \ldots, m$.

□

Definition 50. *As in Theorem 2, m is called the* ***order of the pole*** *and we call $\sum_{k=1}^m c_k (z - z_0)^{-k}$ the* ***principal part*** *of f at z_0. We call c_1 the* ***residue of** f **at*** *z_0 we write*

$$\mathrm{Res}(f; z_0) = c_1$$

3.2.1 Facts

1. It is easy to see that m and c_j are unique.
2. If $f \in H(D \setminus A)$ where A is a set with no limit points in D, then each point of A is an isolated singularity.
3. f has a pole of order m at z_0 if and only if $\frac{1}{f}$ has a zero of order m at z_0.
4. If f has a pole at z_0 with principal part

$$\sum_1^m \frac{c_j}{(z-z_0)^j}$$

then

$$f - \sum_1^m \frac{c_j}{(z-z_0)^j}$$

has a removable singularity at z_0.

Definition 51. *If $f \in H(D)$ except for poles, then f is said to be **meromorphic** in D. We denote this by $f \in M(D)$.*

3.2.2 Examples

1. if $f, g \in H(D)$ and $g \not\equiv 0$ then $\frac{f}{g} \in M(D)$.
2. Rational functions are meromorphic in $\mathbb{C}$.
3. For example, $f(z) = \frac{1}{z(z-1)^2}$ has two poles $z_0 = 0, z_1 = 1$. Let's try and calculate the pp's. At z_0 we have $z \cdot f(z)$ has a removable singularity, so the pole is simple (i.e. order 1). Observe that

$$c_1 = \lim_{z \to z_0} z \cdot f(z) = \frac{1}{(z_0-1)^2} = 1$$

So the principal part is $c_1/z = \frac{1}{z}$ and $\operatorname{Res}(f;0) = 1$.

Now consider $z_1 = 1$. We have

$$\begin{aligned} f(z) &= \frac{1}{(z-1)^2}\left(\frac{1}{1+(z-1)}\right) \\ &= \frac{1}{(z-1)^2}\left(1-(z-1)+\mathcal{O}(z-1)^2\right) \\ &= \frac{1}{(z-1)^2} - \frac{1}{z-1} + \mathcal{O}(1) \end{aligned}$$

Hence the pp is

$$\operatorname{Res}(f;a) = -1$$

Or we can consider the partial fraction decomposition

$$\frac{1}{z(z-1)^2} = \frac{1}{z} + \frac{1}{(z-1)^2} - \frac{1}{(z-1)}$$

Definition 52. *Given $f(z) = g(z) + \mathcal{O}(h(z))$ or $f(z) = g(z) + \mathfrak{o}(h(z))$ with $z \to ?$ we have* ***big*** $\mathcal{O}$ *is*

$$\left|\frac{f(z)-g(z)}{h(z)}\right| \leq M$$

and ***little*** $\mathfrak{o}$ *is*

$$\left|\frac{f(z)-g(z)}{h(z)}\right| \to 0, \qquad z \to \infty$$

Theorem 40 (Theorem 3)**.** *Let $f = P/Q$ be a rational function with P, Q polynomials and $Q \not\equiv 0$ and $\lim_{z\to\infty} f(z) = 0$. Suppose that f has poles $z_1, \dots, z_l$ with respective principal parts $S_k(z) = \sum_1^{m_k} \frac{a_{j,k}}{(z-z_k)^j}$ then for all $z \in \mathbb{C}$*

$$f(z) = \sum_{k=1}^{l} S_k(z)$$

Proof. Let $g = \sum S_k(z)$. So by Fact 4 we have $f - g$ hasa removable singularity at each z_k. So $f - g$ is entire. Also, $\lim_{z\to\infty} g(z) = 0$. Thus $f - g$ is bounded in $\mathbb{C}$. Applying Liouville's Theorem we have $f - g$ is constant in $\mathbb{C}$ and so by computation we have

$$\lim_{z\to\infty} f - g = 0$$

So the constant is 0 and $f = g$.

□

3.2.3 Exercise

Suppose that P/Q with $\deg P \geq Q$. Figure out what to do...

3.3 Residues

Theorem 41 (Theorem 4)**.** *(Residue Theorem for Disk) Let $\Delta = \mathbb{D}(z_0, R)$ and suppose that $f \in M(\Delta)$ with only finitely many poles $z_1, z_2, \dots, z_l \in \Delta$. If γ is any closed path in $\Delta \setminus \{z_1, \dots, z_l\}$ we have*

$$\frac{1}{2\pi i}\int_\gamma f(z) = \sum_{j=1}^{l} n(\gamma; z_j) \operatorname{Res}(f; z_j)$$

Proof. Let S_k be the principal part of f at z_k. As in the proof of Theorem 3 we can write for $z \in \Delta \setminus \{z_0, \dots, z_l\}$ we have

$$f - \sum_1^l S_k = h$$

Where $h \in H(\Delta)$. By Cauchy for a disk we have

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \frac{1}{2\pi i}\sum_{k=1}^{l}\int_\gamma S_k dz = \sum_{j=1}^{l} n(\gamma; z_j) \operatorname{Res}(f; z_j)$$

Where at the last line we use the example in §2.1 because $\int_\gamma S_k dz = \frac{1}{2\pi i} c_1$. □

Remark: If you already know that $n(\gamma, z) = 1$ or 0 then for $z \notin \gamma$ and z_j 'inside γ' we have

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum \operatorname{Res}(f, z_j)$$

3.3.1 Notes

1. Let D be any open set $f \in M(D)$, $z_0 \in D$ be any pole of f and let Δ be an open disk satisfying
$$\Delta \subseteq \overline{\Delta} \subseteq D$$
and Δ_1 be a disk satisfying
$$\overline{\Delta} \subseteq \Delta_1 \subseteq D$$
We can then apply the Residue Theorem to $\gamma = \frac{\partial}{\partial \Delta}$ and obtain
$$\frac{1}{2\pi i}\int_\gamma f(z)d_z = \text{Res}(f; z_0)$$

2. There is a way to compute residues. Suppose f has a pole at z_0 then
$$f(z) = \sum_1^m c_k(z-z_0)^k + \mathcal{O}(1) \qquad z \to z_0$$
then we can write
$$(z-z_0)^m f(z) = c_m + c_{m-1}(z-z_0) + \cdots + c_1(z-z_0)^{m-1} + \mathcal{O}\left((z-z_0)^m\right)$$
So we have
$$\text{Res}(f; z_0) = c_1 = \left(\frac{1}{(m-1)!}\right) D^{m-1}\left((z-z_0)^{,} f(z)\right)\Big|_{z=z_0}$$
If $m = 1$ then we have
$$\text{Res}(f; z_0) = \lim_{z \to z_0}(z-z_0)f(z)$$

3.3.2 Examples

1. Consider $\frac{1}{z(z-1)^2}$ at $z = 0$ we have
$$\lim_{z \to 0}\frac{1}{(z-1)^2} = 1$$
at $z = 1$ we have
$$\frac{1}{1!}D^1(\frac{1}{z})\big|_{z=1} = -1$$

2. Let $f(z) = \cot z = \frac{\cos z}{\sin z}$. We have simple poles at $0, \pm\pi, \ldots$ First consider
$$\text{Res}(f; \pi k) = \lim_{z \to \pi k}(z - \pi k)\frac{\cos z}{\sin z} = \frac{\cos \pi k}{\frac{\partial}{\partial z}(\sin z)|_{z=\pi k}} = 1$$
So the pp is $\frac{1}{z - \pi k}$. Perhaps
$$\cot z = \sum_{k \in \mathbb{Z}} \frac{1}{z - \pi k}$$
(Yes, but we can't yet prove this).

3. $f(z) = \frac{1}{\sin z - z}$ This is a triple pole at $z = 0$. Let $g(z) = z^3 f(z) = \frac{z^3}{\sin z - z}$. We could take

$$\text{Res}(f;0) = \frac{1}{2}g''(0)$$

But there is a better way. First we use the power series expansion

$$\frac{1}{\sin z - z} = \frac{1}{az^3 + bz^5 + \cdots}, \qquad a = -\frac{1}{3!}, b = \frac{1}{5!}$$

$$\begin{aligned}\frac{1}{az^3 + bz^5 + \cdots} &= \frac{1}{az^3}\left(\frac{1}{1 + \frac{b}{a}z^2 + \mathcal{O}(z^4)}\right) \\ &= \frac{1}{az^3}\left(1 - \left(\frac{b}{a}z^2 + \mathcal{O}(z^4)\right) + \mathcal{O}(z^4)\right) \\ &= \frac{1}{az^3} - \frac{b}{a^2 z} + \mathcal{O}(z)\end{aligned}$$

And as $z \to 0$ we have

$$\text{Res}(f;0) = -\frac{b}{a^2 z} = -\frac{3}{10}$$

3.4 Evaluation of Real Integrals

3.4.1 Facts about Improper Integrals

1. If $f : \mathbb{R} \to [0, \infty)$ then we have

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{n\to\infty} \int_{-n}^{n} f(x)dx = \lim_{A\to-\infty} \int_{A}^{0} f(x)dx + \lim_{B\to\infty} \int_{0}^{B} f(x)dx$$

2. Let $f : \mathbb{R} \to \mathbb{C}$ and

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty$$

then we have

$$\lim_{A\to-\infty, B\to\infty} \int_{A}^{B} f(x)dx$$

exists in $\mathbb{C}$ and equals L. We call an absolutely convergent integral if there exists an $L \in \mathbb{C}$ such that for all $\epsilon > 0$ there exist M so that $B > M$, $A < -M$ and

$$\left| \int_{A}^{B} f(x)dx - L \right| < \epsilon$$

3.4.2 Situation A

Let $f = P/Q$, a rational function with (a) no poles on $\mathbb{R}$ and (b) $f(z) = \mathcal{O}(z^{-2})$ at ∞. Then we have for z_i poles in the upper half plane denoted by $\mathbb{H}$

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{z_i} \{\text{Res}(f; z_i) : z_i \in \mathbb{H}\}$$

Proof. First note that (a) and (b) imply that there exists a C such that $|f(x)| \leq \frac{C}{x^2}$ where $x \geq 1$. Thus,

$$\int_{-\infty}^{\infty} |f(x)|dx \leq \int_{-1}^{1} |f|dx + 2C \int_{1}^{\infty} x^{-2}dx < \infty$$

So the integral exists as an absolutely convergent improper integral.

Let $z_1, z_2, \ldots z_l \in \mathbb{H}$ be the poles of f in $\mathbb{H}$ and let $\gamma(R)$ be the semicircle of radius R around 0 oriented counterclockwise. We can choose R large enough so that all of the z_k are inside $\gamma(R)$. Next apply the Residue Theorem in $\mathbb{D}(0, \mathbb{R})$ with $R_1 > R$. We get

$$\int_{\gamma(R)} f(z)dz = 2\pi i \left(\sum_{i=1}^{l} \text{Res}(f; z_i) \right) = 2\pi i s \qquad (2)$$

Let $\gamma(R) = \delta(R) + [-R, R]$ with $\delta(R)$ the upper half circle, we then have

$$\int_{\gamma(R)} f(z)dz = \int_{-R}^{R} f(x)dx + \int_{\delta(R)} f(z)dz = 2\pi i s$$

for large R.

But by (b) there exists R_0 and C such that $|z| \geq R_0$ and $|f(z)| < \frac{C}{|z|^2}$ so we have

$$\left| \int_{\delta(R)} f(z)dz \right| \leq \pi R \frac{C}{|R|^2} = \frac{\pi C}{R}$$

So as $R \to \infty$ we have (by (2))

$$\int_{-R}^{R} f(x)dx \to 2\pi i s$$

□

3.4.3 Example

- Let $I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$, let $f(z) = \frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$. There is one pole at $z = i$ in $\mathbb{H}$. We have

$$\text{Res}(f; i) = \lim_{z \to i} (z - i)f = \frac{1}{(1 + i)} = \frac{1}{2i}$$

  By theorem we have

$$I = 2\pi i (\frac{1}{2i}) = \pi$$

- One could show that $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx = \frac{\pi\sqrt{2}}{2}$ by using the fourth roots of unity.

3.4.4 Situation B

Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{\cos(tx)}{1 + x^2} dx, \qquad t \in \mathbb{R}$$

Now $I(t) = I(|t|)$ so we may assume that $t > 0$ and considering the function $f(z) = \frac{\cos(tz)}{1+z^2}$ we have

$$\frac{\cos(iRt)}{1-(iR)^2} = \frac{e^{Rt}+e^{-Rt}}{2(1+R^2)} \to \infty$$

We need a different approach. Observe that

$$\cos(xt) = \mathrm{Re}(e^{ixt})$$

Hence

$$I = \int_{-\infty}^{\infty} \mathrm{Re}(e^{ixt})\frac{dx}{1+x^2} = \mathrm{Re}\int_{-\infty}^{\infty}\frac{e^{ixt}dx}{1+x^2}$$

We use $\frac{e^{izt}}{1+z^2}$ to obtain

$$\left|e^{izt}\right| = e^{\mathrm{Re}(i(x+iy)t)} = e^{-yt} \leq 1$$

for $t > 0$, $y > 0$ and $z \in \mathbb{H}$. So now we may use method A to write

$$I = \pi e^{-t}$$

3.4.5 Situation C

Consider

$$I = \int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$$

with $a > 1$. We can rewrite using the curve γ which traverses the unit circle once

$$\begin{aligned} I &= \int_0^{2\pi} \frac{d\theta}{a+\frac{1}{2}(e^{i\theta}+e^{-i\theta})} \\ &= \int_0^{2\pi} \frac{2e^{i\theta}d\theta}{2e^{i\theta}a+e^{2i\theta}+1} \\ &= \frac{2}{i}\int_\gamma \frac{dz}{z^2+2az+1} \\ &= \frac{2}{i}\int_\gamma \frac{1}{f(z)}dz \end{aligned}$$

Where $f(z) = z^2+2az+1 = (z-w_1)(z-w_2)$ where

$$w_1 = -a+\sqrt{a^2-1}, \qquad \text{inside } \mathbb{D}$$

$$w_1 = -a-\sqrt{a^2-1}, \qquad \text{outside } \mathbb{D}$$

So we have

$$\mathrm{Res}\left(\frac{1}{f};w_1\right) = \frac{1}{w_1-w_2} = \frac{1}{2\sqrt{a^2-1}}$$

Now we can apply the Residue Theorem

$$I = \frac{2}{i}2\pi i\left(\frac{1}{2\sqrt{a^2-1}}\right) = \frac{2\pi}{\sqrt{a^2-1}}$$

3.4.6 Situation D (PacMan Situation)

Consider

$$I = \int_0^\infty x^{-\alpha} \frac{dx}{1+x}, \qquad \alpha \in (0,1)$$

Define the function $f(z) = z^{-\alpha}$. There are two problems:

1. There is a pole at -1
2. There is a branch cut due to $z^{-\alpha}$ on the ray from the origin along the real line.

The solution is to consider the curve that just about circles the origin on a small circle of radius $R_1 < 1$, then goes out parallel the real line on the line L_1 with distance $\delta > 0$ for a while and circles back on a circle of radius $R_2 > 1$, then parallels the negative side of the real line with distance δ on the line L_2 until it gets back the circle. Call this curve $\gamma(R_1, R_2, \delta)$ and let it run counterclockwise.

The Residue Theorem says that

$$\int_\gamma \frac{f(z)}{1+z} dz = 2\pi i e^{-i\pi\alpha}$$

Now, with some work, one may show that

$$\int_{R_1}^{R_2} \frac{x^{-\alpha}}{1+x} = \lim_{\delta \to 0^+} \int_{L_1} \frac{f(z)dz}{1+z}$$

and

$$-e^{-2\pi i\alpha} \int_{R_1}^{R_2} \frac{x^{-\alpha}}{1+x} dx = \lim_{\delta \to 0^+} \int_{L_2} \frac{f(z)dz}{1+z}$$

So,

$$2\pi i e^{-i\pi\alpha} = \int_{\gamma_{R_1}} \frac{f(z)dz}{1+z} + \int_{\gamma_{R_2}} \frac{f(z)dz}{1+z} + \int_{L_1} \frac{f(z)dz}{1+z} + \int_{L_2} \frac{f(z)dz}{1+z}$$

We then have

$$2\pi i e^{-i\pi\alpha} - (1 - e^{-2\pi i\alpha}) \int_{R_1}^{R_2} \frac{x^{-\alpha}}{1+x} dx = \lim_{\delta \to 0^+} \left[\int_{\gamma_{R_1}} \frac{f(z)dz}{1+z} + \int_{\gamma_{R_2}} \frac{f(z)dz}{1+z} \right]$$

As $\alpha \in (0,1)$ we may show that both terms on the RHS go to 0 as $\delta \to 0^+$, $R_1 \to 0$, $R_2 \to \infty$. Hence

$$I = \frac{2\pi i}{e^{\pi i\alpha} - e^{-\pi i\alpha}}$$

3.5 Argument Principle

Theorem 42 (Theorem 5)**.** *(Argument Principle) Let $f \in M(\Delta)$ with Δ an open disk, and f has at most finitely many zeros and poles in Δ. Denote the zeros by $a_1, \ldots, a_n$ and poles by $b_1, \ldots, b_m$ counting multiplicity. Let γ be a closed path in Δ passing through none of the a_i and b_i. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{j=1}^n n(\gamma, a_j) - \sum_{j=1}^m n(\gamma, b_j)$$

Moreover if all zeros and poles are simple and γ is simple

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \#\text{zeros} - \#\text{poles} = n - m$$

Proof. Note that $\frac{f'}{f} \in H(\delta \setminus$ poles and zeros of f.

Suppose $z_0 \in \Delta$ is a zero of order m then $f(z) = (z - z_0)^m g(z)$ where g has a removable singularity at z_0 and $g(z_0) \neq 0$. Take a disk Δ_1 with $z_0 \in \Delta_1$ and g is holomorphic and zero free and no other poles and zeros of f are in Δ_1. Now we have

$$\frac{f(z)}{f'(z)} = \frac{m(z-z_0)^{m-1}g(z) + (z-z_0)^m g'(z)}{(z-z_0)^m g(z)} = \frac{m}{(z-z_0)} + \frac{g'(z)}{g(z)}$$

Now, $\frac{g'}{g} \in H(\Delta_1)$, so $\frac{f'}{f}$ has a simple pole with residue m at z_0. Now suppose that f has a pole of order m at z_0 with

$$f(z) = \frac{g(z)}{(z-z_0)^m}$$

Continue this process and we eventually see that $\frac{f'(z)}{f(z)}$ has a simple pole of residue $-m$ at z_0. As $\frac{f'}{f} \in M(\Delta)$ we use the Residue theorem and we are done. □

3.5.1 Example

1. $f(z) = \frac{z(z+1)^2}{(z+2)^3(z-1)}$ We have the zeros are $0, -1, -1$ and the poles are $-2, -2, -2, 1$ hence

$$\frac{1}{2\pi i}\int_{\partial \mathbb{D}(0,3)} \frac{f'(z)}{f(z)} dz = 3 - 4 = -1$$

$$\frac{1}{2\pi i}\int_{\partial \mathbb{D}(0,\frac{3}{2})} \frac{f'(z)}{f(z)} dz = 3 - 1 = 2$$

2. $f \in H(\Delta)$ and $f \neq 0$ then on $\partial\Delta$ we have

$$\frac{1}{2\pi i}\int_{\partial \mathbb{D}(0,3)} \frac{f'(z)}{f(z)} dz = \#\text{ of zeros in } \Delta$$

3.5.2 Heuristic Look

If we took

$$\begin{aligned}\int_\gamma \frac{f'}{f} dz &= \int_\gamma \frac{d}{dz}\log f dz \\ &= \text{Change in } \log(f) \text{ as } z \text{ traverses along } \gamma \\ &= \text{net change in } i\arg f \text{ as } z \text{ traverses } \gamma\end{aligned}$$

Theorem 43 (Theorem 6)**.** *(Geometric Form of Argument Principle) With the hypothesis as in Theorem 5 we conclude that*

$$\sum n(\gamma, a_i) - \sum n(\gamma, b_j) = n(f \circ \gamma, 0)$$

Proof. By Theorem 5 we know that the LHS is equal to

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz$$

By definition of $n(f \circ g, 0)$ we know that this is equal to

$$\frac{1}{2\pi i}\int_{f\circ\gamma}\frac{dw}{w}$$

We are done if we can show that

$$\int_{f\circ\gamma}\frac{dw}{w} = \int_\gamma \frac{f'(z)}{f(z)}dz$$

This is a case of the following change of variable theorem:

Hypothesis: $f \in H(D)$, D open, γ a path in D and $\varphi \in C(f \circ \gamma)$.
Conclusion: $\int_{f\circ g}\varphi(w)dw = \int_\gamma \varphi(f(z))f'(z)dz$ □

3.6 Rouche's Theorem

Theorem 44 (Theorem 7)**.** *(Rouche's Theorem) Let $f, g \in H(\Delta)$ and Δ is an open disk, $\gamma \subseteq \Delta$ is a simple closed path. Suppose for $z \in \gamma$ we have*

$$|f(z) - g(z)| < |g(z)| \qquad (1)$$

then (counting multiplicity) f, g have the same number of zeros inside γ.

Proof. Give γ the positive orientation so that $n(\gamma, z) = 0$ or 1 for all z not on γ. Also note that (1) implies that f and g are zero free along γ. By the uniqueness theorem f and g have at most finitely many zeros on a compact subset of Δ. Let $N(g)$ and $N(f)$ denote the number of zeros of g and f inside γ. Then f and g both satisfy the hypothesis of the argument principle. So,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}dz = N(f) - N(g) \qquad (2)$$

Let $h = \frac{f}{g}$. Then h is holomorphic and zero free in a neighborhood of γ. Additionally, (1) implies that

$$|h(z) - 1| < 1, \qquad z \in \gamma$$

So along γ we have $h \circ \gamma \in \mathbb{D}(1, 1)$ where the function $\frac{1}{w}$ is holomorphic. So, by Cauchy's Theorem

$$0 = \int_{h\circ\gamma}\frac{dw}{w} = \int_\gamma \frac{h'(z)}{h(z)}dz, \qquad (3)$$

We compute

$$\frac{h'}{h} = \frac{f'g - g'f}{g^2}\cdot\frac{g}{f} = \frac{f'}{f} - \frac{g'}{g}$$

By (2) the above combined with (3) gives

$$0 = N(f) - N(g)$$

And so f and g have the same number of zeros inside γ.

□

3.6.1 Example

Suppose $p(z) = \sum_{k=0}^{n} a_k z^k$ with $a_n \neq 0$. Then P has n zeros in $\mathbb{C}$. First observe that

$$\frac{p(z)}{a_n z^n} = 1 + \frac{a_{n-1}}{z} + \cdot + \frac{a_0}{z^n}$$

This goes to 1 as $z \to \infty$. So there exists $R > 0$ so that

$$\left| \frac{p(z)}{a_n z^n} - 1 \right| < 1$$

Hence,

$$|p(z) - a_n z^n| < |a_n z^n|$$

So, Rouche's says that p has n-zeros in $\mathbb{C}$.

3.7 Infinity

Notation: Let $\mathbb{D}(\infty, R) = \{z \in \mathbb{C}^* : |z| > R\}$.

Definition 53. *If $f \in H(\mathbb{D}(\infty, R) \setminus \{\infty\})$ for some R we say that f* ***has an isolated singularity at*** ∞*. In this case $g(z) = f(1/z)$ has an isolated singularity at 0.*

Definition 54. .

1. *If g has a removable singularity at 0 then we say f has a* ***removable singularity at*** 0.
2. *If g has a pole of order m at 0 we say f has a* ***pole of order*** m ***at*** ∞.
3. *If g has an essential singularity at 0 we say that f has an* ***essential singularity at*** ∞.

3.7.1 Examples

- e^z has an essential singularity at ∞.
- $P(z) = \sum^n a_k z^k$ has a pole of order n
- $\frac{1}{P(z)}$ has a zero of order n at ∞
- $P(z)/Q(z)$ Rational function with $\deg P = n$ and $\deg Q = m$ if
  1. $n > m$ we have a pole of order $n - m$
  2. $n = m$ removable singularity
  3. $n < m$ then a zero of order $m - n$

3.8 Equidistribution

Definition 55. *Suppose that f is meromorphic in a neighborhood of $z_0 \in \mathbb{C}^*$. Let $w_0 = f(z_0)$. We say that f* ***takes on*** w_0 ***with multiplicity*** m ***at*** z_0 *if $f - w_0$ has a zero of order m at z_0 when w_0 is finite. Or f has a pole of order m at z_0 if $w_0 = \infty$.*

3.8.1 Example

Let $f(z) = z^5 + 5$ this takes on 1 with multiplicity 5 at 0. And takes on ∞ with multiplicity 5 at $z = \infty$ and takes on $1 + z_0^5$ with multiplicity 1 when $z_0 \in \mathbb{C} \setminus \{0\}$.

Definition 56. *If $f = \frac{p}{q}$ with p, q nontrivial polynomials with no common factors. Define the **degree of** f to be*

$$\deg f = \max\{\deg p, \deg q\}$$

Theorem 45 (Theorem 8)**.** *Let $f = \frac{p}{q}$ and $n = \deg f$. Suppose $n \geq 1$. Then for all $w \in \mathbb{C}^*$ the equation $f(z) = w$ has exactly n solutions, counting multiplicity in $\mathbb{C}^*$.*

Proof. Consider $f(z) = \frac{p(z)}{q(z)} = \infty$. If $\deg p \leq \deg q$ then q has n-zeros $\mathbb{C}$ and so $f(z) = \infty$ has n solutions. Also, f is holomorphic at ∞ so f has n poles at ∞. If $\deg q < \deg p$ then f has a pole of order $n - \deg q$ at ∞ and $\deg q$ poles in $\mathbb{C}$. So all together f takes on ∞ with multiplicity n.

Now let $w \in \mathbb{C}$, let $g(z) = \frac{1}{w - f(z)}$. Then f takes on w with multiplicity $m$at $z_0 \in \mathbb{C}^*$ if and only if g has a pole of order m at z_0. So the number of w values of f in $\mathbb{C}^*$ is equal to the number of poles of g which is equal to the degree of g. Since

$$g = \frac{1}{w - \frac{p}{q}} = \frac{q}{qw - p}$$

and one may check that $\deg g = \deg f$.

□

Corollary 10. *For any rational function the number of zeros equals the number of poles.*

3.9 The Local Mapping Theorem

3.9.1 Example

Take $f(z) = z^n$ in $\mathbb{D}(0, \mathbb{R})$ each $w \in \mathbb{D}(0, R^n)$ is taken on n times by z's in $\mathbb{D}(0, R)$.

Theorem 46 (Theorem 9)**.** *(Local Mapping Theorem)*

Let $f \in H(\mathbb{D}(z_0, R))$ for $z_0 \in \mathbb{C}$ and $R > 0$, $f(z_0) = w_0$ and $f(z) - w_0$ has a zero of order n at z_0. Then, there exists an $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ there is a $\delta = \delta(\epsilon)$ such that for each $a \in \mathbb{D}(w_0, \delta)$ the equation $f(z) = a$ has exactly n solutions in $\mathbb{D}(z_0, \epsilon)$. Moreover, if $a \neq w_0$ then the n solutions are distinct.

Remarks:

1. Theorem is also true if f is meromorphic, look at $\frac{1}{f}$.
2. Theorem is also true if $z_0 = \infty$ and f is meromorphic in a neighborhood of ∞. look at $g(z) = f(\frac{1}{z})$.
3. $\mathbb{D}(w_0 \delta) \subseteq f(\Delta(z_0, \epsilon)$

Proof. Since $f \neq c$, the zeros of $f - w_0$ and f' are discrete sets take $\epsilon > 0$ such that $f - w_0$ has no zeros in $\mathbb{D}(z_0, \epsilon_0)$. except at z_0 and f; has no zero in $\mathbb{D}(z_0, \epsilon_0)$ except for possible at z_0. Let

$\epsilon \in (0, \epsilon_0)$ and set $\gamma = \partial\mathbb{D}(z_0, \epsilon)$. Then $f - w_0$ is not zero on γ. So, $\inf_{z\in\gamma} |f(z) - w_0| = \delta > 0$. Choose $a \in \mathbb{D}(w_0, \delta)$, then for $z \in \gamma$ we have

$$|(f(z) - a) - (f(z) - w_0)| = |a - w_0| < \delta \leq |f(z) - w_0|$$

By Rouche's Theorem we have $f - a$ and $f - w_0$ have the same number of zeros inside $\gamma = \mathbb{D}(z_0, \epsilon)$. Since f; has no zeros in $\mathbb{D}(z_0, \epsilon)$ except possibly at z_0 all a-values have multiplicity 1.

□

Corollary 11 (The Open Mapping Theorem)**.** *If $f \in H(D)$, for some open connected D and non-constant, then $f(D')$ is open for all open $D' \subseteq D$.*

Proof. Let $w_0 \in f(D')$, then $w_0 \in f(z_0)$ for some $z_0 \in D'$. Let $R > 0$ be such that $\mathbb{D}(z_0, R) \subseteq D'$. By connectedness of D we know that f is non-constnat on $\mathbb{D}(z_0, R)$. By the local mapping theorem we know that $\mathbb{D}(z_0, \delta) \subseteq f(\mathbb{D}(z_0, \epsilon)$, so $f(D')$ is open.

□

Corollary 12. *If $f \in H(D)$ and real valued then f must be constant.*

Corollary 13 (Another Local Mapping Theorem)**.** *With the hypotheses of Theorem 9 there is an open set $V \subseteq \mathbb{D}(z_0, R)$ with $z_0 \in V$ so that*

(a) $f(V)$ is a disk centered at w_0.

(b) f is an n-to-1 mapping of $V \setminus \{z_0\}$ onto $f(V) \setminus \{w_0\}$.

(c) f' has no zero in $V \setminus \{z_0\}$.

Proof. Let $\epsilon = \epsilon_0$ and let $V = f^{-1}(\mathbb{D}(w_0, \delta(\epsilon_0))) \cap \mathbb{D}(z_0, \epsilon)$, this is open and one can check that (a), (b) and (c) hold.

□

Corollary 14. *If $f \in H(\mathbb{D}(z_0, R))$ and $f'(z_0) \neq 0$, then f is 1-1 on every sufficiently small neighborhood of z_0.*

Proof. See previous Corollary.

□

3.9.2 Question:

Suppose f has multiplicity n at z_0, do there exist arbitrarily small disks centered at z_0 on which f is n to 1?

Theorem 47 (Theorem 10)**.** *(Inverse Function Theorem) If $f \in H(D)$, with D a domain and f is 1-1 on D, then $f^{-1} \in H(f(D))$.*

Proof. If f is 1-1 that implies that f is nonconstant. This means that f is open and so f^{-1} is continuous. Let $w_0 \in f(D)$ and let $w \in f(D) \setminus \{w_0\}$. Also, let $z = f^{-1}(w)$, $z_0 = f^{-1}(w_0)$. Then,

$$\lim_{w\to w_0} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \lim_{w\to w_0} \frac{z - z_0}{f(z) - f(z_0)}$$

So as $w \to w_0$ by continuity we know that $z \to z_0$ and we get the above limit is equal to

$$\lim_{w\to w_0} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \lim_{w\to w_0} \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{f'(z_0)} = (f^{-1})'(w_0)$$

Because $f^{-1} \in C(f(D))$ and $f'(z_0) \neq 0$ or else not 1-1 by the Local Mapping Theorem. Hence $f^{-1} \in H(f(D))$ □

3.9.3 Example

Let D be a domain on which e^z is 1-1. For example, $D = \{x + iy \mid 0 < y < 2\pi\}$. We have e^D is the slit plane, $\{x + iy : y = 0, x > 0\}^c$.

3.10 Maximum Modulus Principle

Definition 57. *if $h : D \to \mathbb{R}$ where D is open in $\mathbb{C}$ then h has a* ***local max at*** *$z_0 \in D$ if there is a neighborhood V of z such that $h(z) \leq h(z_0)$ for all $z \in V$.*

Theorem 48 (Theorem 11)**.** *(Maximum Modulus Principle) Suppose that $f \in H(D)$, f is nonconstant and D is a domain in $\mathbb{C}^*$, then $|f|$ has no local max in D.*

Proof. (Geometric Proof) Let $z_0 \in D$ and let $\Delta_1 \subseteq D$ be an open disk containing z_0. By the open mapping theorem we have $f(\Delta_1)$ is an open set. Thus there is an open disk $\Delta_2 \subseteq f(\Delta)$ with $f(z_0) \in \Delta_2$. Since $|w_0|$ is not the maximum of $|\cdot|$ on Δ_2, z_0 is not a point when $|f|$ has a local max.

□

Proof. (Analytic) Again fix $z_0 \in \mathbb{D}$ and let $\Delta = \mathbb{D}(z_0, R)$. We have

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{f(z_0)}{z - z_0} dz = \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + Re^{it} dt$$

The above formula says that f at z_0 is the average of its value over the circle. Hence

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + Re^{it})| dt \leq \sup_{\partial\Delta} |f|$$

with equality holding if and only if f is constant on the boundary and hence on all of Δ. □

Theorem 49 (Theorem 11')**.** *Suppose D is a domain in $\mathbb{C}^*$ and $f \in H(D)$ and is continuous on $\overline{D}$. Let $M(f) = \sup_{z \in \overline{D}} |f(z)|$, then*

1. *$M(f) = |f(\zeta)|$ for some $\zeta \in \partial D$.*
2. *If $M(f) = |f(z)|$ for $z \in D$ then f is consant.*

Proof. (1) M(f) = —f(z)—4 for $z \in D$, then z is a local max, by Theorem 11 we have f is constant.

(2) $|f|$ continuous from $\overline{D} \to \mathbb{R}$ nd $\overline{D}$ is compact, so it achieves its max and that max cannot be achieved in D.

□

Corollary 15. *The max over ∂D of $|f|$ is equal to the max of $\overline{D}$.*

Corollary 16. *If $|f(z)| \leq C$ for all $z \in \partial D$ then $|f| \leq C$ in D.*

3.11 Schwarz's Lemma

3.11.1 Extremal Problem

Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic and $f(0) = 0$. How large can $|f(z)|$ be? How about $|f'(0)|$.

Theorem 50 (Theorem 12)**.** *(Schwarz's Lemma) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic and $f(0) = 0$, then*

(a) $|f(z)| \leq |z|$ *for all* $z \in \mathbb{D}$.

(b) $|f'(0)| \leq 1$

Moreover, if $|f(z)| = |z|$ for $z \in \mathbb{D} \setminus \{0\}$ or if $|f'(0)| = 1$ then there exists $\alpha \in \mathbb{R}$ so that

$$f(z) = e^{i\alpha} z$$

Proof. We consider

$$g(z0 = \begin{cases} \frac{f(z)}{z}, & z \in \mathbb{D} \setminus \{0\} \\ f'(0), & z = 0 \end{cases}$$

Since $f(0) = 0$ we know that $\frac{f}{z}$ has a removable singularity at 0. So $g \in H(\mathbb{D})$. For $z \in \partial\mathbb{D}(0, R)$, with $0 < R < 1$ we know that

$$|g(z)| \leq \frac{1}{R}$$

By the MMP we have $|g(z)| \leq \frac{1}{R}$ for $z \in \mathbb{D}(o, R)$. Fix $z \in \mathbb{D}$ and let $R \to 1$ we then have

$$|g(z)| \leq 1$$

and thus

$$|f(z)| \leq z, \qquad |f'(0)| = |g(0)| \leq 1$$

Now for uniqueness. Suppose that $|f(z_0)| = |z_0|$ or $|f'(0)| = 1$, then $|g|$ achieves its max in $\mathbb{D}$. So g is constant by MMP. Hence $g = e^{i\alpha}$ for some α. So $f(z) = zg(z) = e^{i\alpha} z$. □

3.11.2 A Generaliztion

If $f : \mathbb{D}(0, R_1) \to \mathbb{D}(0, R_2)$ with $f(0) = 0$ how big can $|f(z)|$ be?

$$|f(z)| \leq \frac{R_2}{R_1}|z|$$

$$|f'(0)| \leq R_2/R_1$$

3.11.3 Extremal Problems for Later

1. $f : \mathbb{D} \to \mathbb{D}$ $f(0) = w_0$ now maximize $|f(z)|$.
2. $f \in H(\{\text{Re}\, z > 0\})$ and $f : \{\text{Re}\, z > 0\} \to \mathbb{D}$ with $f(1) = 0$ what is the max of $|f'(1)|$?

3.12 Simple Connectivity

Definition 58. *The domain $D \subseteq \mathbb{C}^*$ is called simply connected if $\mathbb{C}^* \setminus D$ is connected.*

3.12.1 Examples

1. $\mathbb{D}(z_0, R)$ is sc
2. $D = \{z : R_1 < |z| < R_2\}$ is called doubly connected
3. A disk with two deleted points is triply connected.
4. $\mathbb{C} \setminus \{0\}$ is double connected
5. $D = \mathbb{C} \setminus [0, \infty)$ is simply connected
6. $\mathbb{C} \setminus \mathbb{Z}$ is infinitely connected/

Definition 59. *Let γ_0, γ_1 be closed curves in $\mathbb{D} \subseteq \mathbb{C}^*$. Reparametrizing if necessary we may assume γ_0, γ_1 are defined on $[0,1]$. We say that γ_0, γ_1 are **homotopic in** D if there exists a map*

$$F : [0,1] \times [0,1] \to D$$

with

(a) F is continuous.

(b) $F(s, 0) = \gamma_0(s)$ for all $s \in [0,1]$.

(c) $F(s, 1) = \gamma_1(s)$ for all $s \in [0,1]$.

(d) $F(0, t) = F(1, t)$ for all $1 \in [0,1]$

Theorem 51 (Theorem 13)**.** *Let D be a domain in $\mathbb{C}$ the following are equivalent*

(a) D is simply connected.

(b) $N(\gamma, z) = 0$ for all $z \in \mathbb{C} \setminus D$, $\gamma \subseteq D$.

(c) Every closed curve $\gamma \subseteq D$ is homotopic to a point.

Remark:

1. For general topological space (c) defines simply connected.
2. $\infty \in D$ then $(a) \iff (c)$, but $(a) \not\Rightarrow (b)$.

3.12.2 Example

Convex sets $D \subseteq \mathbb{C}$ are simply connected since we can take a straight line homotopy to a central point.

Theorem 52 (Theorem 14)**.** *Let D be a domain in $\mathbb{C}$ and suppose that $\gamma_0, \gamma_1 \subseteq D$ are closed paths homotopic in D, then for $z \notin D$ we have $n(\gamma_0, z) = n(\gamma_1, z)$.*

Proof. Suppose that the homotopy is smooth and that $\gamma_t'(s)$ converges to $\gamma_{t_0}'(s)$ for $t \to t_0$. So we have $t \to n(\gamma_t, z)$ is continuous on $[0,1]$ and it is integer valued so we have the desired equality. □

3.13 The General Form of Cauchy's Theorem

Theorem 53 (Theorem 15)**.** *(General Cauchy Theorem) Let D be a simply connected domain in $\mathbb{C}$ and let $\gamma \subseteq D$ be a closed path and $f \in H(D)$ then*

$$\int_\gamma f(z)dz = 0$$

Theorem 54 (Theorem 15)**.** *(From Alfhors) Let Ω be simply connected in $\mathbb{C}$ and γ be a closed path in Ω with $f \in H(\Omega)$ then*

$$\int_\gamma f(z)dz = 0$$

Proof. (Sketch of Proof: See Alfhors page 142 for complete argument) Assume Ω is bounded. Given $\delta > 0$ cover Ω by a grid of disjoint squares of side length δ. Let J be the collection of $\{Q_j\}_{j\in J}$ with $Q_j \subseteq \Omega$ a closed square. Define the curve

$$\Gamma_\delta = \sum_{j\in J} \partial Q_j$$

where Γ_δ is a sum of oriented line segments which are sides of exactly one Q_j. Define

$$\Omega_\delta = \bigcup_{j\in J} Int(Q_j)$$

Let γ be a cycle homologous to 0 in Ω. We want δ so small so that $\gamma \subseteq \Omega_\delta$. Consider $\zeta \in \Omega \backslash \Omega_\delta$. We then know that there exists $\zeta_0 \in Q$ with $\zeta \in Q, Q \notin \{Q_j\}$. So we have $\zeta_0 \notin \Omega$. Since $\gamma \subseteq \Omega_\delta$ we have

$$n(\gamma, \zeta) = n(\gamma, \zeta_0) = 0$$

If $z \in Q_{j_0}$ then

$$\frac{1}{2\pi i}\int_{\partial Q_j} \frac{f(\zeta)}{\zeta - z} d\zeta = \begin{cases} f(z), & j = j_0 \\ 0, & j \neq j_0 \end{cases}$$

Since both sides are continuous functions of z, the equation holds in Ω_δ. Hence

$$\int_\gamma f(z)dz = \int_\gamma \frac{1}{2\pi i}\int_{\Gamma_\delta} \frac{f(\zeta)}{\zeta - z} d\zeta dz = 0$$

□

Corollary 17. *(Grand Corollary to Theorem 15) For closed paths γ in a simply connected domain the following are valid*

1. *Cauchy's Theorem*
2. *Residue Theorem*
3. *Argument Principle*
4. *Rouche's Theorem*

Theorem 55 (Theorem 16)**.** *Suppose $f \in H(D)$ with D simply connected, then there is an $F \in H(D)$ such that $F' = f$ in D.*

Proof. For fixed $z_0 \in D$ take another $z \in D$ and let γ be any path from z_0 to z. Define $F(z) := \int_\gamma f(z)\, dz$. This is well defined by the general form of Cauchy's Theorem. Now we need that $F \in H(D)$. For $z \in \mathbb{D}(z,\delta) \subseteq D$ for some $\delta > 0$, then for $z_1 \in \mathbb{D}(z,\delta)$ we have

$$F(z_1) - F(z) = \int_{L(z,z_1)} f(\zeta) d\zeta$$

From this we obtain $F \in C^1(D)$ and $F'(z) = f(z)$ (see previous arguments). □

3.14 Logs and Powers

Theorem 56 (Theorem 17). *Suppose $f \in H(D)$ and $D \subseteq \mathbb{C}$ is simply connected. Also, assume that f has no zeros in D. Then*

(a) There is a $g \in H(D)$ so that $e^g = f$ in D. In other words $g = \log f \in H(D)$.

*(b) For all $\alpha \in \mathbb{C} \setminus \{0\}$ there is a holomorphic function h such that $h(z)$ is one of the values of $(f(z))^\alpha$. We call h a **single valued branch of** f^α. In particular there is an h so that $h^n = f$ in D for $n \in \mathbb{Z}^+$.*

Proof. (a) Let $g_1 \in H(D)$ satisfying $g_1' = \frac{f'}{f}$ in D (this is okay since f is zero free and it exists by Theorem 16). We calculate

$$\frac{d}{dz}(fe^{-g_1}) = f'e^{-g_1} - fg_1'e^{-g_1} = f'e^{-g_1} - f\frac{f'}{f}e^{-g_1} = 0$$

Hence fe^{-g_1} is constant. Say $fe^{-g_1} = c$ in D for some $c \in \mathbb{C} \setminus \{0\}$. So we must have $c = e^a$ for some $a \in \mathbb{C}$. So $f = e^a e^{g_1} = e^{a+g_1}$. Let $g = a + g_1$ we then have

$$g = \log f$$

(b) Note that $z^w = e^{w \log z}$. Since we know that $e^g = f$ we let $h = e^{\alpha g}$ and this does the job. □

Theorem 57 (Theorem 17a). *Let $g \in C(D)$ satisfying $e^g = f$ so that f has no zeros where $f \in H(D)$ and D is* any *domain (not necessarily simply connected) then $g \in H(D)$.*

Proof. It suffices to show that $g \in H(\Delta)$ for open $\Delta \subseteq D$. Take Δ, by Theorem 17 there exists g_1 so that

$$e^{g_1} = f, \qquad \text{in } \Delta \qquad \text{and } g_1 \in H(\Delta)$$

We have

$$\frac{e^{g_1}}{e^g} e^{g_1 - g} = 1 \qquad \text{in } \Delta$$

So $g_1(z) - g(z) = 2\pi i\mathbb{Z}$, but $g \in C(D)$, $g_1 \in C(\Delta)$ so $g_1(z) - g(z) = c \in \Delta$ thus g is holomorphic. □

Theorem 58 (Theorem 17 b). *Let $D \subseteq C$ be simply connected and $f \in C(D)$ having no zeros, then there exists $g \in C(D)$ such that $f = e^g$ in D.*

Proof. We delay the proof because we need a little more machinery. □

3.14.1 Example

1. $D = \mathbb{C} \setminus (-\infty, 0]$, $f(z) = z$. Choose $|\theta| < \pi$, so we have $\log z$ maps D to the infinite strip between πi and $-\pi i$. So $g(z) = \log r + i\theta$ is a holomorphic branch of $\log z$ by Theorem 17a. Also, $h(z) = \sqrt{r}e^{\frac{i\theta}{z}}$ is holomorphic

2. $D = \mathbb{C} \setminus \{$ spiral centered at 0$\}$ and $f(z) = z$. Theorem 17 implies that there is a holomorphic branch g of $\log z$ in D. Also, observe that $\lim_{z\to\infty} \mathrm{Im}(g(z)) = +\infty$.

3. $D = \{R_1 < |z| < R_2\}$ then no single valued holomorphic branch of $\log z$ or $\sqrt{z}$ exists in D (HW problem).

3.15 Multiply Connected Domains

Definition 60. *A domain $D \in \mathbb{C}^*$ is **n-connected** if $\mathbb{C}^* \setminus D$ contains n components.*

Definition 61. *A domain $D \in \mathbb{C}^*$ is **regular** if it is bounded and ∂D consists of finitely many disjoint C^1 closed Jordan Curves. Let γ_n be the **outer boundary curve** and $\gamma_1, \cdots, \gamma_{n-1}$ be the inner boundary curves oriented negatively (relative to γ_n).*

Theorem 59 (Theorem X)**.** *Suppose $f \in H(D)$ and is continuous on $\overline{D}$ with D a regular domain, then*

$$\sum_1^n \int_{\gamma_j} f(z)dz = 0$$

and

$$\frac{1}{2\pi i}\sum_1^n \int_{\gamma_j} \frac{f(\zeta)}{\zeta - z}d\zeta = f(z), \qquad \text{for } z \in D$$

Proof. (Sketch) Let us 'cut' D to become simply connected (we make streams run into the lakes in the domain). Call this new domain D'. We can take a closed path that runs close to the boundary of D', call this path $\tilde{\gamma}$, then

$$\int_{\tilde{\gamma}} f(z)dz = 0$$

Next, take $\tilde{\gamma} \to \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$. We assumed that $f \in C(\overline{D})$ so the value of the integral remains unchanged as $\tilde{\gamma} \to \bigcup \gamma_i$ and we have the result. Note we are glossing over the existence of $\tilde{\gamma}$. ☐

Lemma 7 (Splitting Lemma)**.** *Suppose $f \in H(A(R_1, R_2))$ and $0 \le R_1 < R_2 \le \infty$ (this is an annulus), then there exists a unique $f_1, f_2 \in H(\mathbb{D}(0, R_2)$ and $\mathbb{C}^* \setminus \mathbb{D}(0, R_2)$ respectively with $f_2(\infty) = 0$ such that*

$$f = f_1 + f_2 \qquad \text{in } A$$

Proof. Throughout the proof we will let γ_2 be a negatively oriented curve and γ_1 be a positively oriented curve. Suppose z, R_3, R_4 satisfy

$$R_1 < |z| < R_3 < R_4 < R_2$$

Then $g(\zeta) = \frac{f(\zeta)}{\zeta - z}$ is holomorphic in $\overline{A(R_3, R_4)}$ as a function of ζ. By Cauchy's Theorem (or by Theorem X) we get the following equation

$$\int_{\gamma_2(R_3)} \frac{f(\zeta)}{\zeta - z}d\zeta = \int_{\gamma_2(R_4)} \frac{f(\zeta)}{\zeta - z}d\zeta \qquad (1)$$

So, for $z \in A(R_1, R_2)$ define

$$f_1(z) := \frac{1}{2\pi i}\int_{\gamma(R)} \frac{f(\zeta)}{\zeta - z} d\zeta, \qquad |z| < R < R_2$$

By (1) we have f_1 is well defined, also

$$\frac{1}{\zeta - z} = \sum_0^\infty \frac{z^n}{\zeta^{n+1}}$$

with uniform convergence on $\gamma(R)$ when z is fixed. Then, term by term integration shows that f_1 is represented as a power series on $\mathbb{D}(0, R)$ and thus is holomorphic on this disk $\mathbb{D}(0, R_2)$. So we have f_1.

To find f_2 we define

$$f_2(z) = \frac{1}{2\pi i}\int_{\gamma_1(R)} \frac{f(\zeta)}{\zeta - z} d\zeta, \qquad R_1 < R < |z|$$

Then $f_2 \in H(\mathbb{C}^* \setminus \overline{\mathbb{D}(0, R_1)}$ [via proof of f_1 holomorphic and using the transformation $z \mapsto \frac{1}{z}$, which is fine since the integrand is $\mathcal{O}(\frac{1}{\infty})$]. By Theorem X we have $f = f_1 + f_2$ in A.

Now we seek uniqueness. Suppose

$$f = f_1 + f_2 = g_1 + g_2$$

then in $A(R_1, R_2)$ we have

$$f_1 - g_1 = g_2 - f_2$$

Now, $f_1 - g_1 \in H(\mathbb{D}(0, R_2))$. Extend $f_1 - g_1 = g_2 - f_2$ in $\mathbb{C}^* \setminus \mathbb{D}(0, R_1)$. Then $f_1 - g_1$ is well defined and holomorphic in $\mathbb{C}$ and $g_1 - g_1 \to 0$ as $z \to \infty$. By Liouville we have $f_1 - g_1$ is constant and is 0.

□

3.16 Laurent Series Expansions

Theorem 60 (Theorem 18)**.** *Suppose $f \in H(A)$ with $A = \{z : R_1 < |z - z_0| < R_2\}$ then*

(a) f has a representation

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where $z \in A$ and $\sum_0^\infty$ and $\sum_{-\infty}^{-1}$ converge uniformly and absolutely on compact subsets of A.

(b) $a_n = \frac{1}{2\pi i}\int_\gamma f(z)(z - z_0)^{-n-1} dz$ with $n \in \mathbb{Z}$ and γ a closed path in A with $n(\gamma; z_0) = 1$.

(c) The a_n are unique.

**Note: If no negative powers appear, then f has a removable singularity at z_0. Secondly, if no positive powers appear then f has a removable singularity at ∞.*

Proof. (a) Assume first that $z_0 = 0$. By the splitting lemma we find f_1 and f_2 and a Taylor series representations

$$f_1(z) = \sum_0^\infty a_n z^n$$

$$f_2(z) = \sum_1^\infty b_n z^{-n}$$

So, let $a_n = b_{-n}$ for $n \in \mathbb{Z}^-$. As each series converges respectively on $\mathbb{D}(0, R_2)$ and $\mathbb{C}^* \setminus \mathbb{D}(0, R_1)$ we have the sum converges locally uniformly on the annulus $A(R_1, R_2)$.

(b) We know that

$$z^{-n-1} f(z) = \sum_{m \in \mathbb{Z}} a_m z^{m-n-1}$$

So, term by term integration gives

$$\int \frac{f(z)}{z^{-n-1}} dz = 2\pi i a_n n(\gamma, 0) = e\pi i a_n$$

(c) follows from (b) and the uniqueness proven in the splitting lemma.

To apply to arbitrary z_0 apply to $f(z + z_0) = g(z)$. □

3.16.1 Two Special Cases to Remember

1. Let $f \in H(\mathbb{D}(z_0, R) \setminus \{z_0\}$ then
   - no negative index a_n's then holomorphic on $\mathbb{D}(z_0, R)$.
   - If there are a finite number of negative index a_n's then there is a pole at z_0
   - if there is an infinite number of negative index a_n's then there is an essential singularity at z_0.
2. $f \in H(\mathbb{C}^* \setminus \mathbb{D}(z_0, R))$
   - No positive index a_n's implies holomorphic at ∞
   - Finitely many positive index a_n's implies a pole at ∞
   - Infinitely many positive index implies there is an essential singularity at ∞

3.16.2 Example

1. Let $f(z) = e^z + e^{\frac{1}{z}}$.
   (a) Find the Laurent series in powers of z. Where is it valid?

   Proof. We have

   $$f(z) = \sum_0^\infty \frac{z^n}{n!} + \sum_0^\infty \frac{z^{-n}}{n!} = 2 + \sum_{-\infty}^\infty \frac{z^n}{n!}$$

   which is valid for $0 < |z| < \infty$

   □

(b) Let $z_0 = 1$. We need two representations, one in $\mathbb{D}(1,1)$ and one in $\mathbb{C} \setminus \mathbb{D}(1,1)$.

Start with the $\mathbb{D}(1,1)$ series.

Notice that

$$e^z = e \cdot e^{z-1} = \sum_0^\infty \frac{e}{n!}(z-1)^n$$

Next up we have for some b_n's

$$e^{\frac{1}{z}} = \sum_0^\infty b_n (z-1)^n$$

Hence

$$f(z) = \sum_0^\infty (\frac{e}{n!} + b_n)(z-1)^n, \qquad |z-1| < 1$$

We know that

$$\limsup_{n\to\infty} |b_n|^{\frac{1}{n}} = 1$$

Since if it were less than 1 there would be a removable singularity at 0. We have

$$e^{\frac{1}{z}} = e^{\frac{1}{1+(z-1)}}$$

Letting $\zeta = z - 1$ we then have

$$\begin{aligned} e^{\frac{1}{z}} &= e^{1-\zeta+\zeta^2-\zeta^3+\cdots} \\ &= e \cdot e^{-\zeta+\zeta^2+\mathcal{O}(\zeta^3)} \\ &= e \cdot \left[1 + (-\zeta+\zeta^2) + \frac{(-\zeta+\zeta^2)^2}{2!} + O(\zeta^3)\right] \\ &= e\left[1 - \zeta + \frac{3}{2}\zeta^2 + \mathcal{O}(\zeta^3)\right] \end{aligned}$$

Thus $b_0 = e, b_1 = -emb_2 = \frac{3}{2}e$. It then follows that

$$e^{\frac{1}{z}} + e^z = 2e + 2e(z-1)^2 + \mathcal{O}((z-1)^3)$$

Next we do the $\mathbb{C} \setminus \overline{\mathbb{D}(1,1)}$ case.

$$e^z = e\sum_0^\infty \frac{(z-1)^n}{n!}$$

$e^{1/z}$ is tougher. Let $\zeta = z - 1$, we have

$$\begin{aligned} e^{\frac{1}{z}} = e^{\frac{1}{1+zeta}} &= e^{\zeta^{-1}}\zeta^{-1} + 1 \\ &= e^{\zeta^{-1}(1-\zeta^{-1}+\zeta^{-2}-\cdots)} \\ &= e^{\zeta^{-1}-\zeta^{-2}+\zeta^{-3}-\cdots} \\ &= 1 + (\zeta^{-1} - \zeta^{-2} + \zeta^{-3} + \cdots) + \frac{1}{2}(\zeta^{-1} - \zeta^{-2} + \zeta^{-3} + \cdots)^2 \\ &\quad + \frac{1}{6}(\zeta^{-1} - \zeta^{-2} + \zeta^{-3} + \cdots)^3 + \mathcal{O}(\zeta^{-4}) \\ &= 1 + \zeta^{-1} - \frac{1}{2}\zeta^{-2} + \frac{1}{6}\zeta^{-3} + \mathcal{O}(\zeta^{-4}) \end{aligned}$$

So we have

$$f(z) = e\sum_{1}^{\infty} \frac{(z-1)^n}{n!} + (e+1) + \frac{1}{z-a} - \frac{1}{2}\frac{1}{(z-1)^2} + \frac{1}{6}\frac{1}{(z-1)^3} + \cdots$$

(c) Let $f(z) = \frac{1}{z(z-1)^2} = \frac{1}{z} + \frac{1}{(z-1)^2} - \frac{1}{z-1}$. We will fix $z_0 = 2$. There are three different expansions (Let $\zeta = z - 2$):

i. $0 \le |\zeta| < 1$:

$$f(z) = \frac{1}{2+\zeta} + \frac{1}{(1+\zeta)^2} - \frac{1}{1+\zeta} = \frac{1}{2+\zeta} - \frac{\zeta}{(1+\zeta^2)}$$

So for $0 \le |\zeta| < 1$ we have

$$f(z) = \frac{1}{2}\sum_{0}^{\infty}(-1)^n 2^{-n}\zeta^n - \sum_{1}^{\infty}(-1)^{n+1}n\zeta^n = \frac{1}{2} + \sum_{1}^{\infty}[2^{-n-1}(-1)^{n-1} + (-1)^n n](z-2)^n$$

ii. $1 < \zeta < 2$:

$$\begin{aligned} f(z) &= \frac{1}{2}\frac{1}{1+\zeta/2} - 1\frac{\zeta^{-1}}{(\zeta^{-1}+1)^2} \\ &= \frac{1}{2}\sum_{0}^{\infty} 2^{-n}(-1)^n\zeta^n - \sum_{1}^{\infty}(-1)^{n+1}n\zeta^{-n} \end{aligned}$$

Finish by substituting out ζ for $z - 2$.

iii. $|\zeta| > 2$ we have

$$f(z) = \frac{\zeta^{-1}}{1 + 2\zeta^{-1}} - \frac{\zeta^{-1}}{(1+\zeta^{-1})} \cdots$$

Fill in the rest...

4 Conformal Mappings and Möbius Transformations

4.1 Angles and Stretches

Definition 62. *Let z_0 be in the complex plane and $a \in \mathbb{C} \setminus \{0\}$ the sets of the form*

$$L^+(z_0, a) = \{z_0 + ta : t \in [0, \infty)\}$$

are called ***rays through*** *z_0.*

Definition 63. *For $a_1, a_2 \in \mathbb{C} \setminus \{0\}$ the* ***angle θ between the rays*** *$L^+(z_0, a_1)$* ***and*** *$L^+(z_0, a_2)$ is defined to by* $\arg\left(\frac{a_2}{a_1}\right)$.

Definition 64. *When $a_1 \in \mathbb{R}^+$ then θ is called* ***the angle between*** *$L^+(z_0, a_2)$* ***and the positive real axis****.*

Definition 65. *Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a C^1-path and $\gamma'(t_0) \neq 0$. $L^+(z_0, \gamma'(t_0))$ is called the* ***tangent ray to*** *γ* ***at*** *$z_0 = \gamma(t_0)$.*

Definition 66. *Let γ_1, γ_2 be as in the first definition. With $\gamma_1(t_0) = z_0$ and $\gamma_2'(t_0') = z_0$. The* ***angle between*** *γ_2* ***and*** *γ_1* ***at*** *z_0 is defined as the angle between their tangent rays*

$$\theta = \arg\left(\frac{\gamma_2'(t_0')}{\gamma_1'(t_0)}\right)$$

Theorem 61 (Theorem 1)**.** *Suppose f is holomorphic in a neighborhood of z_0 with $f'(z_0) \neq 0$.*

(a) Let γ e C^1 curve defined in a neighborhood of $t_0 \in \mathbb{R}$ with $\gamma(t_) = z_0$, $\gamma'(t_0) \neq 0$. Set $\Gamma = f \circ g$. Then

$$\arg \Gamma'(t_0) = \arg \gamma'(t_0) + \arg f'(z_0)$$

(b) If γ_1, γ_2 are two such curves, the angle between γ_2 and γ_1 at z_0 is the same as the angle between Γ_2 and Γ_1 at $f(z_0)$.

Proof. Part (a): We know that $\Gamma(t) = f(\gamma(t))$ so

$$\Gamma'(t_0) = f'(\gamma(t_0))\gamma'(t_0) = f'(z_0)\gamma'(t_0)$$

Hence the claim holds.

Part (b): From (a) we have

$$\frac{\Gamma_2'(t_0)}{\Gamma_1'(t_0)} = \frac{\gamma_2'(t_0)}{\gamma_1'(t_0)}$$

□

4.1.1 Notes

1. The converse of Theorem 1.b is also true. We may use angle conservation as a characterization of holomorphic functions. This will be a future homework.

2. Suppose that f is holomorphic in a neighborhood of z_0 and f assumes $q_0 = f(z_0)$ with multiplicity m. Then f expands angles by a factor of m.

3. Holomorphic f preserves not only angles but also their sense.

4. Also coming in future homework... consider the significance of $|f'(z_0)|$. We know that

$$f(z) = w_0 + f'(z_0)(z - z_0) + \mathcal{O}((-z_0)^2)$$

   So we can write

$$|f(z) - w_0| = |f'(z_0)||z - z_0|(1 + \mathcal{O}(|z - z_0|))$$

   So holomorphic maps map circles to almost circles.

4.1.2 What about non-holomorphic but still smooth maps?

The classic example is $f(z) = az + b\overline{z}$.

1. We know that f takes circles to ellipses.
2. Stretch depends on the ratio between $|a|$ and $|b|$.
3. Here angles are not preserved.

4.1.3 Example

$f(z) = 2z + \overline{z}$ We have

$$f(e^{\frac{i\pi}{4}}) = \frac{3}{\sqrt{2}} + \frac{i}{\sqrt{2}}, \qquad \arg\left(f(e^{\frac{i\pi}{4}})\right) \approx .32 \text{ radians}$$

Definition 67. *A* ***conformal mapping*** *of a domain $D \subseteq \mathbb{C}$ is a holomorphic 1-1 function in $\mathbb{D}$.*

4.1.4 Notes

1. If f is conformal in D, then $f'(z_0) \neq 0$ for $z_0 \in D$. So, f is conformal at each point of D if it preserves angles.
2. $f(z) = e^z$ has $f'(z) \neq 0$ for all $z \in \mathbb{C}$, but e^z is not 1-1. It is only locally conformal.
3. If $D \subseteq \mathbb{C}^*$ and f is $1-1$ and meromorphic in D, then we still call f a conformal mapping.

4.2 Möbius Transformations

Definition 68. *A Möbius Transformation is a function $S : \mathbb{C}^* \to \mathbb{C}^*$ of the form*

$$S(z) = \frac{az+b}{cz+d}$$

where $ad - cb \neq 0$. If $c \neq 0$, $S(\infty) = \frac{a}{a}$, $S(\frac{-d}{c}) = \infty$. If $c = 0$ then $S(\infty) = \infty$ and we have an affine mapping of C^* onto itself.

Definition 69. *Let $\mathfrak{M}$ be the* ***set of all Möbius transformations****.*

Theorem 62 (Theorem 2)**.** .

(a) If $S \in \mathfrak{M}$ then S is a meromorphic 1-1 function from $\mathbb{C}^*$ to $\mathbb{C}^*$.

(b) Conversely if $f \in M(\mathbb{C}^*)$ and $1-1$ on $\mathbb{C}^*$ then $f \in \mathcal{M}$.

Proof. (a) If $S \in \mathcal{M}$ then S is rational of degree 1. By the equidistribution theorem it is 1-1 on $\mathbb{C}^*$.

(b) If f is 1-1 then the degree is 1 so $f \in \mathcal{M}$. □

Theorem 63 (Theorem 3)**.** *If $S_1, S_2 \in \mathcal{M}$ then so are $S_1 \circ S_2$ and S_1^{-1}. Moreover, $\mathcal{M} \cong SL(2,\mathbb{C})/\{\pm I\}$.*

4.2.1 Some subgroups of $\mathcal{M}$

1. $S(z) = z + b$, **translations** of $\mathbb{C}^*$ isomorphic to $(\mathbb{C}, +)$
2. $S(z) = ze^{i\varphi}$ **rotations** isomorphic to S^1.
3. $S(z) = \rho z$, **dilations** isomorphic to $((0,\infty), \cdot)$.
4. $S(z) = \frac{1}{z}$ an **inversion** isomorphic to $\{1, \frac{1}{z}\} \cong \mathbb{Z}_2$.
5. $S(z) = az$, $a \in \mathbb{C} \setminus \{0\}$ complex dilations.

Theorem 64 (Theorem 4)**.** *Every $S \in \mathcal{M}$ can be obtained via composition of elements* $(1), (2), (3)$ *and* (4).

Proof. We know that $S(z) = \frac{az+b}{cz+d}$. Suppose $c = 0$, then

$$S(z) = \frac{a}{d}z + \frac{b}{d} = \tilde{a}z + \tilde{b} = \text{ rotation followed by a translation}$$

If $c \neq 0$ we write

$$z \mapsto (cz + d) \mapsto \frac{1}{cz + d} \mapsto \frac{\lambda}{cz + d} \mapsto \frac{\lambda}{cz + d} + A = \frac{\lambda + A(cz + d)}{cz + d}$$

If we let $Ac = a$ and $Ad + \lambda = b$ then we have $A = \frac{a}{c}, \lambda = b - \frac{ad}{c}$. So it is a composition as stated. □

Theorem 65 (Theorem 5)**.** *Suppose $S \in \mathcal{M}$.*

(a) Let K be a circle in $\mathbb{C}$. If $\infty \notin S(K)$ then $S(K)$ is a circle in $\mathbb{C}$. If $\infty \in S(K)$ then $S(K) - \{\infty\}$ is a line in $\mathbb{C}$.

(b) Let L be a line in $\mathbb{C}$. Write $L^* = L \cup \{\infty\}$. Then if $\infty \notin S(L^*)$ then $S(L^*)$ is a circle in $\mathbb{C}$. If $\infty \in S(L^*)$ then $S(L^*) \setminus \{\infty\}$ is a line in $\mathbb{C}$.

Proof. It suffices to prove Theorem 5 only for the four primitive elements. For types $(1), (2)$ and (3) the proof is left as an exercise. So, we turn our attention to inversion, $S(z) = \frac{1}{z}$. A line or circle has the equation

$$Az + By + C(x^2 + y^2) = D$$

Assume first that K is a circle and $0 \notin K$. Then C, D are not 0. Let $w = \frac{1}{z}$ then we get (using the substitution $x = \frac{z+\overline{z}}{2}$ and $y = \frac{z-\overline{z}}{2i}$

$$\frac{1}{2}A(\frac{1}{w} + \frac{1}{\overline{w}}) + \frac{1}{2i}B(\frac{1}{w} - \frac{1}{\overline{w}}) + c|w|^{-2} = D$$

Multiply this by $\overline{w}w = |w|^2$, with $w = u + iv$ and we obtina

$$Au - Bv - D|w|^2 = -C$$

Call this set K'. Hence the image of K is a circle and $0 \notin S(K)$. (Note that we technically only showed $S(K) \subseteq K'$, but we could take the inverse to prove the other containment.

Other cases are left as an exercise. □

4.2.2 Example: $\frac{1+z}{1-z}$

Let $S(z) = \frac{1+z}{1-z}$.

(a) Consider $S(\partial\mathbb{D})$. We know that $S(1) = \infty$, $S(-1) = 0$ and $S(i) = i$. Hence $S(\partial\mathbb{D}) = i\mathbb{R} \cup \{\infty\}$.

(b) Since $S(0) = 1$ we have the right half plane contains the image of $S(\mathbb{D})$ vian connectivity considerations. But $S(\infty) = -1$ and so the left half plane contains the image of $S(\mathbb{D}^c)$. But since $S(\mathbb{C}^*) = \mathbb{C}^*$ it must be onto. So $S : \mathbb{D}$ to the RHP.

(c) $S(\mathbb{R})$ we have $S(\infty) = -1$, $S(0) = 1$, $S(1) = \infty$ so S is an automorphism of the upper half plane.

In general $\frac{az+b}{cz=d}$ *takes the upper half plane to itself if* $a, b, c, d \in \mathbb{R}$ *and* $ad-bc > 0$. *Future HW: You may need to compose with the inverse to get the RHP into the circle to apply Schwarz's Lemma*

(d) We consider $S(i\mathbb{R})$ and $S(\{\mathrm{Re}(z) > 0\})$. So

$$S(0) = 1, S(\infty) = -1, S(i) = i$$

Hence $S(i\mathbb{R}) = \partial\mathbb{D}$. And since $S(1) = \infty$ we have the right half plane maps to the outside of $\mathbb{D}$ and the left half plane maps to the inside of $\mathbb{D}$.

(e) We compute S^{-1}. Note that if $w = \frac{1+z}{1-z}$ then we have $w(1-z) = 1+z$, so

$$S^{-1}(w) = \frac{w-1}{w+1}$$

4.3 Möbius Transformations II

Theorem 66 (Theorem 6)**.** *(Three on Three Theorem) Let* z_1, z_2, z_3 *and* w_1, w_2, w_3 *be distinct in letter points in* $\mathbb{C}^*$. *There exists a unique Möbius transformation* S, *such that*

$$S(z_i) = w_i, \qquad \text{for } i = 1, 2, 3$$

Proof. Supposing no $w_i = \infty$ or $z_i = \infty$ let

$$S_1(z) = \frac{z-z_1}{z-z_2} \cdot \frac{z_3-z_2}{z_3-z_1}$$

$$S_2(z) = \frac{w-w_1}{w-w_2} \cdot \frac{w_3-w_2}{w_3-w_1}$$

So $S_2^{-1} \circ S_1$ does the trick. Even if z_i or w_i is ∞ this still works if you are careful.

It remains to prove uniqueness. First, suppose that $S(0) = 0$, $S(1) = 1$ and $S(\infty) = \infty$. As $S(\infty) = \infty$ then we must have $c = 0$. And if $S(0) = 0$ then $b = 0$ and since $S(1) = 1$ there is no rotation so $a = d$. Thus S is the identity.

Now let S_1 and S_2 map $z_1, z_2, z_3 \mapsto w_1, w_2, w_3$. We want to prove that $S_1 = S_2$. Let $T = S_1^{-1} \circ S^2$. Then T fixes z_1, z_2 and z_3. Let U take z_1, z_2, z_3 to $0, 1, \infty$. Then $U \circ T \circ U^{-1}$ fixes 0, 1 and ∞. So it is equal to the identity map, I. So $T = I$ and thus $S_1^{-1} \circ S_2 = I$ and $S_1 = S_2$.

□

4.3.1 Example

Find $S \in \mathcal{M}$ so that $S(\mathbb{D}(3,2)) = \{\operatorname{Im} w > 1\}$. There are two ways to do this

1. Send $\mathbb{D}(3,2) \to \mathbb{D}(0,1) \to \text{RHP} \to \text{UHP} \to \text{UHP} + 1$.

$$S(z) = \left(\frac{1 + \frac{2-3}{2}}{1 - \frac{z-3}{2}} \right) i + 1$$

2. Or we could choose three points on the circle and write a Möbius Transformation

$$5 \mapsto \infty$$

$$3 + 2i \mapsto i$$

$$1 \mapsto 1 + i$$

Definition 70. *Let $z_1, z_2, z_3, z_4 \in \mathbb{C}^*$ be distinct points. Define*

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

*to be the **cross ratio** of these points.*

For a fixed z_2, z_3, z_4 we can define a Möbius Transformation

$$S(z) = (z, z_2, z_3, z_4)$$

$$S(z_2) = 1$$

$$S(z_3) = 0$$

$$S(z_4) = \infty$$

We conclude that

1. The unique Möbius Transformation taking z_2, z_3, z_4 to $1, 0, \infty$ is (z, z_2, z_3, z_4).
2. If z_1, z_2, z_3, z_4 are fixed then (z_1, z_2, z_3, z_4) is never $0, 1, \infty$.
3. One way to find S taking

$$z_1 \mapsto w_1, z_2 \mapsto w_2, z_3 \mapsto w_3$$

is to solve the equation

$$(w, w_1, w_2, w_3) = (z_1, z_2, z_3, z_4)$$

for w.

Theorem 67 (Theorem 7)**.** *If $S \in \mathcal{M}$ and z_1, z_2, z_3, z_4 are distinct points in $\mathbb{C}^*$ then*

$$(S(z_1), S(z_2), S(z_3), S(z_4)) = (z_1, z_2, z_3, z_4)$$

Proof. First suppose that
$$S(z_1) = 1, S(z_2) = 0, S(z_3) = \infty$$
then we have
$$(S(z_1), S(z_2), S(z_3), S(z_4)) = S(z_1)$$
but $S(z_1) = (z_1, z_2, z_3, z_4)$. Now for a general S let
$$S = S_2^{-1} \circ S_1$$
where
$$S_1 : z_2, z_3, z_4 \to 1, 0, \infty$$
$$S_2 : w_2 w_3, w_4 \to 1, 0, \infty$$
As S_2 and S_1 preserve cross ratios so does S.

□

Theorem 68 (Theorem 8)**.** *Let z_1, z_2, z_3, z_4 be distinct points of $\mathbb{C}^*$ then*
$$(z_1, z_2, z_3, z_4) \in \mathbb{R} \iff z_1, z_2, z_3, z_4 \text{ lie on a line or a circle}$$
Proof. Homework Assignment.

□

Theorem 69 (Theorem 9)**.** *Let $\mathbb{D} = \mathbb{D}(0, 1)$. Then*

1. *For $a \in \mathbb{D}$ and $\varphi \in \mathbb{R}$ we have*
$$T(z) = e^{i\varphi} \frac{z - a}{1 - \overline{a}z}$$
maps $\mathbb{D}$ 1-1 and onto $\mathbb{D}$ with $T(a) = 0$.

2. *Every conformal map of $\mathbb{D}$ onto $\mathbb{D}$ is of this form.*

Proof. (1) First consider
$$|T(e^{i\theta})| = \left| \frac{e^{i\theta} - a}{1 - \overline{a}e^{i\theta}} \right| = \left| \frac{e^{i\theta} - a}{e^{-\theta} - \overline{a}} \right| = 1$$
So T maps $\partial\mathbb{D}$ onto $\partial\mathbb{D}$. Since T is holomorphic in $\mathbb{D}$ by the MMP we know that $\mathbb{D}$ maps into $\mathbb{D}$. To show that T is onto, note that T maps $\mathbb{C}^*$ 1-1 onto $\mathbb{C}^*$. Since $T(a) = 0$we must have T maps $\mathbb{D} \to \mathbb{D}$ as it maps $\partial\mathbb{D} \to \partial\mathbb{D}$.

(2) Suppose first that $f(0) = 0$. By the Schwartz Lemma we have
$$|f(z)| \leq |z|, \qquad \forall z \in \mathbb{D}$$
Also,
$$|f^{-1}(z)| \leq |z|$$
So we must have $|f(z)| = |z|$. By the uniqueness part of the Schwartz Lemma we have
$$f = e^{i\varphi} \cdot z, \qquad \text{for some } \varphi \in \mathbb{R}$$
Next assume that $f(0) = a$ with $a \in \mathbb{D} \setminus \{0\}$. Let
$$T(z) = \frac{z - a}{1 - z\overline{a}}$$

then $T \circ f : \mathbb{D} \to \mathbb{D}$ with $(T \circ f)(0) = 0$. So we have $T \circ f = e^{i\varphi} \cdot z$. Then via some algebra

$$f(z) = e^{i\varphi} \frac{z + ae^{-i\varphi}}{1 - \overline{a}e^{i\varphi}z}$$

□

4.3.2 Notes

- Aut $(\mathbb{D}) = \left\{ e^{i\varphi} \frac{z-a}{1-\overline{a}z} : a \in \mathbb{D}, \varphi \in \mathbb{R} \right\}$ is a 3-dimensional lie group under composition.

$$\text{Aut } (\mathbb{D}) = S^{-1} \circ \frac{SL(2, \mathbb{R})}{\{\pm\}} \circ S$$

  with $S = i \left(\frac{1+z}{1-z} \right)$.

- For A an annulus then we have Aut (A) is generated by rotations and 1 inversions.
- If D has connectivity $n \geq 3$ then Aut(D) is a finite group, usually trivial by has order less than or equal to $84(n-1)$.

4.4 Fixed Points of a Möbius Transformation

Definition 71. *$z_0 \in \frac{1}{2\pi i}^*$ is a **fixed point** of $T \in \mathcal{M}$ if $T(z_0) = z_0$.*

Lemma 8 (Lemma 1)**.** *Suppose $T \in \mathcal{M}$, $T \neq Id$ then T has exactly 1 or 2 fixed points.*

Proof. If $T = az + b$ then ∞ is fixed and

$$z = az + b$$

$$z = \frac{b}{1-a}$$

is another fixed points. This is it if $a \neq 1$.

If $T = \frac{az+b}{cz+d}$ and if $c \neq 0$ then

$$R(z) = \frac{az + b}{cz + d} - z$$

is a function with $R(-\frac{d}{c}) = \infty$ and $R(\infty) = \infty$. As R is rational and it is of degree 2. Hence it has 2 zeros via the Equidistribution Theorem. These zeros are the fixed points.

□

Definition 72. *If $S, T \in \mathcal{M}$ we say that S and T are **conjugate** if there exists a $U \in \mathcal{M}$ such that*

$$S = U^{-1}TU$$

Lemma 9 (Lemma 2)**.** *Let S and T be conjugate. Then z_0 is a fixed point of S if and only if $U(z_0)$ is a fixed point of T.*

Proof. Left to reader.

□

Lemma 10 (Lemma 3)**.** *Conjugate Möbius Transformations have the same number of fixed points.*

Proof. Follows immediately from U being 1-1. □

Lemma 11 (Lemma 4)**.** *If $b_1, b_2 \in \mathbb{C} \setminus \{0\}$. Let $S = z + b_!$ and $T = z + b_2$, then S and T are conjugate.*

Proof. Let $U = \frac{b_2}{b_1} z$ and go from there. □

Lemma 12 (Lemma 5)**.** *Let $S = Az$, $T = Bz$ where $A, B \in \mathbb{C} \setminus \{0\}$. Then S and T are conjugate if and only if $A = B$ or $A = \frac{1}{B}$.*

Proof. ($\Rightarrow$) Suppose $U \in \mathcal{M}$, $S = U^{-1}TU$. If $T = I, S = I$. So if either A or B is equal to 1 then the other is also 1. Now assume that $A, B \in \mathbb{C} \setminus \{0, 1\}$. Then S, T have only 2 fixed points, 0 and ∞. From Lemma 2 we have $U(0)$ and $U(\infty)$ are also fixed points we either have

(a) $U(0) = 0, U(\infty) = \infty$

(b) $U(0)\infty, U(\infty) = 0$

In case (a) there exists $C \in \mathbb{C} \setminus \{0\}$ so that $U(z) = C \cdot z$ and thus

$$Az = \frac{1}{C} Bz$$

and so $A = B$

In case (b) we have $U(z) = \frac{C}{z}$ for some $C \in \mathbb{C} \setminus \{0\}$.

($\Leftarrow$) This direction follows from the last part of the if direction. □

Theorem 70 (Theorem 10)**.** *Let $T \in \mathcal{M}$ with $T \neq Id$.*

1. *If T has one fixed point then T is conjugate to a translation.*

2. *If T has 2 fixed points then T is conjugate to a transformation.*

$$S(z) = az, \qquad a \in \mathbb{C} \setminus \{0\}$$

Proof. (1) Let $U \in \mathcal{M}, U(\infty) = z_0$ where z_0 is a fixed point of T, let $S = U^{-1}TU$. We have $S(\infty) = \infty$. So, $S = az + b$ for some $a, b \in \mathbb{C}$. Since S has only 1 fixed point we must have $a = 1$. Thus T is conjugate to $z + b$.

(2) Let

$$0 \xrightarrow{U} z_1 \xrightarrow{T} z_1 \xrightarrow{U^{-1}} 0$$

$$\infty \xrightarrow{U} z_2 \xrightarrow{T} z_2 \xrightarrow{U^{-1}} 0$$

Then $S = U^{-1}TU$ and $S(0) = 0$ and $S(\infty) = \infty$. So we must have $S(z) = az$ for some $A \in \mathbb{C} \setminus \{0\}$. □

Corollary 18. *If $T \in \mathcal{M}$ and $T \neq Id$ then there exists a unique $i \in \{1, 2, 3, 5\}$ such that T is conjugate to a Möbius Transformation of type i.*

Definition 73. *Let $T \in \mathcal{M}$*

- *T is called **parabolic** if it is conjugate to a translation.*
- *T is called **hyperbolic** if it is conjugate to a positive dilation.*
- *T is called **elliptic** if it is conjugate to a rotation.*
- *T is called **loxodromic** if it is conjugate to a complex dilation.*

4.4.1 Intuition

1. Translation, $z \to z + b$. This sends the whole plane along lines with angle $\arg b$.

   On the sphere this is mapping points to circles through the north pole with only one being a great circle.

2. Positive Dilation, $T(z) = \rho z$ and $T^n = \rho^n z$. Then the flow lines radiate from the origin if $\rho > 1$ and shrink in to the origin if $\rho < 1$.

   On the sphere we get all of the great circles through the north pole.

3. T is a rotation, $T - e^{i\varphi}z$. Then the flow lines are circles centered at the origin.

   And on the sphere we get lines of latitude.

4. T is a complex dilation. $T(z) = az$ and $T^n = a^n z$. THe flow lines look like spirals radiating from the origin.

   On the sphere we get a spiral that tightens as it approaches the north pole.

4.5 Other Conformal Maps

4.5.1 Examples

1. $D(\theta_1, \theta_2) = \{re^{i\theta} : \theta_1 < \theta < \theta_2, r > 0\}$. Assume $0 < \theta_2 < \theta_1 < 2\pi$ for $\alpha > 0$ we let

   $$f(z) = z^\alpha = r^\alpha e^{i\alpha\theta}$$

   We have $f \in H(D(\theta_1, \theta_2))$ and f maps $\rho(\theta) \to \rho(\alpha\theta)$. If $\alpha(\theta_2 - \theta_1) \leq 2\pi$ then f maps $D(\theta_1, \theta_2)$ to $D(\alpha\theta_1, \alpha\theta_2)$. On the other hand if $\alpha(\theta_2 - \theta_!)? > 2\pi$ then f maps $D(\theta_1, \theta_2)$ to $\mathbb{C} \setminus \{0\}$ and is not 1-1.

2. **Exponentials:** Let $S(a, v) = \{z = x + iy : y \in (a, b)\}$. We have $f(z) = e^z = e^x e^{iy}$ and f maps $\{x + iy : y_0 \in \mathbb{R}, x \in \mathbb{R}\}$ is 1-1 and onto the ray $\rho(y_0)$. So

   $$f : S(a, b) \to D(a, b)$$

   bijectively if $b - a \leq 2\pi$.

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$$f : S(a, b) \to D(a, b)$$

bijectively if $b - a \leq 2\pi$.

4.6.2 Example 1

We want to map $\mathbb{D}$ 1-1 onto $\mathbb{C} \setminus (-\infty, 0]$ the map $f(z) = \frac{1+z}{1-z}$ and $g(z) = z^2$ the composition

$$h(z) = (g \circ f)(z) = \left(\frac{1+z}{1-z}\right)^2$$

works.

4.6.3 Example 2

We want to map $\mathbb{D}(0, R)$ bijectively to $|\arg w| < \alpha$ with $\alpha \in (0, \pi]$. We use the map

$$f_1(z) = \frac{z}{R}, \qquad f_2(z) = \frac{1+z}{1-z}, \qquad f_3(z) = z^{\frac{2\alpha}{\pi}}$$

The map

$$g(z) = (f_3 \circ f_2 \circ f_1)(z) = -\left(\frac{1 + \frac{z}{R}}{1 - \frac{z}{R}}\right)^{\frac{2}{\pi}(\pi - \alpha)}$$

4.6.4 Example 3

We can map $\mathbb{D} \cap \mathbb{H}$ bijectively onto $\mathbb{D}$. We use the maps

$$f_1(z) = \frac{1+z}{1-z}, \qquad f_2(z) = z^2, \qquad f_3(z) = -iz, \qquad f_4(z) = \frac{z-1}{z+1}$$

Then we take

$$f(z) = (f_4 \circ f_3 \circ f_2 \circ f_1)(z) = \frac{\left(\frac{1+z}{1-z}\right)^2 - i}{\left(\frac{1+z}{1-z}\right)^2 + i}$$

4.6.5 Example 4

Map $|\operatorname{Im} z| < b$ onto $\mathbb{D}$. We use the maps

$$f_1(z) = \frac{\pi}{2b}z, \qquad f_2(z) = e^z, \qquad f_3(z) = \frac{z-1}{z+1}$$

$$f(z) = (f_3 \circ f_2 \circ f_1)(z) = \frac{e^{\frac{\pi}{2b}z} - 1}{e^{\frac{\pi}{2b}z} + 1}$$

4.6.6 Example 6

Consider the map $f(z) = \sin z = \sin x \cosh y + i \cos x \sinh y$. We have f maps the half strip $[-\frac{\pi}{2}, \frac{\pi}{2}] \times (\mathbb{R}^+ \cup \{0\})$ bijectively to $\mathbb{H}$. and maps rectangles to half ellipses. So we have $\sin z$ maps the whole strip onto the plane minus rays on the real line starting at -1 and 1.

4.6.7 Example 7

Consider the map $f(z) = z + \frac{1}{z} = \frac{z^2+1}{z}$ which is a rational map of degree 2. This has simple poles at 0 and ∞, with $f'(z) = 1 - z^{-2}$. So, $z = 1$ and $z = -1$ are double points.

$$f(1) = 2, \quad f(-1) = -2 \quad \text{both with multiplicity 2}$$

Each $w \in \mathbb{C}^* \setminus \{2, -2\}$ has two distinct preimages

$$f(z) = f(\frac{1}{z})$$

And also,

$$f(e^{i\theta}) = 2\cos\theta$$

Hence, f maps $\{z : |z| = 1\}$ $2-1$ and onto $[-2, 2]$. So, f maps $\mathbb{D}$ 1-1 and onto $\mathbb{C}^* \setminus [-2, 2]$. Also the exterior of $\mathbb{D}$ maps to the same set. If $r < 1$ then

$$\mathbb{D}(0, r) \to re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

By a previous homework problem we know that $\partial\mathbb{D}(0, r)$ is taken to the boundary of an ellipse and orientation is reversed. So f maps small disks around the origin to the exterior of ellipses.

To map $\mathbb{D}$ to the interior use the special function sn

$$w = \sqrt{k} \cdot \operatorname{sn}\left(\frac{2k}{\pi}\sin^{-1} z, \rho\right)$$

5 Harmonic Functions I

5.1 Harmonic Functions

Let D always be an open set in $\mathbb{C}$. Recall if $f \in H(D)$ and $f = u + iv$ where u, v are real then

$$u_x = v_y$$

$$u_y = -v_x$$

If $u, v \in C^1(D)$ and the CR are satisfied then $f \in H(D)$.

Definition 74. *The **Laplacian of** u is given by*

$$\Delta u := u_{xx} + u_{yy}$$

*where $u \in C^2(D)$. Δ is called the **Laplace Operator**.*

Definition 75. *$u : D \to \mathbb{R}$ is **harmonic in** D denoted by $u \in Ha(D)$ if $u \in C^2(D)$ and $\Delta u = 0$.*

Theorem 71 (Theorem 1)**.** *If $f \in H(D)$ then $f = u + iv$, $u, v \in Ha(D)$.*

Proof. Suppose we know $u, v \in C^2(D)$, then $u_{xx} = v_{yx}$ and $u_{yy} = -v_{xy}$. So we have

$$\Delta u = v_{yx} - v_{xy} = 0$$

and thus $u \in Ha(D)$, same for v. It remains to show that $u, v \in C^2(D)$. We know that $f' \in C(D)$ and

$$f' = \frac{\partial f}{\partial x} = u_x + iv_x = v_y - iu_y$$

via the CR equations. So we must have $u, v \in C^1(D)$. Since $f' \in H(D)$ we may do this again, so $u, v \in C^2(D)$.

□

Corollary 19. *If $u, v \in C^1(D)$ and satisfy the CR equations in D, then $u, v \in Ha(D)$ and in particular we have $u, v \in C^2(D)$.*

5.1.1 Examples

1. Consider $u = x^2 - y^2$. We have $u_{xx} = 2$, $u_y y = -2$, and so $\Delta u = 0$, $u = \text{Re}(z^2)$.

2. $u = x^2 + y^2$, then we have $\Delta u = 4$ so $u \notin H(\mathbb{C})$.

3. $u(z) = \log |z| = \text{Re}(\log z) = \frac{1}{2} \log(x^2 + y^2)$. So we have

$$u_x = \frac{x}{x^2 + y^2}$$

$$u_{xx} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$u_{yy} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

   and thus $\Delta u = 0$ away from $z = 0$.

4. Alternate Proof of 3. Let $z_0 \in \mathbb{C} \setminus \{0\}$ and let $\mathbb{D}(z_0, \epsilon) \subseteq \mathbb{C} \subseteq \{0\}$. Let f be a branch of the log holomorphic in $\mathbb{D}(z_0, \epsilon)$. Then we have $\log |z| = \text{Re} |f|$ in $\mathbb{D}(z_0, \epsilon)$, and so $\log |z|$ is harmonic since real and imaginary parts of holomorphic functions are harmonic.

Definition 76. *If $f = u + iv \in H(D)$ then v is called a **conjugate harmonic functions to** u.*

5.1.2 Remarks

1. If v is the harmonic conjugate of u then $v+c$ is also for all $c \in \mathbb{R}$.
2. If D is connected and v_1, v_2 are conjugate to u then in D we have $v_1 - v - 2 = c \in \mathbb{R}$

Proof. (1) is clear. (2) follows from the fact that we have

$$\begin{aligned}(u+iv_1)-(u+iv_2) &= i(v_1-v_2)\\ &= ic\end{aligned}$$

For some $c \in \mathbb{R}$ by uniqueness. □

5.1.3 Example

1. $u = x^2 - y^2$ then $v = 2xy$. We have $u = \operatorname{Re}(z^2), v = \operatorname{Im}(z^2)$
2. $u = \log|z|$. If there was a harmonic conjugate it would be $v = \arg z$. However $\arg z$ is not even continuous, so this does not work.

Theorem 72 (Theorem 2)**.** *Let D be simply connected and $u \in Ha(D)$. Then there exists $v \in Ha(D)$ such that v is a conjugate harmonic function of u.*

Proof. If we found some $f = u + iv \in H(D)$ then by the CR

$$f' = u_x + iv_x = u_x - iu_y$$

So we look for an antiderivative of u_y. Let $g = u_x - iu_y$. The fact that u is harmonic implies that g is continuous and differentiable. Additionally, harmonicity of u implies that

$$(u_x)_x = (-u_y)_y$$

$$(u_x)_y = -(-u_y)_y$$

This means that g satisfies CR and thus is holomorphic. Since we are in a simply connected domain there exists $f \in H(D)$ such that $f' = g$ in D. We compute

$$\frac{\partial}{\partial x}(\operatorname{Re} f) = \operatorname{Re} f' = u_x$$

$$\frac{\partial}{\partial y}(\operatorname{Re} f) = \operatorname{Re}(if') = \operatorname{Re}(ig) = -\operatorname{Im} g = u_y$$

So, $\nabla(\operatorname{Re} f) = \nabla u$. This means that there exists $c \in \mathbb{R}$ such that

$$\operatorname{Re} f = u + c, \qquad \text{in } D$$

now $f - c \in H(D)$ and $\operatorname{Re}(f-c) = u$. This means that

$$\operatorname{Im}(f-c) =: v$$

is a harmonic conjugate. □

5.1.4 Example

If a domain is not simply connected we may not be able to find a harmonic conjugate. Consider

$$D = \mathbb{C} \setminus \{0\}, \qquad u = \log|z|$$

If $f = u + iv \in H(\mathbb{C} \setminus \{0\})$ then $g := f'$. Hence,

$$\int_\gamma g(z)dz = 0, \qquad \gamma = \partial\mathbb{D}(0, R)$$

Because g has an anti-derivative. But if $u = \log|z|$ then $u_x = \frac{x}{x^2+y^2}, u_y = \frac{y}{x^2+y^2}$. So,

$$u_x - iu_y = \frac{x - iy}{x^2 + y^2} = \frac{\overline{z}}{z\overline{z}} = \frac{1}{z}$$

This means that $g = \frac{1}{z}$. However,

$$\int_\gamma \frac{dz}{z} = 2\pi i$$

So, no such f exists.

Theorem 73 (Theorem 3)**.** *Suppose that $u \in Ha(D)$, $F \in H(D')$ and $F(D') \subseteq D$, then $u \circ F \in Ha(D')$.*

Proof. Take $z_0 \in D'$. Let Δ be a disc with $F(z_0) \in \Delta$. Let $\Delta' \subseteq D'$ so that $F(\Delta') \subseteq \Delta$ and $z_0 \in \Delta'$. As Δ is simply connected there exists $f \in H(\Delta)$ so that $u = \operatorname{Re} f$. Then $f \circ F \in H(\Delta')$ and $u \circ F = \operatorname{Re}(f \circ F)$. This means that $u \circ F \in Ha(D')$ as being harmonic is a local property. □

There is an alternate proof.

Proof. We know that $u \in C^2(D), F \in H(D')$. We have

$$\Delta(u \circ F) = \Delta u \circ F \cdot |F'|^2 = 0$$

□

5.2 Mean Value Property and Series Representation

Theorem 74 (Theorem 4)**.** *Let $u \in Ha(D)$ and suppose $\overline{\mathbb{D}}(z_0, r) \subseteq D$. We then have*

$$u(z_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(z_0 + re^{i\theta})d\theta$$

Proof. Let Δ be an open disk with $\overline{\mathbb{D}}(z_0, r) \subseteq \Delta \subseteq D$ and let $f = u + iv$ be in $H(\Delta)$. By Cauchy's Formula we know that

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}(z_0,r)} f(z)\frac{dz}{z - z_0}$$

Take the parametrization $z = z_0 + re^{i\theta}$ and $dz = ire^{i\theta}d\theta$. So we have

$$f(z_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z_0 + re^{i\theta})d\theta$$

taking the real part gives

$$\begin{aligned} u(z_0) &= \mathrm{Re}(f(z_0)) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Re}(f(z_0 + re^{i\theta}))d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(z_0 + re^{i\theta})d\theta \end{aligned}$$

□

There again is an alternate proof.

Proof. We use Green's Theorem, which states that if $P, Q \in C^1(D \cup \partial D)$ then

$$\int_{\partial D} Pdx + Qdy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Our hypotheses tell us that

$$\begin{aligned} 0 &= \int\int_{\mathbb{D}(z_0,r)} \Delta u \; dx \; dy \\ &= \int\int_{\mathbb{D}(z_0,r)} (u_{xx} + u_{yy})dxdy \\ &= \int_{\partial\mathbb{D}(z_0,r)} (-u_y)dx + (u_x)dy \end{aligned}$$

If we let

$$dx = -r\sin\theta d\theta$$

$$dy = r\cos\theta d\theta$$

Then we have the above is equal to

$$\begin{aligned} 0 &= r\left(\int_0^{2\pi}\left(\frac{\partial u}{\partial y}\sin\theta + \frac{\partial u}{\partial x}\cos\theta\right)d\theta\right) \\ &= r\int_0^2 \pi\frac{\partial u}{\partial r}\left(z_0 + re^{i\theta}\right)d\theta \\ &= r\frac{\partial}{\partial r}\left(\int_0^{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta\right) \end{aligned}$$

So we must have $\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta$ is constant for some $r \in (0, R)$. But as $r \to 0$ continuity implies that

$$\lim_{r\to 0}\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta = u(z_0)$$

□

5.2.1 Remark

For $u, v \in C^2(\overline{D})$ with D bounded. Green's Theorem says that

$$\int_{\partial D} u\frac{\partial v}{\partial n}|dz| = \int_D \nabla u \cdot \nabla v dxdy + \int_D u\delta v dxdy$$

So we have

$$\int_{\partial D}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)|dz| = \int_D u\Delta v - v\Delta u dxdy$$

Let $v = 1$ then we have

$$\int_{\partial D}\frac{\partial u}{\partial n}|dz| = \int_D \Delta u\, dxdy$$

Theorem 75 (Theorem 5)**.** *Let $u \in Ha(\mathbb{D}(0,R))$. Then u has a representation of the form*

$$u(re^{i\theta)\frac{1}{[}2}a_0 + \sum_1^\infty (a_n\cos(n\theta) + b_n\sin(n\theta))r^n$$

for $o < r < R$ and $0 \le \theta \le 2\pi$. The series converges uniformly on all smaller disks. Moreover,

$$a_n := \frac{1}{\pi r^n}\int_0^{2\pi} u(re^{i\theta})\cos(n\theta)d\theta, \qquad n \ge 0$$

$$b_n := \frac{1}{\pi r^n}\int_0^{2\pi} u(re^{i\theta})\sin(n\theta)d\theta, \qquad n \ge 1$$

Proof. Let $f \in H(\mathbb{D}(0,R))$ such that $u = \text{Re}(f), f(0) = u(0)$. Then $f(z) = \sum_0^\infty c_n z^n$ for all $|z| < R$. Let $c_0 = \frac{1}{2}a_0$ and write

$$c_n = a_n - ib_n, \qquad n \ge 1$$

We have

$$\begin{aligned} Re(c_n z^n) &= \text{Re}[(a_n - ib_n)r^n(\cos n\theta + i\sin n\theta)] \\ &= a_n r^n \cos n\theta + b_n r^n \sin n\theta \end{aligned}$$

So,

$$u(z) = \frac{1}{2}a_0 + \sum_1^\infty (a_n\cos n\theta + b_n\sin n\theta)r^n$$

And we get uniform convergence via the uniform convergence theorem for f.

Since we have

$$\int_0^{2\pi}\cos n\theta\cos m\theta d\theta = \pi\delta_{m,n}$$

$$\int_0^{2\pi}\sin n\theta\sin m\theta d\theta = \pi\delta_{m,n}$$

$$\int_0^{2\pi}\cos n\theta\sin m\theta d\theta = \int_0^{2\pi}\sin\theta d\theta = \int_0^{2\pi}\cos\theta d\theta = 0$$

So the formula for a_n, b_n follow via uniform convergence.

□

5.2.2 Notes

1. If $u \in Ha(A(R_1, R_2))$ then $z = re^{i\theta}$ and we have

$$u(z) = A\log r + B + \sum_1^\infty (z_n \cos n\theta + b_n \sin n\theta) r^n + \sum_1^\infty (c_n \cos n\theta + d_n \sin n\theta) r^{-n}$$

2. If $u \in Ha(|x| < 1)$ in $\mathbb{R}^3$ then we can write

$$u(r, \theta, \varphi) = \sum_0^\infty r^n \left(\sum_{k=-n}^\infty A_{n,k} Y_{n,k}(\theta, \varphi) \right)$$

5.3 The Poisson Kernel

5.3.1 First a Calculation

$$\begin{aligned}
\operatorname{Re}\left(\frac{1+z}{1-z}\right) &= \operatorname{Re}\left(\frac{(1+z)(1-\overline{z})}{|1-z|^2}\right) \\
&= \operatorname{Re}\left(\frac{1+z-\overline{z}+\overline{z}z}{|1-z|^2}\right) \\
&= \frac{1-r^2}{|1-z|^2} \\
&= \frac{1-r^2}{|1-re^{i\theta}|^2} \\
&= \frac{1-r^2}{1+r^2-2r\cos\theta}
\end{aligned}$$

Definition 77. *Let $P(r, \theta) = \operatorname{Re}\left(\frac{1+z}{1-z}\right)$ this is called the* ***Poisson kernel*** *for the unit disc $\mathbb{D}$.*

Theorem 76 (Theorem 6)**.** *(Poisson Representation) Suppose $u \in Ha(\mathbb{D}) \cap C(\overline{\mathbb{D}})$, then*

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{i\varphi}) P(r, \theta - \varphi) d\varphi, \qquad 0 < r < 1$$

Proof. Assume first that $u \in Ha(\mathbb{D}(0, R))$ for some $R > 1$. Then we have $u = \operatorname{Re} f$ with $f \in h(D(0, R))$. We can then apply Cauchy's Formula

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z} d\zeta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(e^{i\varphi})}{e^{i\varphi} - z} d\varphi \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\varphi}) \frac{e^{i\varphi}}{e^{i\varphi}} \frac{1}{1 - ze^{-i\varphi}} d\varphi
\end{aligned}$$

Notice that

$$\begin{aligned}
\frac{1}{1 - ze^{-i\varphi}} + \frac{\overline{z}e^{i\varphi}}{1 - \overline{z}e^{i\varphi}} &= \frac{1 - \overline{z}e^{i\varphi} + \overline{z}e^{i\varphi} - z\overline{z}}{|1 - ze^{\imath\varphi}|^2} \\
&= \frac{1 - r^2}{|1 - ze^{-i\varphi}|^2} \\
&= \frac{1 - r^2}{|1 - re^{-i\varphi}|^2} \\
&= \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta 0 \varphi)} \\
&= P(r, \theta - \varphi)
\end{aligned}$$

We then have

$$f(z) = f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\varphi})\frac{1}{1 - ze^{i\varphi}}d\varphi$$

Next note that

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\varphi})\frac{\overline{z}e^{i\varphi}}{1 - \overline{z}e^{i\varphi}}d\varphi &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{i\zeta}f(\zeta)\frac{\overline{z}\zeta}{1 - \overline{z}\zeta}d\zeta \\
&= \frac{1}{2\pi i}\int_{|\zeta|=1} f(\zeta)\frac{\overline{z}}{1 - \overline{z}\zeta}d\zeta \\
&= 0
\end{aligned}$$

Because $\overline{z}$ is fixed and the expression is holomorphic in $\overline{\mathbb{D}}$ because the pole is at $\frac{1}{z}$. So

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\varphi})P(r, \theta - \varphi)d\varphi$$

As P is rea-valued we may take real parts to get

$$u(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\varphi})P(r, \theta - \varphi)d\varphi, \qquad z = re^{i\theta} \in \mathbb{D}$$

Now to remove the assumption that $u \in Ha(\mathbb{D}(0, R))$ for some $R > 1$. Let $u_R = u\left(\frac{z}{R}\right)$. Then we have $u_R \in Ha(\mathbb{D}(0, R))$ so

$$u\left(\frac{1}{R}z\right) = u_R(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\frac{1}{R}e^{i\varphi})P(r, \theta - \varphi)d\varphi, \qquad z = re^{i\theta}$$

Since u is uniformly continuous on $\overline{D}$, $u\left(\frac{1}{R}e^{i\varphi}\right) \to u(e^{i\varphi})$ as $R \searrow 1$ uniformly, so we may pass to the limit. □

5.3.2 Notes

(a) For $u \in Ha(\mathbb{D}(0, R)) \cap C(\overline{\mathbb{D}}(0, R))$ we have a similar result for $0 < r < R$:

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(Re^{i\varphi})\frac{R^2 - r^2}{R^2 + r - 2rR\cos(\theta - \varphi)}d\varphi$$

(b) For $u \in Ha(\mathbb{D}(z_0, R)) \cap C(\overline{\mathbb{D}}(z_0, R))$ we also have a similar result for $u(z_0 + re^{i\theta})$ left to reader.

5.4 Dirichlet Problem Overview

Start with a domain D. Knowing the temperature on the boundary of D, what is the temperature inside D? In other words given a temperature function $f \in C(\partial D)$ we want a $u \in Ha(D) \cap C(\overline{D})$ so that $u = f$ on ∂D.

Lemma 13 (Modulus of Continuity Lemma)**.** *Suppose $g, h \in C(\mathbb{R})$ and are 2π-periodic. Define for $\delta > 0$*

$$\omega(\delta, g) = \sup\{|g(\theta + t) - g(\theta)| : \theta \in \mathbb{R}, |t| \leq \delta\}$$

If g is uniformly continuous then $w \to 0$ as $\delta \to 0$. Also, if we let $g_1(\theta) = g * h(\theta)$, then

$$\omega(\delta, g_1) \leq \omega(\delta, g) \cdot \int_{-\pi}^{\pi} |h(\varphi) d\varphi$$

Proof. We have

$$\begin{aligned} g_1(\theta + t) - g_1(\theta) &= \int_{-\pi}^{\pi} [(g(\theta + t) - \varphi) - g(\theta - \varphi)] h(\varphi) d\varphi \\ &\leq \int_{-\pi}^{\pi} |[(g(\theta + t) - \varphi) - g(\theta - \varphi)]| \, |h(\varphi)| d\varphi \\ &\leq \omega(\delta, g) \int_{-\pi}^{\pi} |h(\varphi)| d\varphi \end{aligned}$$

□

Theorem 77 (Theorem 7)**.** *Let $f \in C(\partial \mathbb{D}, \mathbb{R})$ by $u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\varphi}) P(r, \theta - \varphi) d\varphi$ and $z = re^{i\theta}$ with $0 < r < 1$ and $-\pi \leq \theta \leq \pi$ and $u(z) = f(z)$ if $|z| = 1$. Then*

(a) $u \in Ha(\mathbb{D})$

(b) $u \in C(\overline{\mathbb{D}})$

and u is called the ***Poisson integral*** *of f.*

Proof. First we prove (a). Since f is real valued we have

$$u(z) = \text{Re}\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\varphi}) \frac{1 + ze^{-i\varphi}}{1 - ze^{-i\varphi}} d\varphi \right)$$

Let $G(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\varphi}) \frac{1+ze^{-i\varphi}}{1-ze^{-i\varphi}} d\varphi$. Also, let T be a triangle in $\mathbb{D}$. We seek to apply Morera's Theorem, first note that

$$\int_T G(z) dz = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\varphi}) \left(\int_T \frac{1 + ze^{-i\varphi}}{1 - ze^{-i\varphi}} dz \right) d\varphi$$

this is allowed because the interand is in $C([-\pi, \pi] \times T)$. For $\varphi \in \mathbb{R}$ we have

$$\frac{1 + ze^{-i\varphi}}{1 - ze^{-i\varphi}} \in H(\mathbb{D})$$

Cauchy's implies

$$\int_T G(z) = 0$$

Also if $z_n \to z_0 \in \mathbb{D}$ then we have

$$\lim_{n\to\infty} \frac{1+z_n e^{-i\varphi}}{1-z_n e^{-i\varphi}} = \frac{1+z_0 e^{-i\varphi}}{1-z_0 e^{-i\varphi}}$$

So $G(z_n) \to G(z_0)$ and thus $G \in C(\mathbb{D})$. Morera's Theorem tells us then that $G \in H(\mathbb{D})$ and thus $\text{Re}(G) = u \in Ha(\mathbb{D})$.

Next up we prove (b). First some facts before we begin.

- Suppose $g \in L^1(0,2\pi)$ abd $g(\theta+2\pi) = g(\theta)$ for all $\theta \in \mathbb{R}$, then

$$\int_0^{2\pi} g(\theta)d\theta = \int_a^{2\pi+a} g(\theta)d\theta \qquad (1)$$

- Suppose $g, h \in L^1(0,2\pi)$ and are 2π periodic then **convolution** is commutative

$$\int_{-\pi}^{\pi} g(\varphi)h(\theta-\varphi)d\varphi =: g * h(\theta) = h * g(\theta) \qquad (2)$$

Also, we list some **Properties of the Poisson Kernel**.

(3) $P(r,\theta) > 0$

(4) $\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta)d\theta = 1$ for all $r < 1$. (To see this use $u = 1$ and $\delta u = 0$.)

(5) For $\delta > 0$ we have

$$\lim_{r\to 1}\left\{\sup_{\delta\le|\theta|\le\pi} P(r\theta)\right\}$$

Observe that (5) is true since

$$P(r,\theta) = \frac{1-r^2}{|1-re^{i\theta}|^2}$$

and so we have $|1-re^{i\theta}| \ge c\delta$ for some c and for $|\theta| \in [\delta,\pi]$. Thus,

$$P(r,\theta) \le \frac{1-r^2}{(c\delta0^2} \to 0, \qquad r \to 0$$

It remains to prove that $u \in C(\overline{\mathbb{D}})$. Suppose u is continuous at $z = 1$, then for $e^{i\varphi_0} \in \partial\mathbb{D}$ set

$$f_1(e^{i\varphi}) = f(e^{i(\varphi+\varphi_0)})$$

then the theorem for f_1 takes care of

$$\lim_{z\to\varphi+\varphi_0} u(z)$$

Let $y \in (0,1)$, we have

$$|u(1)-u(r)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(1)-f(e^{i\varphi})]P(r,\varphi)d\varphi\right|$$

Let $\epsilon > 0$, choose δ so that

$$|\varphi| \le \delta \implies |f(1)-f(e^{i\varphi})| < \epsilon$$

We then have using the triangle inequality

$$\begin{aligned}|u(1)-u(r)| &\le \left|\frac{1}{2\pi}\int_{-\delta}^{\delta}(f(1)-f(e^{i\varphi}))P(r,\varphi)d\varphi\right| + \left|\int_{|\varphi|>\delta}(f(1)-f(e^{i\varphi}))P(r,\varphi)d\varphi\right| \\ &\le \epsilon^{-\delta} + M\int_{|\varphi|>\delta}P(r,\varphi)d\varphi\end{aligned}$$

By (5) as $r \to 0$ we know that $P(r,\varphi) \to 0$ uniformly for $|\varphi| > \delta$. Hence

$$\lim_{r\to 1} u(r) = 1$$

Hence u is radially continuous at $z = 1$. We still want continuity. Let $\{z_n\}$ be any sequence in $\mathbb{D}$ with $z_n \to 1$. Let $h_r(\varphi) = P(r,\varphi)$ Using the modulus continuity lemma with $z_n = r_n e^{i\theta_n}$. We have

$$\begin{aligned}|u(z_n)-u(1)| &\le |u(z_n)-u(r_n)| + |u(r_n)-u(1)| \\ &= |f * h_{r_n}(\theta_n) - f * h_{r_n}(0)| \\ &< \omega(\theta_n, f) + |u(r_n)-u(1)|\end{aligned}$$

We know that $\omega(\theta_n, f) \to 0$ since $f \in C(\partial\mathbb{D})$. The other term goes to 0 by above. Hence $\lim_{n\to\infty} u(z_n) = u(1)$. □

5.4.1 Remarks about Dirichlet Problem

1. With $\mathbb{D}(z_0, R)$ we can solve the corresponding problem

$$u(z_0 + re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z_0 + Re^{i\theta})\frac{1-\left(\frac{r}{R}\right)^2}{|1-\frac{r}{R}e^{i(\theta-\varphi)}|^2}d\varphi$$

2. $D = \mathbb{H}$ then

$$u(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)P(x-t,y)dt$$

and

$$P(x,y) = \frac{y}{x^2+y^2} = -\operatorname{Im}\left(\frac{1}{z}\right)$$

3. For bounded domains Ω with $\partial\Omega \in C^1$ we have

$$u(z) = \frac{1}{2\pi}\int_{\partial\Omega} f(\zeta)P_\Omega(z,\zeta)|d\zeta|$$

$$P_\Omega(z,\zeta) = -\frac{\partial g}{\partial n}(z,\zeta)$$

Where g if Green's Function

$$\Delta_\zeta(g(z,\zeta)) = -2\pi\delta_z$$

and is zero on ∂R.

4. For general D we solve with the 'Perron Process', which we will learn about in the spring.

5. For some D and f the Dirichlet problem cannot be solved. Consider, for example, $D = \mathbb{D}\backslash\{0\}$ and
$$f(z) = \begin{cases} 0, & z \in \partial\mathbb{D} \\ 1, & z = 0 \end{cases}$$

6. For simply connected Jordan domains we can solve via the Riemann Mapping Theorem and the Caratheodory Extension Theorem.

7. Solve via probability using Brownian motion.
$$u(z) = \mathbb{E}^z(f(B_z))$$
One can show that $u(z)$ is harmonic on u on ∂D is f.

8. Consider the p-Laplacian
$$\text{div}[|\nabla u|^{p-2}\nabla u] = 0$$
Solutions to this are called p-harmonic. Also defined for $p = \infty$. There is a paper called 'Tug-of-War and $\infty - \Delta$'.

5.5 The Converse to the Mean Value Property

Theorem 78 (Theorem 8)**.** *Let $u \in C(D)$ and assume that for each $z_0 \in D$ there is an $r_0 > 0$ such that $\mathbb{D}(z_0, r) \subseteq D$ and*
$$u(z_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(z_0 + re^{i\theta})d\theta, \forall r \in [0, r_0] \qquad (1)$$
then $u \in Ha(D)$.

Proof. Let Δ be a disk with $\overline{\Delta} \subseteq D$. It suffices to show $u \in Ha(\Delta)$. Let u_1 be the solution of the Dirichlet Problem in Δ with $u_1 = u$ on $\partial\Delta$. Then for all $z_0 \in \Delta$ (1) holds for u_1. We wish to show that $u = u_1$. Let $w = u - u_1$. Let
$$M = \sup_{\partial\Delta} w$$
Suppose that $M > 0$ and let
$$E = \{z \in \partial\Delta : w(z) = M\}$$
We have $w \in C(\overline{\Delta})$ and $w = 0$ on $\partial\Delta$. So E is a non-empty compact subset of Δ. Take $z_0 \in E$ with
$$d(z_0, \partial\Delta) = \inf_{z\in E} d(z, \partial\Delta)$$
Let $r > 0$ be small, then by simple geometry, more than half of the circle $|z - z_0| = r$ lies outside of E. We have $w \leq M$ on Δ. Since $w \in C(\overline{\Delta})$ we know that $w < M$ on $\Delta \setminus E$. So,
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} w(z_0 + re^{i\theta})d\theta < M$$
This implies that $w(z_0) = M > \frac{1}{2\pi}\int_{-\pi}^{\pi} w(z_0 + re^{i\theta})d\theta$. This is a contradiction, as u and u_1 satisfy the mean value property. Hence $w \leq 0$. Similarly, by considering $-w$, we can show that $w \geq 0$. Hence $w = 0$ and $u = u_1$ on Δ and $u \in Ha(\Delta)$. □

5.6 Limits of Harmonic Functions

Theorem 79 (Theorem 9)**.** *Let $\{u_n\}$ be a sequence of functions in $Ha(D)$. If $u_n \to u$ locally and uniformly then $u \in Ha(D)$.*

Proof. As u is a local uniform limit is also obeys the mean value property. An so by the converse of the mean value property the result holds. □

5.7 The Harnack Theorems

Theorem 80 (Theorem 10)**.** *(Harnack's Inequality)*

Suppose $D \subseteq \mathbb{C}$ is a domain and $K \subseteq D$ is compact. Then there exists a constant $\alpha = \alpha(K, D)$ such that for all $u \in Ha(D)$ with $u > 0$.

$$\sup_K u \leq \alpha \inf_K u \qquad (1)$$

Proof. Future homework problem. □

Theorem 81 (Theorem 11)**.** *(Harnack's Theorem) Let $\{u_n\}$ be a sequence in $Ha(D)$ and suppose that $0 \leq u_1 \leq u_2 \leq \cdots$ then either u_n converges locally uniformly in D or $\lim_{n\to\infty} u_n(z) = \infty$.*

Proof. Future homework problem. □

Corollary 20. *If $\{u_n\}$ is a sequence of non-negative harmonic functions in a domain D then $\sum_1^\infty u_n$ is either in $Ha(D)$ or is identically equal to ∞.*

5.8 The Maximum Principle

Theorem 82 (Theorem 12)**.** *Suppose $u \in Ha(D)$. If u has a local max or min at some point in a domain D then u is constant.*

Proof. Also on the homework. □

Theorem 83 (Theorem 13)**.** *Suppose D is bounded and $u \in C(\overline{D}) \cap Ha(D)$. Then*

$$\max_{\overline{D}} u = \max_{\partial D} u$$

$$\min_{\overline{D}} u = \min_{\partial D} u$$

And if u is nonconstant then we get

$$\min_{\overline{D}} u < u(z) < \max_{\overline{D}} u, \qquad z \in \mathbb{D}$$

Proof. Immediately follows from Theorem 12. □

Corollary 21 (Comparison Principle)**.** *If $u_1, u_2 \in D$ satisfy the hypothesis of Theorem 13 and $u_1(z_0) \leq u_2(z_0)$ for $z_0 \in \partial D$, then $u_1 \leq u_2$ on D.*

Proof. Examine $u_1 - u - 2$ which is less than or equal to 0 on ∂D. □

Corollary 22 (Uniqueness of Solutions to Dirichlet Problem with Continuous Boundary Data)**.** *If $u_1, u_2 \in Ha(D)$ as in Corollary 1 and $u_1 = u_2$ on ∂D then $u_1 \equiv u_2$ in D.*

5.9 Schwarz Reflection Principle

We start with some setup...

5.9.1 The Setup

The setup is as follows. Let

$$\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

define D to be a domain in $\mathbb{H}$ and let I be a nonempty open set in $\mathbb{R}$ which has the property that every $t \in I$ is the center of a disc Δ_t such that $\Delta_t \cap \mathbb{H} \subseteq D$. We let

$$D^* = \{z : \overline{z} \in D\}$$

$$\hat{D} = D \cup I \cup D^*$$

Theorem 84 (Theorem 14)**.** *Suppose $u \in Ha(D)$ is a harmonic function and $\lim_{n\to\infty} u(z_n) = 0$ whenever $\{z_n\} \subseteq D$ converges to a point in I. Extend u to $\hat{D}$ by setting*

$$u(z) = \begin{cases} u(z), & z \in D \\ 0, & z \in I \\ -u(\overline{z}), & z \in D^* \end{cases}$$

Then $u \in Ha(\hat{D})$.

Proof. If $z \in D^*$ then the Laplacian satisfies

$$\Delta u(z) = i(\Delta u)(\overline{z}) = 0$$

since $u \in Ha(D)$. Thus, $u \in Ha(D^* \cup D)$ and satisfies the mean value property at all discs centered at points of $D \cup D^*$ and contained in $D \cup D^*$. Clearly, $u \in C(\hat{D})$. It remains to check the mean value property on small disks centered at I. This is trivial since the values of u on each side of I are negatives of each other so the integral over any circle centered at I is zero.

□

Theorem 85 (Theorem 15)**.** *(Schwarz Reflection Principle) Suppose $f \in H(D)$ with D a domain as defined in the set up. Also, assume that*

$$\lim_{n\to\infty} (\operatorname{Im} f)(z_n) = 0$$

whenever $\{z_n\} \subseteq D$ converges to a point in I. Then there exists $F \in H(\hat{D})$ such that $F = f$ on D. Moreover, for $z \in \hat{D}$ we have

$$F(\overline{z}) = \overline{F(z)} \qquad (1)$$

Before we prove note that

(i) we made no assumption on the boundary behavior of $\operatorname{Re} f$.

(ii) If z is real and in $\hat{D}$ then $F(z)$ is real.

(iii) F is the unique holomorphic extension to $\hat{D}$.

Proof. Assume first that $\hat{D}$ is simply connected. This means that I is an open interval. Let $v = \operatorname{Im} f$. Setting $v = 0$ on I and $v(z) = -v(\overline{z})$ for $z \in D^*$ we get $v \in Ha(\hat{D})$ from Theorem 14. Take $F \in H(\hat{D})$ such that $\operatorname{Im} F = v$ and $F(z_0) = f(z_0)$ for some $z_0 \in D$. We can do this because $\hat{D}$ is simply connected an so there exist harmonic conjugates. We have $\operatorname{Im}(F - f) = 0$ in D. By the open mapping theorem it holds that $F - f$ is constant in D. So, $F = f$ in D. By a future homework problem we have $F(\overline{z}) = \overline{F(z)}$ for $z \in \hat{D}$.

Now we prove the result for a general $\hat{D}$. For each $t \in I$ let Δ_t be an open disc centered at t with $\Delta_t \subseteq \hat{D}$. Let F_t be the extension of f from $\Delta_t \cap H$ to Δ_t. Suppose that $\Delta_{t_1} \cap \Delta_{t_2} \neq \emptyset$. Then it holds that

$$F_{t_1} = F_{t_2} = f, \qquad \text{in } \Delta_{t_1} \cap \Delta_{t_2} \cap \mathbb{H}$$

and this set is non-empty and open in $\mathbb{H}$. So $F_{t_1} = F_{t_2}$ in $\Delta_{t_1} \cap \Delta_{t-2}$ by uniqueness. So we define the function $F : \hat{D} \to \mathbb{C}$ by

$$F(z) = \begin{cases} f(z), & z \in D \\ F_t(z), & z \in I \cap \Delta_t \\ F(z) = \overline{F(\overline{z})}, & z \in D^* \end{cases}$$

Then F is well defined because the Δ_t taken does not matter. By Homework 12 Problem 1 we know that $F \in H(D^*)$. Clearly, $F \in H(D)$. If $z \in D^*$ then

$$F(z) = \overline{f(\overline{z})} = \overline{F(\overline{z})}$$

Also, if $z \in D$ then $\overline{z} \in D^*$. So $F(\overline{z}) = \overline{F(z)}$. Hence

$$F(z) = \overline{F(\overline{z})}, \qquad z \in D \cup D^*$$

So $F \in H(D \cup D^*)$ and obeys (1) in $D \cup D^*$. Take $z_0 \in I$, $t \in I$ with $z_0 \in \Delta_t$. Then in $\Delta_t \cap D$ we have $F = f$ and $F_t = f$. So

$$F = F_t, \qquad \text{in } \Delta_t \cap D$$

So in $\Delta_t \cap D^*$ it holds that

$$F(z) = \overline{f(\overline{z})} = \overline{F(\overline{z})} = \overline{F_t(\overline{z})}$$

BY applyin the first part of the proof to F_t we have $F_t(z) = \overline{F_t(\overline{z})}$. Thus $F = t_t$ in $D^* \cap \Delta_t$. We now know that $F = T_t$ in δ_t. So as $F_t \in H(\Delta_t)$ then $F \in H(\Delta_t)$. So $F \in H(\hat{D})$ and (1) holds on $\hat{D}$. ☐

5.9.2 Other Reflection Situations

1. If $f : D \to \mathbb{C}$ such that $\lim_{z \in \partial D} f(z)$ is always on a line L we extend by mapping L to $\mathbb{R}$ with a conformal map, extending via the reflection principle then going back.
2. If $f : \mathbb{D} \to \mathbb{H}$ and as $\{z_n\} \to \partial D$ we have $\{f(z_d)\} \to \partial\mathbb{H}$. Let $z^* = \frac{1}{\overline{z}}$ for $z \in \mathbb{D}$. Then let $f(z^*) = \overline{f(z)}$. *This is the original case that Schwarz considered*
3. Suppose that $D \subseteq \mathbb{D}(0, R_1)$ and $f : \mathbb{D}(0, R_2)$ with $R_2 \geq R_1$. We let $z^{*_1} = \frac{R_1^2}{\overline{z}}$ and $z^{*_2} = \frac{R_2^2}{\overline{z}}$. We have $f(z^{*-1}) = f(z^{*_2})$.
4. The property still holds for analytic curves as the boundary.

6 Normal Families and the Riemann Mapping Theorem

6.1 Equicontinuity

Theorem 86 (Bolzano-Weierstrauss Theorem)**.** *In $\mathbb{R}^n$ every bounded set is relatively compact.*

Definition 78. *Let (K, ρ) be a compact metric space. Let $C(K)$ be all $\mathbb{C}$-valued continuous functions on K. The **distance between** f **and** g is given by*

$$d(f, g) := \sup_{x \in K} |f(x) - g(x)|$$

Definition 79. *We say that a family of functions $\mathcal{F} \subseteq C(K)$ is **bounded** if there exists M such that*

$$\sup_K |f| < M, \qquad \forall f \in \mathcal{F}$$

*We often say that $\mathcal{F}$ is **uniformly bounded**.*

Definition 80. *A family $\mathcal{F} \subseteq C(K)$ is **equicontinuous** if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $f \in \mathcal{F}$ it holds that*

$$\rho(x, y) < \delta \implies |f(x) - f(y)| < \epsilon$$

Theorem 87 (Theorem 1)**.** *(Arelia-Asioli) Let K be a compact metric space and $\mathcal{F}$ a uniformly bounded equicontinuous family of functions in $C(K)$. Then each sequence in $\mathcal{F}$ has a subsequence which converges uniformly.*

Proof. Let $\{x_i\} \in K$ be a countable dense subset. Compact implies separable. Now run a diagonalization argument.

□

6.1.1 Remarks

- The limit is in $C(K)$ but not necessarily in $\mathcal{F}$.
- The converse is also true. If $\mathcal{F}$ is not uniformly bounded or equicontinuous then there exists a sequence in $\mathcal{F}$ which has no convergent subsequences.

Definition 81. *Let $D \subseteq \mathbb{C}$ be open and $\mathcal{F} \subseteq C(D)$. We say that $\mathcal{F}$ is **locally uniformly bounded** if $\mathcal{F}$ is uniformly bounded on each compact subset of D. We say that $\mathcal{F}$ is **locally equicontinuous** if $\mathcal{F}$ is equicontinuous on each compact subset of D.*

Theorem 88 (Theorem 2)**.** *Let D be open in $\mathbb{C}$ with $\mathcal{F} \subseteq C(D)$. If $\mathcal{F}$ is locally uniformly bounded and locally equicontinuous on D then each sequence in $\mathcal{F}$ contains a locally uniformly convergent subsequence..*

Proof. Exhaust D by increasing compact sets K_m. Diagonalize after applying Theorem 1.

□

6.2 Normal Familes

Definition 82. *Let $D \subseteq \mathbb{C}$ be open. A **normal family** is a set $\mathcal{F} \subseteq H(D)$ such that each sequence has a locally uniformly convergent subsequence.*

Theorem 89 (Theorem 3)**.** *(Montel's Theorem I) Let $\mathcal{F} \subseteq H(D)$ with $D \subseteq \mathbb{C}$ open. Suppose also that $\mathcal{F}$ is locally uniformly bounded. Then $\mathcal{F}$ is normal.*

Proof. Let $\overline{\Delta} = \overline{\mathbb{D}(z_0, r)}$ with $\overline{\Delta} \subseteq D$. Also, let $\overline{\Delta}_1 = \overline{\mathbb{D}(z_0, r_1)}$ with $r_1 > r_0$ and $\overline{\Delta}_1 \subseteq D$. Let $z_1, z_2 \in \Delta$. By Cauchy's formula we know that

$$f(z_1) - f(z_2) = \frac{1}{2\pi i} \int_{\partial\Delta_1} f(\zeta) \left(\frac{1}{\zeta - z_1} - \frac{1}{\zeta - z_2} \right) d\zeta$$

Letting M be the uniform bound on $\partial\Delta_1$ for the family we have

$$|f(z_1) - f(z_2)| \leq \frac{1}{2\pi}(2\pi r_1)M(r_1 - r_0)^{-2}|z_1 - z_2|$$

Given $\epsilon > 0$ let $\delta = \frac{\epsilon(r_1 - r_0)^2}{r_1 M}$. If $|z_1 - z_2| < \delta$ the $|f(z_1) - f(z_2)| < \epsilon$. Thus $\mathcal{F}$ is equicontinuous on $\overline{\Delta}$. via a compactness argument $\mathcal{F}$ is locally equicontinuous. Theorem 2 then implies that $\mathcal{F}$ is normal.

□

6.2.1 Example

1. $\mathcal{F} = \{$ all holomorphic f in a domain D with $|f| < 1\}$ is norma since uniformly bounded.
2. $\mathcal{F} = \{f \in H(D) : \text{Re}(f) > 0, f(z_0) = w_0\}$ with D simply connected is normal via a Schwartz lemma type argument.
3. $\mathcal{F} = \{f \in H(D) : f(z_0) = w_0, \exists a, b$ such that $a, b \notin f(D)\}$ is normal.
4. $\mathcal{F} = \{f \in M(D) : \exists a, b, c \in \mathbb{C}^*$ such that $a, b, c \notin f(D)\}$ is normal.
5. $\mathcal{F} = \{f \in H(D) : \exists a \notin f(D), f(z_0) = w_0\}$ is not normal. Consider $\mathcal{F} = \{f_n(z) = e^{nz} : n \in \mathbb{N}\}$ we have $f(0) = 1$ for all n and f omits 0, but this the f_n cleary do not have a convergent subsequence.

6.3 Riemann Mapping Theorem

Theorem 90 (Theorem 4)**.** *(Riemann Mapping Theorem) Let D be a simply connected domain with $D \neq \mathbb{C}$, $z_0 \in D$. Then there exists a unique $F \in H(D)$ such that*

1. *F maps D 1-1 onto $\mathbb{D}$*
2. *$F(z_0) = 0$*
3. *$F'(z_0) > 0$*

6.3.1 Notes

1. Suppose $D = \mathbb{C}$. Then Liouville is violated.
2. True for simply connected domains in $\mathbb{C}^*$ provided that $\mathbb{C}^* \setminus D$ has at least 2 points.
3. For $z_0 = \infty$ then (c) become $F(z) = \frac{c_1}{z} + O(z^{-2})$ when $c_1 > 0$.
4. Theorem 4 implies that any two simply connected domains in $\mathbb{C}^*$ with at least two complimentary points are conformally equivalent.

Proof. We first prove uniqueness. Assume that F_1, F_2 exist. Let $G = F_2 \circ F_1^{-1} : \mathbb{D} \to \mathbb{D}$. Our hypotheses guarantee that G is 1-1 and onto and so $G(z) = e^{i\alpha}\frac{z-a}{1-\bar{a}z}$ for some $a \in \mathbb{D}$ and $\alpha \in \mathbb{R}$. But $G(0) = 0$ so $a = 0$ and thus $G(z) = e^{i\alpha}z$. But $G'(z) = e^{i\alpha}$ and $G'(z)$ is real. So, $\alpha = 2\pi z$ and thus $G(z) = z$ and $F_1 = F_2$.

Let $\mathcal{F} = \{f \in H(D) : f \text{ is } 1-1 \text{ in } D, f(D) \subseteq \mathbb{D}\}$. Let $m = \sup_{f\in\mathcal{F}} |f'(z_0)|$. We will prove

(i) the family $\mathcal{F}$ is nonempty

(ii) m is actually achieved by some $f \in \mathcal{F}$

(iii) this achiever is the conformal map we are looking for

Proof of (i): Let $a \in \mathbb{C} \setminus D$. Then the map $z \mapsto z - a$ belongs to $H(D)$ and is zero free. Since D is simply connected there exists a $\varphi \in H(D)$ such that $\varphi^2(z) = z - a$. We will show that φ is 1-1. Suppose that $\varphi(z_1) = \varphi(z_2)$ with $z_1, z_2 \in D$, then

$$z_1 - a = \varphi(z_1)^2 = \varphi(z_2)^2 = z_2 - a$$

and so $z_1 = z_2$. By the same argument $\varphi(z_1) = \varphi(z_2)$ only if $z_1 = z_2$. If $w \in \varphi(D)$ then $-w \notin \varphi(D)$. Let $w_0 \in \varphi(D)$. The open mapping theorem implies that there exists $r > 0$ so that $\overline{\mathbb{D}}(w_0, r) \subseteq \varphi(D)$. Thus $\mathbb{D}(-w_0, r) \cap \varphi(D) = \emptyset$. So $|\varphi(z) + w_0| > r$ for all $z \in D$. Let $f(z) = \frac{r}{\varphi(z)+w_0}$, then $f \in H(D)$ and is 1-1 on D and $f(D) \subseteq \mathbb{D}$). So $f \in \mathcal{F}$ and $\mathcal{F} \neq \emptyset$.

Proof of (ii): By the definition of sup there exists a sequence $\{f_n\}_1^\infty \subseteq \mathcal{F}$ such that $\lim_{n\to\infty} |f'(z_0)| = m$. The family $\mathcal{F}$ is uniformly bounded. So $\{f_n\}$ has a convergent subsequence $\{f_{n_k}\}$. Say f_{n_k} converges to g locally uniformly. Since the functions f_{n_k} are holomorphic we know that $g \in H(D)$. Also, Hurwitz's theorem implies that g is either 1-1 or constant. But since each $|f'(z_0)| > 0$ it follows $|g'(z_0)| > 0$ and so m is positive and g is not constant and thus is 1-1 on D. As $f_{n_k}(D) \subseteq (\mathbb{D})$ we know from continuity that $g(D) \subseteq \overline{\mathbb{D}}$. However, by the open mapping theorem we have $g(D)$ is open and thus $g(D) \subseteq \mathbb{D}$ which implies $g \in \mathcal{F}$. As $g'(z_0) = me^{i\alpha}$ set

$$F(z) = e^{-i\alpha}g(z)$$

We will show that F is the desired function. First let's brainstorm what could go wrong

- $F(D) \subsetneq \mathbb{D}$
- $F(z_0) \neq 0$

To deal with this we need a timely lemma.

Lemma 14. *Let D' be a simply connected subdomain of $\mathbb{D}$ and $w_0 \in D'$. If $D' \neq \mathbb{D}$ or if $w_0 \neq 0$ then there is a $\varphi \in H(D')$ such that φ is 1-1 in D' and $|\varphi'(w_0)| > 1$.*

Proof. (of Lemma 2) For any $a \in \mathbb{D}$ let

$$T_a(z) = \frac{z-a}{1-\overline{a}z}$$

then

$$|T_a'(z)| = \frac{1-|a|^2}{|1-\overline{a}z|^2} = \frac{1-|T_a(z)|}{1-|z|^2}$$

First suppose that $w_0 = 0$, then $D' \subsetneq \mathbb{D}$. Let $a \notin D' \setminus \{0\}$. Then a is not zero so T_a has no zero in D. Since D' is simply connected there exists $g \in H(D')$ so that

$$g^2(z) = T_a(z)$$

Also $g : D' \to \mathbb{D}$ is 1-1 as T_a is 1-1. Let $b = g(0)$. Then

$$b^2 = g^2(0) = T_a(0) = -a$$

whence $|b|^2 = a$. Let $\varphi = T_b \circ g$. We have

$$D' \xrightarrow{g} \mathbb{D} \xrightarrow{T_b} \mathbb{D}$$

Then $\varphi \in H(D')$ and φ maps D' 1-1 into $\mathbb{D}$. we must show that $|\varphi['(0)| > 1$ with $2g \cdot g' = T_a'$. This means that

$$|\varphi'(0)| = \frac{1}{1-|b|^2} \cdot \frac{1-|a|^2}{2|b|} \overset{*}{=} \frac{1+|b|^2}{2|b|}$$

Where at $*$ we use $|b|^2 = |a|$. So, if $|b| < 1$ then the above is greater than 1 because $1 + x^2 - 2x = (x-1)^2 > 0$. Now suppose that $w_0 \neq 0$, let $\varphi(z) = T_{w_0}(z)$ then

$$|\varphi'(w_0)| = \frac{1}{1-|w_0|^2} > 1$$

So the lemma is proved. □

Proof of (iii): Assume $w_0 = F(z_0) \neq 0$ or $D' = F(D) \neq \mathbb{D}$. Then take φ according to the lemma. We have $\varphi \circ F \in \mathcal{F}$. Additionally $||\varphi \circ F)'(z_)| = m|\varphi'(z_0)| > m$ which contradicts maximality. So we are done.

□

6.3.2 Remarks

- Is $\mathbb{C}^n$ $\mathbb{D} \times \mathbb{D}$ -biholomorphically equivalent to $B^2 \left\{(z_1, z_2) : |z_1^2 + |z_2|^2 < 1\right\}$?

6.4 Boundary Behavior of Conformal Maps

Theorem 91 (Theorem 5)**.** *(Caratheodory, Osgood, Taylor, Extension Theorem -1913) Suppose D is a simply connected domain in $\mathbb{C}^*$ whose boundary is a Jordan curve in $\mathbb{C}^*$ homeomorphic to S^1. Then every conformal map from D into $\mathbb{D}$ can be extended to a homeomorphism of $\overline{D}$ into $\overline{\mathbb{D}}$.*

6.4.1 Remarks

- This is false for $\mathbb{C}^n$.
- However, it is conjectured to be true if $F : D \to D'$ and $\overline{D}$ and $\overline{D}'$ are topological balls with smooth boundary.
- In $\mathbb{R}^n$ the analogue is true for quasi-conformal maps if $n = 2$ but false for $n = 3$. Proven by (kuusalu, 1985). For a prof see J. Garnett and D. Marshall's book Harmonic Measure Ch. 1.

Now let D be a bounded and simply connected domain with ∂D a Jordan curve.

Definition 83. *The Jordan curve ∂D is given the positive orientation. When for $z_0 \in D$ we choose $\gamma : S^1 \to \partial D$ so that γ is homotopic to $z_0 + re^{i\theta}$ in $\mathbb{C} \setminus \{z_0\}$.*

Definition 84. *Distinct points $z_1, z_2, z_3 \in \partial D$ are said to be in* ***positive cyclic order*** *if when γ is as above we have*

$$\gamma(e^{i\theta_j + \theta_0}) = z_j, \qquad j = 1, 2, 3$$

for some $\theta_0 \in [0, 2\pi)$ and $\theta_1 < \theta_2 < \theta_3 < \theta_1 + 2\pi$.

6.4.2 Facts

1. Let $F : D \to D'$ be a conformal mapping of domains in $\mathbb{C}$, thn

$$J_f = \begin{vmatrix} (\operatorname{Re} f)_x & (\operatorname{Im} f)_x \\ (\operatorname{Re} f)_y & (\operatorname{Im} f)_y \end{vmatrix} = |F'|^2 > 0$$

   If D and D; are bounded, $J_f > 0$ implies that F preserves orientation of the boundaries. So F preserves cycle order.

2. If D, D' are Jordan domains in $\mathbb{C}^*$ with $\overline{D} \subseteq \mathbb{C}$ and $\infty \in D'$ and $F : D \to D'$ is conformal than cyclyic order is reversed (see $f(z) = \frac{1}{z}$). We can show this by writing $f = f_1 \circ f_2$ where $f_1 = \frac{1\zeta}{+}a$, $f_2 = \frac{1}{F-a}$ with $a \notin \overline{D}$.

Theorem 92 (Theorem 6)**.** *(3 on 3) Let D, D' be bounded simply connected Jordan domains and z_1, z_2, z_3 be distinct points on ∂D and w_1, w_2, w_3 be distinct points on $\partial D'$. Also assume that z_1, z_2, z_3 and w_1, w_2, w_3 have the same cyclic order. Then there exists a unique conformal map $f : D \to D'$ such that $f(z_j) = w_j$ for $j = 1, 2, 3$.*

Proof. We will work with the disk and define Möbius transformations that move the points as we want

$$\begin{array}{ccc} D & \xrightarrow{f} & D' \\ \uparrow & & \uparrow \\ S^1 & \longrightarrow & S^1 \end{array}$$

By the Riemann-Mapping Theorem and Theorem 5 we reduce the problem to the case of $D = D' = \mathbb{D}$. There is a Möbius map

$$T(z_j) = w_j, \qquad j = 1, 2, 3.$$

Since Möbius map circles to circles we know that T maps $\overline{D}'$ to either $\overline{D}$ or $\overline{\mathbb{D}}^c$. If the latter held that $\frac{1}{T} : \mathbb{D} \to \mathbb{D}$ conformally and would thus preserve cyclic order, but then T would have reversed the cyclic order. However, $T(z_j) = w_j$, a contradiction so $T : \mathbb{D} \to \mathbb{D}$.

For uniqueness if two such f_1, f_2 exist then $f_2^{-1} \circ f$ maps $\mathbb{D} \to \mathbb{D}$ fixing z_1, z_2, z_3 and since this fixes more than 2 points we have $f_1 = f_2$.

□

6.5 Schwarz-Christoffel Form

6.5.1 The Problem

We want to conformally map $\mathbb{H}$ onto the interior of an n-sided polygon.

We will denote the vertices of the polygon D by $w_1, \dots, w_n$ and we will write interior angle at each w_i as $\pi\alpha_j$ and the exterior angle at each w_i by $\pi\beta_j$. We know that $\beta_j = 1 - \alpha_j$ and that

$$\sum \beta_j = 2.$$

So if we walk around the boundary of D in the positive direction the β_j are the 'turn at each vertex. If we turn left the $\beta_j < 0$ and right $\beta_h > 0$. By the Riemann Mapping Theorem there is such a map $f : \mathbb{H} \to D$ onto. And by the Caratheodory Extension Theorem we know that f extends to a homeomorphism of $\overline{\mathbb{H}}$ onto $\overline{D}$. Take such an f and let $z_j = f^{-1}(w_j)$. Since we can always compose with a Möbius transformation, we assume that all of the z_j are finite.

Theorem 93 (Theorem 7)**.** *There exist complex constants $A, B \in \mathbb{C}$ such that for $z \in \mathbb{H}$ it holds that*

$$f(z) = A \int_{z_0}^{z} \prod_{j=1}^{n} (\zeta - z_j)^{-\beta_j} d\zeta + B$$

Where $(\zeta - z_j)^{-\beta_j}$ denote fixed branches of the powers which are holomorphic in $\mathbb{H}$. Also, the integration is along any path in $\mathbb{H}$.

Note: If $f(z)$ is as above then

$$f'(z) = A \prod_{1}^{n} (z - z_j)^{-\beta_j}$$

and

$$\log\left(f'(z)\right) = \log A - \sum_{1}^{n} \beta_j \log(z - z_j)$$

$$\frac{f''}{f'} = \sum_{1}^{n} \frac{\beta_j}{z - z_j}$$

Lemma 15. *Let f_1, f_2 be meromorphic and non-constant in a domain D. Then $f_2 = Af_1 + b$. in D for A, B if and only if*

$$\frac{f_2''}{f_1'} = \frac{f_1''}{f_1'} \qquad \text{in } D$$

.

Proof. ($\Rightarrow$) If $f_2 = Af_1 + B$ then we get the result immediately.

($\Leftarrow$) Take $z_0 \in D$ such that $f_1'(z_0) \neq 0$ or ∞ and $f_2'(z_0) \neq 0, \infty$. Let $\log f_1'$ and $\log f_2'$ denote holomorphic branches of the log in a small disc, Δ, around z_0. The in Δ we have

$$(\log f_1')' = \frac{f_1''}{f_1'} = \frac{f_2''}{f_1'} = (\log f_2')'.$$

Therefore in Δ we have

$$\log f_1' = \log f_2' + c$$

which means that in Δ it holds that $f_1' = f_2' e^c$. Integrating yields

$$f_1 = e^c f_2 + B, \qquad \text{in } \Delta$$

Then we use uniqueness. □

PROOF OF THEOREM 7

Proof. Let f be the map guaranteed by Riemann Mapping Theorem. Let

$$f_1 = \int_{z_0}^{z} \prod (z - z_j)^{-\beta_j} \, dz$$

Let $f_2 = f$. We may apply the Lemma if we can show that

$$g(z) := \frac{f_2''}{f_2'} = -\sum \frac{\beta_j}{(z - z_j)} \qquad (*)$$

To prove (*) we require two main steps

(1) g extends holomorphically to $\mathbb{C}^* \setminus \{z_1, z_2, \ldots, z_n\}$ with $g(\infty) = 0$.

(2) At each z_j, g has a pole of order 1 with residue $-\beta_j$.

Once we prove (1) and (2) it follows that

$$g(z) + \sum \frac{\beta_j}{z - z_j} \to 0, \qquad z \to \infty$$

Thus g is as requested by Liouville's Theorem.

Proof of (1): For each j we may extend f from $\mathbb{H}$ to $\mathbb{H} \cup \{(z_j, z_{j+1}\} \cup \mathbb{H}^*$ via Schwarz reflection in (w_j, w_{j+1}) (essentially we are reflecting the domain over each line (w_j, w_{j+1}) and extending by Schwarz). Call each extension f_j. The extended f_j are possibly only locally 1-1 on $\mathbb{H} \cup \{z_j, z_{j+1}\} \cup \mathbb{H}^*$ which maps to $\hat{D}_j = D \cup \{(w_j, w_{j+1})\} \cup D^*$. To deal with the point at ∞ we employ the convention

$$(z_n, z_1) = \mathbb{R} \cup \{\infty\} \setminus (z_1, z_n), \qquad z_1 < z_n$$

For each j there are constants A_j, B_j so that

$$f_j(\overline{z}) = f(z)^* := A_j(\overline{f(z)}) + B_j$$

Also, for each pair j, k there are constants A_{jk} and B_{jk} so that

$$f_j = A_{jk} f_k + Bjk, \qquad \text{in } \mathbb{H}^*$$

This means that the n different extensions differ only via an affine transformation. The lemma implies that

$$\frac{f_j''}{f_j'} = \frac{f_k''}{f_k'}, \qquad z \in \mathbb{H}^*, \forall j, k.$$

So, we may extend g to $\mathbb{C} \setminus \{z_1, \dots, z_n\}$ by setting g to be any $\frac{f_j''}{f_j'}$ in $\mathbb{H}^*$ and on the axis itself equal to the unique one on which $\frac{f_j''}{f_j'}$ is defined. It remains to prove that $g(\infty) = 0$.

Since each f_j is locally 1-1, f_j' has no zeros in $\mathbb{H} \cup \{z_j, z_{j+1}\} \cup \mathbb{H}^*$. Moreover, the last extension f_n has a removable singularity at ∞. Thus,

$$f_n(z) = f(\infty) +_1 z^{-1} + c_2 z^{-2} + O(z^{-3})$$

in a neighborhood of ∞. So we have the derivatives satisfy

$$f_n'(z) = -c_1 z^{-2} - 2c_2 z^{-3} + O(z^{-4})$$

$$f_n''(z) = 2c_1 z^{-3} + 6c_2 z^{-4} + O(z^{-5})$$

So,

$$\frac{f_n''}{f_n'} = \frac{-2}{z} + O(z^{-2}) = g$$

Hence $g(z) \to 0$ as $z \to \infty$. So let $g(\infty) = 0$. We now have

$$g \in H(\mathbb{C}^* \setminus \{z_1, \dots, z_n\}0, \qquad \square$$

Proof of (2): Assume that $z_j = w_j = 0$ and $(w_j, w_{j+1}) \in \mathbb{R}$. In $\mathbb{H}$ define $h = f^{\frac{1}{\alpha_j}}$. We define the branch of $\frac{1}{\alpha_j}$ to be positive real on $(w_j, w_{j+1}0$. Then h satisfies the conditions of the Schwarz reflection principle. So, extend h to be holomorphic in a neighborhood of 0 and h is 1-1 in a neighborhood of 0. Notice that h is 1-1 in a neighborhood of 0. So $h'(0) \neq 0$ and $f = h^{\alpha_j}$ satisfies

$$f' = \alpha_j h^{\alpha_{j-1}} \cdot h'$$

$$\log f' = \log \alpha_j + (\alpha_j - 1) \log h + \log h'$$

Now we have

$$g = \frac{f''}{f'} = -\beta_j \frac{h'}{h} + \frac{h''}{h}$$

Let us write

$$h(z) = c_1 z + c_2 z^2 + O(z^3)$$

$$h'(z) = c_1 + 2c_2 z + O(z^2)$$

$$h''(z) = 2c_2 + O(z)$$

$$\frac{h'}{h} = \frac{1}{z} + O(1)$$

and thus

$$\text{Res}(g; 0) = -\beta_j$$

If z_j, w_j are arbitrary we can define the function

$$\varphi(z) = (f(z+z_j) - w_j)e^{-i\arg(w_{j+1}-w_j)}$$

Then $\varphi(0) = 0$ and $(0, z_0)$ gets mapped to $(0, y_0)$. The multiplication term goes aways in $\frac{\varphi''}{\varphi'}$. So $\text{Res}(g; z_j) = -\beta_j$. $\square$

$\square$

6.5.2 Comments

1. If $z_n = \infty$, the formula just lose the $(\zeta - z_n)^{-\beta_n}$ factor. To prove this consider $f(a - 1/z)$ with $a \in \mathbb{R}$. We know that $f(a)$ is no a vertex, and so $a - 1/z : \mathbb{H} \to \mathbb{H}$ and $g(\infty) = f(a)$.
2. Limiting argument can show that we ma take $z_0 \in \mathbb{R}$ or even ∞.
3. An f of the Schwarz-Christoffel form is sometimes $1-1$ in $\mathbb{H}$ and sometimes not. This depends on β_j and the map is 1-1 onto a convex polygon.
4. We could take $\mathbb{D}$ as the fundamental domain. In fact the same formula holds where $z_j \in \partial\mathbb{D}$ and $z_0 \in \overline{\mathbb{D}}$.
5. In $\mathbb{H}$ we have $\arg f'(x) = \arg(w_{j+1} - w_j)$ for all $x \in (z_j, z_{j+1})$. Solving the Dirichlet problem with this data actually solves again.

6.6 Particular Cases of Schwarz-Christoffel

6.6.1 Mapping onto a Triangle

We can map $\mathbb{H}$ onto the triangle (w_1, w_2, w_3). To do this let $f(0) = w_1, f(1) = w_2, f(\infty) = w_3$. We write

$$f(z) = A\int_0^z \zeta^{\alpha_1-1}(1-\zeta)^{\alpha_2-1}d\zeta + w_1$$

where the powers are positive real for $\zeta \in (0,1)$. Letting $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$.

$$\begin{aligned} w_2 &= A\int_0^1 t^{\alpha_1-a}(1-t)^{\alpha_2-1}dt + w_1 \\ &= A\frac{\Gamma(\alpha_1)\Gamma(\alpha-2)}{\Gamma(\alpha_1+\alpha_2)} \end{aligned}$$

We use some know-how to write

$$A = (w_2 - w_1)\frac{\Gamma(1-\alpha_3)}{\Gamma(\alpha-1)\Gamma(\alpha_2)}$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

$$A = \frac{\pi(w_2 - w_1)}{\sin(\pi\alpha_3)\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}$$

6.6.2 Mapping onto a Rectangle

Let $R \subseteq \mathbb{H}$ be a rectangle resting on the real line with base of lenght $2A$ and height B. Assign $0, 1, \infty \mapsto 0, A, -A$. We claim that the map is symmetric with respect to the imaginary axis. Reflect in this axis, the three on three theorem says that this must be the identity. We also have that $f(-x) = f(x)^*$ for all real x, $f(\infty)iB$. Let M be so that

$$f(M) = A + iB$$

. Write $k = \frac{1}{M}$. Then $k \in (0,1)$. All of the $\alpha_j = \frac{1}{2}$ and thus all of the $\beta_j = \frac{1}{2}$. We obtain the formula

$$\begin{aligned} f(z) &= c\int_0^z (1-\zeta)^{-\frac{1}{2}}(1+\zeta)^{-\frac{1}{2}}(1/k-\zeta)^{-\frac{1}{2}}(1/k+\zeta)^{-\frac{1}{2}}d\zeta \\ &= c\int_0^z \frac{1}{\sqrt{(1-\zeta^2)(\frac{1}{k^2}-\zeta^2)}} \\ &= c'\int_0^z \frac{1}{\sqrt{(1-\zeta)^2(1-k^2\zeta^2)}} \end{aligned}$$

We know that $f(x) > 0$ on $(0,1)$ and so $c > 0$. Note that $f'(0) = c$ and so the derivative is positive real. Also, if we had another similar rectangle with parameters (A', B') with $\frac{B}{A} = \frac{B'}{A'}$. So there exists $\lambda > 0$ with $B'\lambda B$ and $A'\lambda A$. This means that $k = k(B/A)$.

We define the map

$$\begin{aligned} K(k) &:= \int_0^1 \frac{dx}{[(1-x^2)(1-k^2x^2)]^{(1/2)}} \\ \tilde{K}(x) &:= \int_0^{1/k} \frac{dx}{[(1-x^2)(1-k^2x^2)]^{(1/2)}} \end{aligned}$$

If we choose $c = 1$ then f maps $\mathbb{H}$ to the rectangle $(K, \tilde{K})$. As k increases we have K increases and $\tilde{K}$ decreases. So the ratio $\frac{\tilde{K}}{K}$ decreases. We define two functions

$$\begin{aligned} \operatorname{sn}(w,k) &:= f^{-1}(w), \qquad \text{when } c = 1 \\ \tau &:= i\frac{B}{A} = i\frac{\tilde{K}}{K} \\ q &:= e^{\pi i \tau} \end{aligned}$$

So q, τ are function of k for a given rectangle to find the value of c. One forms

$$q = e^{-\pi\frac{B}{A}}$$

then finds k and $K(k)$, then

$$c = \frac{A}{K}$$

More details can be found in Hille, Anlaytic Function Theory Vol. 2. We have the following formula

$$k^2 = 16q\prod_{n=1}^{\infty}\left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^8 \qquad (2)$$

$$K = \frac{\pi}{2}\left(\sum_{-\infty}^{\infty} q^{n^2}\right), \qquad (3)$$

There are alternate formulae for K

$$\begin{aligned} K &= \int_0^{\pi/2} \frac{dt}{1-k^2\sin^2 t} \\ &= \frac{\pi}{2}F(\frac{1}{2},\frac{1}{2},1;k^2) \\ &= \frac{1}{4}\int_{\partial\mathbb{D}} \frac{|dz|}{|z-k|} \\ &= \frac{\pi}{2}\sum_{m=0}^{\infty} \frac{k^{2m}[(2m)!]^2}{2^{4m}(m!)^2} \\ &= \text{ Complete Elliptic Integral of the First Kind} \end{aligned}$$

Where F is a hypergeometric function.

6.6.3 A Limiting Case

Let $k \to 0$ and $f(z) = c\int_0^z \frac{d\zeta}{(1-\zeta^2)^{1/2}} = \arcsin(z)$.

6.7 Multiply Connected Domains

6.7.1 2-connected domains

Theorem 94. *Each 2-connected domain $D \subseteq \mathbb{C}^*$ can be conformally mapped onto exactly one of the following domains:*

(a) $\mathbb{C}^* \setminus \{0\}$

(b) $\mathbb{D} \setminus \{0\}$

(c) $A(1,R) = \{z : 1 < |z| < R\}$

Proof. The idea of the proof is as follows. The uniqueness of R is a homework problem. For existence we prove an extremal condition. We want to find $u \in Ha(D)$ with $u = 0$ on K_1 and $u = 1$ on K_2 with $\tilde{u}$ the harmonic conjugate we have

$$f = e^{(u+i\tilde{u})\frac{2\pi}{T}}$$

Where $|f| = 1$ on K_1 and $e^{\frac{2\pi}{T}}$ on K_2. Then show that f is 1-1 and maps D onto $A(1,\frac{2\pi}{T})$. □

6.7.2 n-connected domains

Let $D \subseteq \mathbb{C}^*$, n-connected with $n \geq 3, n < \infty$. There is no single standard domain but some different flavors. (See Nehari, Ch. 7).

- Parallel slit domains
- Circular slit domains

- Radial Slit domains
- Disc with circular slits
- Annulus with circular slits

The existence of such maps is obtained with various extremal problems.

6.7.3 Infinitely Connected Domains

Unsolved Conjecture: Each D which is ∞-connected can be conformally mapped onto a 'circle domain': meaning the compliment is a union of disjoint discs and points.

Best Result: In 1990 it was proven true if D^c has countably many components.

7 Infinite Products

Definition 85. *Let $\{a_n\} \subseteq \mathbb{C}$, set $P_n = \prod_{j=1}^n a_j$. If $P = \lim_{n\to\infty} P_n$ exists in $\mathbb{C}^*$ we write $P = \prod_1^\infty a_j$. If $P \in \mathbb{C}$ we say that the product converges.*

Lemma 16 (Lemma 1)**.** *.*

(a) $1 + x \leq e^x$ *for all* $x \in \mathbb{R}$

(b) $e^x \leq 1 + 2x$ *for all* $x \in \mathbb{R}$.

Lemma 17 (Lemma 2)**.** *Let $\{u_k\}_1^n$ be a finite set in $\mathbb{C}$. Set $P_n = \prod_1^n (1+u_k)$ and $P_n^* = \prod_1^n (1+|u_k|)$. Then*

(a) $|P_n| \leq P_n^* \leq e^{\sum_1^n |u_k|}$

(b) $|P_n - 1| \leq P_n^* - 1$

Proof. (a) Just apply the Δ-inequality for the first part. For the second inequality use Lemma 1.

(b) Subtract 1's. □

Theorem 95 (Theorem 1)**.** *Let $\{f_n\}$ be a sequence of bounded complex valued functions on $S \subseteq \mathbb{C}$ such that $\sum_1^\infty |1 - f_n|$ converges uniformly on S then*

$$f = \prod_1^\infty f_n$$

converges uniformly on S. Moreover,

(a) $x_0 \in S$ *then*

$$f(x_0) = 0 \iff f_n(x_0) = 0, \qquad \text{for some } n$$

(b) For every permutation $\{n_1, n_2, \dots\}$ of $\{1, 2, \dots\}$ we have

$$f = \prod_{k=1}^\infty f_{n_k}$$

Proof. (a): Let u_n, P_n on S be defined by $f_n = 1 + u_n$ and $P_n = \prod_{k=1}^{n} f_k$. Then we will show that $\{P_n\}$ is a uniform Cauchy sequence. Let $1 \leq m < n$.

$$(1) \qquad P_n - P_m = P_m \left[\left(\prod_{k=n+1}^{n} (1 + u_k) \right) - 1 \right]$$

By Lemma 2 we know that

$$|P_n - P_m| < |P_m| \left(\prod_{k=m+1}^{n} (1 + |u_k|) \right) - 1 \leq |P_m| \left(e^{(\sum_{k=m+1}^{n} |u_k|) - 1} \right)$$

Since $\sum_1^\infty |u_k|$ converges uniformly, the u_k are bounded there exists B such that

$$\sum_{1}^{\infty} < B, \qquad \forall x \in S$$

Lemma 2a implies that for all $x \in X$ and $m \in \mathbb{Z}^+$ we have

$$(2) \qquad |P_m(x)| \leq e^{\sum_1^m |u_k(x)|} \leq e^B$$

Referencing (1) we obtain

$$|P_n - P_m| \leq e^B \left(e^{\sum_{k=m+1}^{n} |u_k| - 1} \right)$$

Given $\epsilon > 0$ choose $\delta > 0$ so that

$$(3) \qquad e^B(e^\delta - 1) < \epsilon$$

Then choose m_0 such that

$$\sup_{x \in S} \left(\sum_{m_0+1}^{\infty} |u_k| \right) < \delta$$

Thus $\{P_n\}$ is uniformly Cauchy. So $\prod f_n$ converges uniformly.

Let $f(x_0) = 0$. Choose l such that

$$\sup_{x \in S} \left(e^{\sum_l^\infty |u_n(x)|} - 1 \right) < \frac{1}{2}$$

So for $n \geq l$ and (1) we have

$$|P_n(x) - P_l(x)| < |P_l(x)| \frac{1}{2}$$

So for $x \in S$ and $n \geq l$ it holds that

$$(4) \qquad |P_n(x)| > \frac{1}{2} |P_l(x)|$$

If each $f_k(x_0) \neq 0$ then $|P_l(x_0)| \neq 0$. Then (4) implies that

$$|f(x_0)| > \frac{1}{2} |P_l(x_o)| > 0$$

(b): Fix a permutation $\{n_k\}$ and $0 < \epsilon < 1$. Choose l so large that

$$(5) \qquad \sup_{x \in S} |f(x) - P_l(x)| < \epsilon/2$$

$$(6) \qquad \sup_{x\in S}\left|\sum_{l}^{\infty}|u_k(x)|\right| < \frac{\epsilon e^{-B}}{4} := \delta_1$$

Increasing B if necessary, assume that $\delta_1 < 1$. Choose k_0 so that

$$\{1,2,\ldots,l\} \subseteq \{n_1,n_2,\ldots,n_{k_0}\}$$

So for $k \geq k_0$ we have $\{1,2,\ldots,l\} \subseteq \{n_1,\ldots,n_k\}$. Let

$$I = \{n_1,\ldots,n_k\} \setminus \{1,2,\ldots,l\}$$

Let

$$\begin{aligned} Q_k &:= \prod_{j=1}^{k} f_{n_j} \\ &= \prod_{j=1}^{l} f_j \cdot \prod_{I} f_j \\ &= P_l \prod_{I} f_j \end{aligned}$$

This means that

$$\begin{aligned} |Q_k - P_l| &\leq |P_l| \left|\prod_{j\in I}(1+u_j) - 1\right| \\ &\leq |P_l| \left(e^{\sum_I |u_j|} - 1\right) \\ &\leq e^B \left(e^{\delta_1} - 1\right) \\ &< 2\delta_1 e^B \\ &< \frac{\epsilon}{2} \end{aligned}$$

Combined with (5) this gives

$$\sup_S |f - Q_k| < \epsilon.$$

Hence $\prod_1^\infty f_{n_k} = f$ and the convergence is uniform.

□

Corollary 23. *Let $\{a_n\} \subseteq \mathbb{C}$ with $\sum |1-a_n| < \infty$ then $\prod a_n < \infty$ and is not zero if the a_n's are never zero. The product is independent of the order.*

7.0.4 Example

A counterexample of the converse is

$$\prod_2^\infty [1 + \frac{(-1)^n}{n}]$$

This converges and is not 0 but the sums $\sum |a_n - 1| = \sum \frac{1}{n} = \infty$.

7.0.5 A Partial Converse

Proposition 7. *If* $\{a_n\} \subseteq (0,1]$, $P_n = \prod_1^n a_k$ *then* $P_n \subseteq (0,1]$ *and* $P - n$ *decreases. So* $\lim P_n$ *must exist and is greater than or equal to zero. Theorem 2 says that (with these hypotheses) we have* $\prod a_n > 0 \iff \sum(1-a_n) < \infty$.

Proof. ($\Leftarrow$) follows from the Corollary since $|1-a_n| = (1-a_n)$.

($\Rightarrow$) We know that $1 - x \leq \log \frac{1}{x}$ when $0 < x < \infty$ (true since $x \leq e^{x-1}$). Fix n, then

$$\sum_1^n (1-a_k) \leq \sum_{k=1}^n \log \frac{1}{a_k} = \log\left(\prod_1^n \frac{1}{a_k}\right) = \log\left(\frac{1}{\prod_1^\infty a_k}\right)$$

□

Theorem 96 (Theorem 3)**.** *Let* $\{f_n\}$ *be a sequence in* $H(D)$, $D \subseteq \mathbb{C}^*$ *with none of the* $f_n \equiv 0$. *Assume that* $\sum |1-f_n|$ *converges locally uniformly on* D. *Then*

(a) $f = \prod f_n$ *converges locally uniformly and* $f \in H(D)$.

(b) $f \not\equiv 0$ *on* D.

(c) Let $m(z,f)$ *be the multiplicity of the zero of* f *at* z, *then*

$$m(z,f) = \sum_1^\infty m(x, f_n)$$

Proof. (a) Theorem 1 on all compact sets and local uniform limits of holomorphic functions are holomorphic.

(b) Let $A = \{z \in D : f_n(z) = 0 \text{for some } n\}$. We know A is countable (by the uniqueness theorem) and from Theorem 1 this is all of the zeros of f.

(c) For finite products this is true by factoring. Let $f = \prod^\infty f_n$ and $z_0 \in D$ take a disc Δ such that $z_0 \in \Delta \subseteq \overline{\Delta} \subseteq D$. There is an N such that

$$\sup_{a \in \overline{\Delta}} \left(\sum_N^\infty |1 - f_n(z)|\right) < 1$$

Thus $f_n(z) \neq 0$ for $n \geq N$ and $z \in \overline{\Delta}$. Therefore

$$f = \prod^{N-1} f_n \cdot \prod_N^\infty f_n$$

The conclusion follows. □

7.1 Functions with Assigned Zeros

Weierstrass asked the question: Given $\{a_n\} \subseteq \mathbb{C}$ with $\lim_{n\to\infty} a_n = \infty$, does there exist an $f \in H(\mathbb{C})$ which has zeros precisely at the a_n's with the chosen multiplicity.

7.1.1 First Try

Let $f(z) = \prod_1^\infty (z - a_n)^{m_n}$. Unfortunately this only works for finite sequences.

7.1.2 Second Try

How about $f(z) = \prod (1 - z/a_n)^{m_n}$? But we have

$$\left|1 - (1 - \frac{z}{a_n})\right| \leq \frac{|z|}{|a_n|}$$

converges if $\sum \frac{1}{|a_n|} < \infty$. So not good enough.

7.1.3 Third and Final Attempt

Weierstrass proceeded thusly:

$$\log(1 - z) = \sum_1^\infty -\frac{z^n}{n}$$

This means that

$$\log(1 - z) + \sum_{k=1}^p \frac{z^k}{k} = O(z^{p+1})$$

Exponentiating gives

$$E_p(z) := \begin{cases} (1 - z)e^{\sum_1^p \frac{z^k}{k}}, & p \geq 1 \\ (1 - z), & p = 0 \end{cases}$$

We then take

$$\boxed{\prod_1^\infty E_{p_n}\left(\frac{z}{a_n}\right)}$$

We would like to prove that this converges.

Lemma 18. *For $z \in \overline{\mathbb{D}}$ it holds that*

$$|E_p(z) - 1| \leq |z|^{p+1}, \qquad p \geq 0$$

Proof. It is clear for $p = 0$. Suppose that $p \geq 1$ and let $g(z) = \frac{E_p(z)-1}{z^{p+1}}$. We would like to prove that $|g(z)| \leq 1$ for $z \in \overline{\mathbb{D}}$. Since $\log(E_p(z)) = O(z^{p+1})$ we have

$$E_p(z) = e^{O(z^{p+1})} = 1 + O(z^{p+1})$$

therefore g has a removable singularity near 0 and thus $g \in H(\mathbb{C})$. We can calculate $E_p'(z)/E_p(z)$. We obtain

$$\frac{E_p'(z)}{E_p(z)} = \frac{-1}{1 - z} + \sum_1^p z^{k-1} = -\sum_{k=p}^\infty z^k = -\frac{z^p}{1 - z}$$

We then have

$$-E_p'(z) = z^p e^{\sum_1^p z^k/k} = \sum_{k=p}^{\infty} b_k z^k, \qquad b_k > 0$$

So

$$1 - E_p(z) = -\int_0^z E_p'(\zeta)d\zeta = \sum_{k=p}^{\infty} \frac{b_k}{k+1} z^{k+1}$$

This means for $z \in \partial\mathbb{D}$ it holds that

$$|1 - E_p(z)| \leq \sum_1^{\infty} \frac{b_k}{k+1} = 1 - E_p(1) = 1$$

The maximum principle now says that $g \leq 1$ in $\mathbb{D}$. So, $|E_p(z) - 1| \leq |z|^{p+1}$ for $z \in \overline{\mathbb{D}}$. □

Theorem 97 (Theorem 4)**.** *Let $\{a_n\}$ be a sequence of not necessarily distinct nonzero complex numbers such that $\lim_{n\to\infty} a_n = \infty$. Let $\{p_n\}$ be a sequence of non negative integers such that for any $R > 0$ it holds that*

$$\sum_1^{\infty} \left(\frac{R}{|a_n|}\right)^{1+p_n} < \infty$$

Then the product

$$f(z) = \prod_1^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

is an entire function which has zeros at each a_n and no other zeroes in $\mathbb{C}$. (If a_n appears m-times then its multiplicity is m).

7.1.4 Remark

1. $\sum_1^{\infty} \left(\frac{R}{|a_n|}\right)^n < \infty$ whenever $a_n \to \infty$. So we may choose $p_n = n - 1$.
2. If you want zeroes at 0 then we have

$$f(z) = z^k \prod_{n=1}^{\infty} E_{p_n}(z/a_n)$$

Proof. By lemma, for $z \in \overline{\mathbb{D}}(0, |a_n|)$ we have

$$|1 - E_{p_n}(z/a_n)| \leq \left(\frac{|z|}{|a_n|}\right)^{p_n+1}$$

Let $K \subseteq \mathbb{C}$ be compact, let $R > 0$ be such that $K \leq \mathbb{D}(0, R)$. So, there exists N such that $|a_n| > R$ for $n \geq N$. So

$$|1 - E_{p_n}(z/a_n)| \leq \left(\frac{R}{|a_n|}\right)^{p_n+1}, \qquad n \geq N$$

By our hypothesis and Weierstrauss M-test,

$$\sum_1^{\infty} |1 - E_{p_n}(z/a_n)| < \infty$$

uniformly on K. Theorem 3 applies to prove theorem 4.

□

Theorem 98 (Theorem 5)**.** *(Weierstrass Factorization Theoerem) Let f be entire with $f \not\equiv 0$. Let $a_1, a_2, \dots$ be the nonzero zeros of f listed according to mulitplicity. Then there exists $g \in H(\mathbb{C})$ and a sequence $\{p_n\}$ of non-negative integers such that*

$$f(z) = z^k \prod_1^\infty E_{p_n}(z/a_n) e^{g(z)} \tag{2}$$

where k is the order of the zero at 0.

Proof. Let $p_n = n-1$ and let $f_1(z) = \prod E_{p_n}(z/a_n)$. Theorem 4 implies that $z^k f_1$ has the same zero set. So $f_2 = \frac{f(z)}{z^k f_1}$ is entire in $\mathbb{C}$ and zero free. Since $\mathbb{C}$ is simply connected and f_2 has no zeros this implies that there exists $g \in H(\mathbb{C})$ such that

$$f_2 = e^g.$$

□

Theorem 99 (Theorem 6)**.** *(See Rudin pg 326) Let $D \subsetneq \mathbb{C}^*$. Let $A \subseteq D$ have no limit points in D. Associate to each $a \in A$ a number $m(a) \in \mathbb{Z}^+$. Then there exist $f \in H(D)$ all of whose zeros lie in A such that f has a zero at $a \in A$ of order $m(a)$.*

Proof. Assume first that $\infty \in D$ but $\infty \notin A$. If A is finite and some $b \in \mathbb{C} \setminus D$, then

$$f = \prod_{a \in A} (z-a)^{m(a)} (z-b)^{-\sum_{a\in A} m_a}$$

If A is infinite then A is countable. Index the $\{a_n\}$ and let each term appear $m(a)$ times. Note that $\mathbb{C} \setminus D$ is compact for each $n \geq 1$ there is a $b_n \in \mathbb{C}^* \setminus D$ which is a closest point to a_n. We try the function

$$f(z) = \prod_1^\infty E_n\left(\frac{a_n - b_n}{z - b_n}\right)$$

By a future homework problem this converges nicely. To get the result for other domains use Möbius transformations. □

Theorem 100 (Theorem 7)**.** *Let $f \in M(D)$ where $D \subsetneq \mathbb{C}^*$. Then $f = \frac{f_1}{f_2}$ with $f_1, f_2 \in H(D)$.*

Proof. Let $A = \{z \in D : f(z) = \infty\}$ then A has no limit point in D. Let f_2 be the f from Theorem 6. Let

$$f_1 = f \cdot f_2$$

. □

7.1.5 Caution

If $D = \mathbb{C}^*$ remember that $M(C^*)$ contains only the rational functions and $H(\mathbb{C}^*)$ are the constants.

7.2 Functions with Assigned Principal Parts

7.2.1 Problem

Given distinct $a_i \in \mathbb{C}$ and $\lambda_i \in \mathbb{C}$ not necessarily distinct with $i = 1, \ldots, N$. Does there exist $f \in M(\mathbb{C})$ such that f has poles only at the a_i and

$$f(z) - \sum_1^N \frac{\lambda_i}{z - a_i}$$

is holomorphic near the a_i's?

7.2.2 Answer

Yes. We let $f(z) = \sum_1^N \frac{\lambda_i}{z-a_i}$.

Now suppose that we have infinitely many a_i.

7.2.3 Example

1. Suppose that $a_n = n^2$ and $\lambda_n = 1$. Here the previous answer suffices since the sum will converge.
2. Suppose that $a_n = n$ and $\lambda_n = 1$. Then the function
$$f(z) = \sum_1^\infty \left(\frac{1}{z-n} + \frac{1}{n} \right) = \sum_1^\infty \frac{z}{(z-n)n}$$
3. How about $a_n = n$ and $\lambda_n = (n!)^{n!}$

Theorem 101 (Theorem 8)**.** *(Mittag-Leffler) Suppose that $A \subseteq \mathbb{C}$ and A has no limit point. For each $a \in A$, let there be given $m(a) \in \mathbb{Z}$ and a rational function $p(a) = \sum_{j=1}^{m(a)} c_{j,a}(z-a)^{-j}$. Then there exists an $f \in M(\mathbb{C})$ whose principle part at each $a \in A$ is P_a and f has no other poles.*

Proof. If A is finite the $f = \sum_{a \in A} p_a$ works. Suppose that A is infinite. Since A has no limit points we know that A is countable. Choose a sequence $\{a_n\} = A$ with the a_n distinct and 0 if in A, is listed first. For convenience write $p_{a_n} := p_n$. For $n \geq 2$ we know that $p_n \in H(|z| < |a_n|)$. So, we may write $p_n = \sum_{m=0}^\infty b_{n,m} z^m$. This series converges locally uniformly on $\mathbb{D}(0, |a_n|)$. Let S_n be the partial sum of the series with

$$(1) \qquad \sup_{|z| < \frac{|a_n|}{2}} |S_n - p_n| \leq 2^{-n}$$

Also, set $S_1 = 0$. We claim that $f(z) = \sum_1^\infty (p_n(z) - S_n(z))$ does the trick. Each term in this series is in $H(\mathbb{C} \setminus A)$. Let K be compact then there exists N so that for all $n \geq N$ it holds that $K \subseteq \mathbb{D}(0, \frac{1}{2}|a_n|)$. From (1) it follows that the series defining f converges uniformly on K.

Fix an $a_k \in A$. Let $a_k \in \Delta$ a disk with $A \cap \overline{\Delta} = \{a_k\}$. For $x \in \Delta \setminus \{a_k\}$ we have

$$(2) \qquad f(z) - p_k(z) = \left[\sum_{n=1, n \neq k}^\infty p_n(z) - S_n(z) \right] - S_k(z)$$

Each term on the right is in $H(\Delta)$ then (1) implies that this series converges uniformly in Δ. So $f - P_k$ has a removable singularity at a_k. So the principle part of f at a_k is p_k. ☐

7.2.4 Remark

Theorem 8 is also true for arbitrary domains. For proof see Rudin.

7.2.5 Recall

If g is rational then this implies that g is the sum of the principle parts plus the principal part at ∞ (which is a polynomial)

Corollary 24. *If $g \in M(\mathbb{C})$ has poles at $\{a_n\}$ with respective principal parts*

$$p_n(z) = \sum_{j=1}^{m(n)} C_{j,n}(z - a_n)$$

then there exists a sequence of polynomials S_n and an $h \in H(\mathbb{C})$ such that

$$g(z) = \sum_{1}^{\infty} [P_n(z) - S_n(z)] + h(z)$$

for $z \in \mathbb{C}$ and the series converges locally uniformly in $\mathbb{C} \setminus \{a_n\}$.

Proof. Let $h = g - f$ where f is defined via the proof of Mittag-Leffler. Since g, f have the same principle parts we know that $h \in H(\mathbb{C})$.

□

7.2.6 Example

A function that is the sum of its principle parts is given by

$$g(z) = \cot(\pi z) = \frac{\cos \pi z}{\sin \pi z}$$

We have poles at $\mathbb{Z}$ and each pole is simple with $1/\pi$ the residue. So the principle part is $\frac{1}{\pi}\frac{1}{z-n}$ for all $n \in \mathbb{Z}$. It would be nice if

$$\cot \pi z = \frac{1}{\pi} \sum_{\mathbb{Z}} \frac{1}{z - n}$$

This is true, but is hard to prove.

7.2.7 Interpolation Problem

Given sequences $\{z_n\}, \{w_n\} \subseteq \mathbb{C}$ we want an entire function $f \in \mathbb{C}$ with $f(z_n) = w_n$ for all n. We suppose that the z_n are distinct and that $\lim_{n \to \infty z_n} = \infty$.

If there are finitely many z_n's then let $g(z) = \prod(z - z_n)$ and

$$f(z) = \sum_{1}^{N} w_n \frac{g(z)}{z - z_n} \frac{1}{g'(z_n)}$$

Theorem 102 (Theorem 9)**.** *(Interpolation Theorem) Let $\{z_n\}, \{w_n\} \subseteq \mathbb{C}$ be sequences as above. Then there exists $f \in H(\mathbb{C})$ such that $f(z_n) = w_n$ for all $n \geq 1$.*

Proof. By Theorem 4 there exists a $g \in H(\mathbb{C})$ such that g has a simple zero at each z_n. By Theorem 8 there exists an $h \in M(\mathbb{C})$ with no poles except at z_n with principle part

$$\frac{c_n}{z - z_n}, \qquad c_n = \frac{w_n}{g'(z_n)}$$

For $z \in \mathbb{C} \setminus \{z_n\}$ we know that $f(z) = \frac{g(z)}{z - z_n}[h(z)(z - z_n)]$. As $z \to z_n$ we have

$$f(z) \to g'(z_n)c_n = w_n.$$

Therefore, f has a removable singularity at z_n and we define f to be w_n at z_n. □

7.2.8 Remark

This is also true on arbitrary $D \subseteq \mathbb{C}$.

7.3 Jensen's Formula

(1859 - 1925)

Lemma 19. *If* $f \in H(\mathbb{D}(0, R))$ *and* $f \not\equiv 0$ *then*

(a) $\log |f(re^{i\theta})| \in L^1([-\pi, \pi]), \quad 0 < r < R.$

(b) $r \mapsto \int_{-\pi}^{\pi} \log |f(re^{i\theta})| d\theta$ *is continuous on* $(0, R)$.

Proof. (a) If f has no zero on $|z| = r$ then $\log |f(re^{i\theta})|$ is continuous on $[-\pi, \pi]$ and hence integrable. Suppose f has zeros on $|z| = r$. Then passing to $f_1(z) = f(rz)$ we may assume that $r = 1$. Now we let the zeros of f on $|z| = 1$ be denoted by $z_1, \ldots, z_p$ counting mulitplicity (there are only finitely many since $f \not\equiv 0$. Then

$$\log |f(z)| = \sum_{1}^{p} \log |z - z_j| + \log |g(z)|, \qquad g \in H(\mathbb{D}(0, R)) \text{ and } g \text{ has no zeros on } |z| = 1$$

By scaling we can consider the circle $|z| = 1$. So it suffices to prove that

$$\int_{-\pi}^{\pi} \log |e^{i\theta} - 1| d\theta$$

is integrable. A future homework problem will prove that

$$|log|e^{i\theta} - 1|| \leq \log \frac{\pi}{|\theta|} + C$$

We can write

$$\int_{-\pi}^{\pi} \log \frac{\pi}{|\theta|} d\theta = 2\pi \int_0^1 \log \frac{1}{x} dx = 2\pi$$

(b) As in (a) it suffices to show continuity at $r = 1$ when $f(z) = z - 1$. We show that

$$(*) \qquad \lim_{r \to 1} \int_{-\pi}^{\pi} \log |re^{i\theta} - 1| d\theta = \int_{-\pi}^{\pi} \log |e^{i\theta} - 1| d\theta$$

We apply the Lebesgue Dominated Convergence theorem and conclude that there exist C_1, C_2 satisfying

$$C_1|\theta| \leq |re^{i\theta} - 1| \leq C_2$$

Taking the log yields

$$|\log|1 - re^{i\theta}|| \leq \log\frac{\pi}{||\theta|} + C_3$$

for $r \in [1/2, 2], \theta \in [-\pi, \pi]$. Since

$$\int_{-\pi}^{\pi} |\log\frac{\pi}{|\theta|} + C_3| d\theta < \infty$$

we may apply the dominated convergence theorem.

□

Let $f \in H(\mathbb{D}(0, R))$,

Definition 86. *Define the function*

$$n(r, 0, f) = n(r) := \text{ \# of zeros of } f \text{ in } \overline{\mathbb{D}}(0, r) \text{ counted with multiplicity.}$$

Note that n is a step function which is continuous from the right. Also, define the function

$$N(r, 0, f) = N(r) := \sum_{0 \leq |z_j| \leq r} \log\frac{r}{|z_j|}, \qquad z_j \text{ zeros of } f$$

Assume f has no zero at the origin. Then $n(t) = 0$ in a neighborhood of 0. We can write

$$\begin{aligned} N(r) &= \int_{[0, r^-]} \log\frac{r}{t} dn(t) \\ &= (\log\frac{rt}{n}(t)\Big|_{t=0^+}^{t=r^+} - \int_0^r n(t)\frac{d}{dt}n(t)dt \\ &= \int_0^r n(t)\frac{dt}{t} \end{aligned}$$

If f has a zero at 0 then $N(r) := \int_0^r \frac{n(t) - n(0)}{t} dt$.

Theorem 103 (Theorem 10)**.** *(Jensen's Formula) Let $f \in H(\mathbb{D}(0, R))$ and $f \not\equiv 0, r \in (0, R)$. Then*

(a) If $f(0) \neq 0$ then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})| d\theta = N(r, 0, f) + \log|f(0)|$$

(b) $f(z) = cz^k g(z)$ where $k \geq 1, g(0) = 1, c \neq 0$. We have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})| d\theta = N(r, 0, f) + k\log r + \log|c|$$

Proof. (a) Both sides are continuous functions of r by the lemma. So, it suffices to verify the formula when f has no zero on $|z| = r$. We may assume that $R > 1$ and no zeros on $r = 1$. There are two cases

- Case 1: If f is zero free on $\overline{\mathbb{D}}(0,1)$ then $\log|f|$ is harmonic and $N(1,0,f)=0$. Thus (a) holds by the mean value property.
- Case 2; Suppose that f as zeros $z_1,\ldots,z_m$ in $\mathbb{D}$. Let

$$B(z)=\prod_{j=1}^{m}\frac{z-z_0}{1-\overline{z}_j z}.$$

Then $f=B\cdot g$ where $g\in H(\mathbb{D}(0,1+\epsilon))$ and g s zero free in $\overline{\mathbb{D}}$. Also $|B|=1$ on $|z|=1$. We then have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(e_{i\theta}|d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|g(e^{i\theta})d\theta\\ &=\log|g(0)|\\ &=\log|f(0)|-\log|B(0)|\\ &=\log|f(0)|+\sum_{0\le|z_j|\le 1}\log\frac{1}{|z_j|}\\ &=\log|f(0)|+N(1,0,f)\end{aligned}$$

(b) Left to reader. □

7.3.1 Example

Consider the function $f(z)=z-a$ with $a\neq 0$. We have

$$n(t)=\begin{cases}1, & t\ge|a|\\ 0, & \text{otherwise}\end{cases}$$

$$N(t)=\begin{cases}\log\frac{t}{|a|}, & t\ge|a|\\ 0, & t<|a|\end{cases}$$

Observe that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|re^{i\theta}-a|d\theta=N(r)+\log|a|=\begin{cases}\log r, & r\ge|a|\\ \log|a|, & \text{otherwise}\end{cases}$$

In particular,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|e^{i\theta}-1|=0$$

7.4 Blashke Condition

We will use the notation $\log^+(x)=\max(\log x,0)$. This means we can write

$$N(r)=\sum_{z_j}\log^+\frac{r}{|z_j|}$$

Definition 87. *For $f\in H(\mathbb{D}(0,R))$ set $T(r,f)=T(r):=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+|f(re^{i\theta})|d\theta$. We call T the* ***Nevanlinna characteristic****.*

Note that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|g(re^{i\theta})|d\theta \leq T(r,f)$$

Definition 88. *Let $f \in H(\mathbb{D})$*

(a) We say that $f \in H^\infty$ if $\sup_{\mathbb{D}}|f| < \infty$.

(b) We say that $f \in H^p$ if $\sup_{r\in(0,1)}\int_{-\pi}^{\pi}|f(re^{i\theta})|^p d\theta < \infty$.

(c) $f \in N$ (Nevanlinna class) if $\sup_{r\in(0,1)}\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+|f(re^{i\theta})|d\theta < \infty$.

7.4.1 Remarks

Note that for $0 < x < \infty, p \in (0,\infty)$ we have

$$0 \leq \log^+ x \leq \frac{1}{p}x^p$$

So we have the containment

$$H^\infty \subseteq H^p \subseteq N$$

Theorem 104 (Theorem 11)**.** *If $f \in N$, $f \not\equiv 0$ then $\sum_1^\infty 1-|z_j| < \infty$ where $|z_j|$ are zeros of f counting multiplicity.*

Proof. (proof of theorem) Suppose $f(0) \neq 0$ and list the zeros $0 < |z_1| \leq |z_2| \leq \cdots$. We then have

$$\log|f(0)| + N(r,0,f) \leq T(r,f) < \sup_{0<r<1} T(r,f) \leq C < \infty$$

We fix some large $k \in \mathbb{Z}^+$ and choose r so that $|z_k| < r < 1$. Look at

$$\sum_{j=1}^{k}\log\frac{r}{|z_j|} = N(r,0,f) \leq C - \log|f(0)|$$

Now we let $r \to 1$ and $k \to \infty$. We get

$$\sum_{j=1}^{\infty}\log\frac{1}{|z_j|} \leq C - \log|f(0)|$$

But $1-x \leq \log\frac{1}{x}$, so $\sum_{j=1}^\infty 1-|z_j| < \infty$. Suppose f has a zero of order m at 0, then $f(z) = z^m f_1(z)$, one easily shows that $f_1 \in N$. So we may add k to this sum to get $\sum_{zeros} 1-|z_j| < \infty$. ☐

Corollary 25. *If $f \in H^\infty$, $f \not\equiv 0$ then $\sum_j 1-|z_j| < \infty$.*

Theorem 105 (Theorem 12)**.** *Let $\{z_j\}_1^\infty \subseteq \mathbb{D}$ with $\sum 1-|z_j| < \infty$. Then there exists $f \in H^\infty$ such that z_j's are precisely the zeros of f with multiplicity given by $\{z_j\}$.*

Proof. Homework Problem. (try Blashke products) ☐

7.5 Comparisson of Functionals

For $f \in H(\mathbb{D}0, R)$ we've defined some functionals, $M(r,f), T(r,F), n(r,0,f), N(r,0,f)$.

Theorem 106 (Theorem 13)**.** *Suppose* $0 < r < s < R$.

(a) $T(r,f) \leq \log^+(M(r,f)) \leq \frac{s+r}{s-r} T(s,f)$

(b) $n(r) \log \frac{s}{r} \leq N(s)$ *if* $f(0) \neq 0$.

(c) $N(r) \leq T(r) - \log |f(0)|$ *if* $f(0) \neq 0$.

Proof. (a) First is trivial (use Jensen's inquality). Second is a homework problem.

(b) We know that $N(s) = \int_0^2 \frac{n(t)}{t} dt$ and $n(t)$ is increasing. So we have

$$N(s) - N(r) = \int_r^s \frac{n(t)}{t} dt \geq \int_r^s \frac{n(r)}{t} dt = n(r) \log \frac{s}{r}$$

(c) We proved in Theorem 11. □

Lemma 20. *Suppose* $u \in Ha(\overline{\mathbb{D}})$.

$$u(re^{i\theta}) \leq \frac{1+r}{1-r} \frac{\partial}{\partial u}^{+} (e^{i\varphi}) d\varphi$$

Proof.

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{i\varphi}) P(r, \theta - \varphi) d\varphi \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} u^+(e^{i\varphi}) P(r, \theta - \varphi) d\varphi = \frac{1+r}{1-r} \frac{1}{2\pi} \int_{-\pi}^{\pi} u^+(e^{i\varphi}) d\varphi$$

□

7.6 Bloch and BMOA Space

Definition 89. *A function f is a **Bloch** function if $f \in \mathcal{B}$ if $f \in H(\mathbb{D})$*

$$\|f\|_{\mathcal{B}} = \sup_{\mathbb{D}} (1 - |z|^2) |f'(z)| < \infty$$

Definition 90.

$$\beta(z,f) := \sup\{R : \mathbb{D}(f(z), R) \text{ is 1-1 image of subdomain of } \mathbb{D}\}$$

$$\beta(z,f) = 0 \text{ if } f'(z) = 0$$

$$\beta(f) := \sup_{\mathbb{D}} \{\beta(z,f)\}$$

Theorem 107. *For $f \in H(\mathbb{D})$ the following are equivalent.*

(a) $f \in \mathcal{B}$

(b) $\{f \circ A - f(A(0)) : A \in \text{Aut}(\mathbb{D})\}$ *is a normal family.*

(c) $|f(z_1) - f(z_2)| \leq C_f d_h(z_1, z_2)$

(d) $\beta(f) < \infty$.

Theorem 108 (Bloch's Theorem (1926))**.** *There exists $B > 0$ such that $\beta(f) \geq B$ for all $f \in H(\mathbb{D})$, $|f'(0)| = 1$. B is called **Bloch's Constant**.*

7.6.1 Remark

The exact value of B is unknown. Alfhor's proved that $\sqrt{3}/4 < B$. Alfhors and Grunsky proved in 1937 that $B < \Gamma(1/3)\Gamma(11/12)/\sqrt{1+\sqrt{3}}\Gamma(1/4)$. In 1996 it was proven that $\sqrt{3}/4 + 2 \cdot 10^{-4}$. So $.43002 < B < K$.

Theorem 109 (Landau's Theorem)**.** *Let $\sigma(f)$ be the in radius of $f(\mathbb{D})$ for fixed $f \in H(\mathbb{D}), |f'(0)| = 1$. Then there is an absolute constant $L > 0$ so that $\sigma(f) \geq L$. This L is called **Landau's Constant**.*

7.6.2 Remarks

L is also not known. However it was proven in 1943

$$\frac{1}{2} \leq L \leq \frac{\Gamma(1/2)\Gamma(5/6)}{\Gamma(1/6)}$$

The upper bound is the conjectured value of L In 1995 it was improved by Xangahoru

$$\frac{1}{2} + 10^{-335} \leq L \leq \frac{\Gamma(1/2)\Gamma(5/6)}{\Gamma(1/6)}$$

Definition 91. *A function f is **BMOA** ($f \in BMOA$) if $f \in H(\mathbb{D})$ and*

$$\sup_{0<r<1, A\in \mathrm{Aut}(\mathbb{D})} T(f \circ A - (f \circ A)(0), r) < \infty$$

7.6.3 Remarks

$$H^\infty \subseteq BMOA \subseteq \bigcap_{0<p<\infty} H^p$$

$(H^1)^* = BMOA$

7.6.4 Example

$\log \frac{1+z}{1-z} \in BMOA \setminus H^\infty$ and $\sum z^{2^n} \in \mathcal{B}$ but not in $BMOA$.

Theorem 110. *For $f \in H(\mathbb{D})$ the following are equivalent.*

(a) $f \in BMOA$

(b) $f \in H^1$ and $\sup_I \frac{1}{I} \int |f(e^{i\theta}) - f_I| d\theta < \infty$ where f_I is the average value of f on I.

(c) There exists $f = f_1 + f_2$ such that $\|\|\| \operatorname{Re} f_1 \|\|\|_\infty < \infty$ and $\|\|\| \operatorname{Im} f \|\|\|_\infty < \infty$.

8 Analytic Continuation

8.1 Introduction

Definition 92. *Let $D \subseteq C^*$ and $f \in H(D)$, $D_1 \supsetneq D$. We say that f has an **analytic continuation to** D_1 if there exists $g \in H(D_1)$ such that $f = g$ on D.*

8.1.1 Example

1. $\xi(s) = \sum_1^\infty n^{-s}$ has an analytic continuation from $\mathrm{Re}(s) > 1$ to $\mathbb{C} \setminus \{1\}$.

2. $z \in \mathbb{D}(1,1)$ and $f(z) = \sum_1^\infty \frac{(z-1)^n(-1)^{n+1}}{n} = \log z$ Then f is a branch of $\log(z)$ which is equal to 0 when $z = 1$. Then f has an analytic continuation to every simply connected domain in $\mathbb{C} \setminus \{0\}$. To see this we let

$$D_1 = \mathbb{C} \setminus \{\text{positive imaginary axis}\}, \qquad D_2 = \mathbb{C} \setminus \{\text{negative imaginary axis}\}$$

Define the functions

$$g_1(z) = \log|z| + i \arg z, \qquad -\frac{3\pi}{2} < \arg z < \frac{\pi}{2}$$

$$g_2 = \log|z| + i \arg z, \qquad -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$$

Notice that $g_1(-1) = -\pi i$ and $g_2(-1) = \pi i$. This shows if f has an analytic continuation g_1, g_2 to D_1, D_2 then $g_1 \neq g_2$ on $D_1 \cap D_2$.

Lemma 21. *If $D_1 \cap D_2$ is connected then $g_1 = g_2$ in the intersection.*

Proof. Let $g_1 = g_2$ in $D \subseteq D_1 \cap D_2$. Hence $g_1 = g_2$ on $D_1 \cap D_2$ by uniqueness. □

Lemma 22 (3 disks lemma)**.** *Let $\Delta_1, \Delta_2, \Delta_3$ be open disks in $\mathbb{C}$ and $f_i \in H(\Delta_i)$. Suppose $f_1 = f_2$ in $\Delta_1 \cap \Delta_2$ and $f_1 = f_3$ in $\Delta_1 \cap \Delta_3$ and that $\Delta_1 \cap \Delta_2 \cap \Delta_3 \neq \emptyset$. Then $f_2 = f_3$ on $\Delta_2 \cap \Delta_3$.*

Proof. Since $\Delta_2 \cap \Delta_3$ is open and connected we know that $f_2, f_3 \in H(\Delta_2 \cap \Delta_3)$ and equal in $\Delta_1 \cap \Delta_2 \cap \Delta_3$. Finish by the previous lemma. □

8.2 Regular and Singular Boundary Points

Definition 93. *Let $f \in H(D)$ and $z_0 \in \partial D$. We say that z_0 is a **regular point** if there is an open disk $\Delta \ni z_0$ with $g \in H(\Delta_0)$ and $g = f$ on $\Delta \cap D$. Otherwise z_0 is a **singular point**.*

Theorem 111 (Theorem 1)**.** *Suppose that $f = \sum_0^\infty a_n z^n$ is in $H(\mathbb{D}(0,R))$ where $0 < R < \infty$ and $(1/R) = \limsup |a_n|^{1/n}$. Then f has at least 1 singular point on$\partial\mathbb{D}(0,R)$.*

Proof. Suppose not. Then by compactness of $\partial\mathbb{D}(0,R)$ we can extract a finite subcover $\Delta_1, \ldots, \Delta_n$ centered on $\partial\mathbb{D}(0,R)$ and $f_j \in H(\Delta_j)$ with $f_j|_{\Delta_j \cap \partial\mathbb{D}(0,R)} \equiv f|_{\Delta_j \cap \partial\mathbb{D}(0,R)}$. By the 3-disk lemma if $i \neq j$ and $\mathbb{D}(0,R) \cap \Delta_i \cap \Delta_j \neq \emptyset$ then $f_i = f_j$.

If $z = re^{i\theta} \in \Delta_i, r > R$ then $|re^{i\theta} - 1| > |Re^{i\theta} - 1|$ so $z' = Re^{i\theta} \in \Delta_i$, where Δ_i is centered at 1. Since Δ_i is open there exists $\epsilon > 0$ so that $z'' = (1-\epsilon)Re^{i\theta} \in \Delta_i$. So $z'' \in \Delta_i$ and in $\mathbb{D}(0,R)$.

From this we conclude that if $\Delta_i \cap \Delta_j \neq \emptyset$ then $\Delta_i \cap \Delta_j \cap \mathbb{D}(0,R) \neq \emptyset$ and the 3 disk lemma applies. Let $D = \mathbb{D}(0,R) \cup \bigcup \Delta_j$ and let $g : D \to \mathbb{C}$ by

$$g(z) = \begin{cases} f(z), & z \in \mathbb{D}(0,R) \\ f_i(z), & z \in \Delta_i \end{cases}$$

which is well-defined by our previous work. Also, $g \in H(D)$¿ Now

$$\overline{\mathbb{D}}(0,R) \subseteq D$$

so there exists $R - 1 > R$ so that $\mathbb{D}(0,R_1) \subseteq D$. So the Taylor series expansion of g around $z = 0$ has radius of convergence at least R_1. By Hadamard's Theorem and the uniqueness of the power series we have a contradiction because $R_1 > R$. □

Definition 94. *Let $f \in H(D)$ we say that ∂D is a* ***natural boundary for*** *f if every $z_0 \in \partial D$ is singular for f.*

Theorem 112 (Theorem 2)**.** *For every $D \subseteq \mathbb{C}^*$ there is an $f \in H(D)$ such that ∂D is a natural boundary.*

Proof. Let $\{a_n\} \subseteq D$ with each $z_0 \in \partial D$ a limit point $\{a_n\}$, but $\{a_n\}$ has no limit points in D. Let $f \in H(D)$ be such that $f(a_n) = 0$ for all $n \geq 1$ but $f(z) \neq 0$ otherwise. Then f has no analytic continuation.

□

Theorem 113 (Theorem 3)**.** *(Fabry-Gap Theorem) Let $f(z) = \sum_1^\infty a_k z^{n_k}$ where $\{n_k\} \subseteq \mathbb{Z}^+$ is increasing with radius of convergence $R \in (0, \infty)$. If $\lim_{k\to\infty} \frac{n_k}{k} = \infty$ then every point of $|z| = R$ is singular for f.*

Proof. The proof is difficult and long. See Hille's Analytic Function Theory Volume II. □

8.2.1 Remarks

1. Theorem 3 is close to sharp in the following sense. Consider the function $f(z) = \sum_{k=1}^\infty z^{kN} = \frac{z^N}{1-z^N}$ observe that $\lim_{n\to\infty} \frac{kN}{k} = N$ and N^{th} roots of unity are singular points.
2. Special Case: When $\frac{n_{k+1}}{n_k} > q > 1$ then $n_k \geq q^{k-1}$ so $\lim_{k\to\infty} \frac{n_k}{k} = \infty$. This is called the *Hademard Gap Theorem.*
3. $\sum z^{2^n}$ has a singular point at every point of $\partial\mathbb{D}$.
4. $f(z) = \sum e^{-(2^n)^{1/2}} z^{2^n}$ has $\partial\mathbb{D}$ is natural while this series converges uniformly on $\partial\mathbb{D}$ and $f(e^{i\varphi}) \in C^\infty(\partial\mathbb{D})$.
5. In $\mathbb{C}^n$ for $n \geq 2$ Theorem 2 is false. This is known as Hartog's Phenomenon. "Domains of holomorphy" are in correspondence with pseudoconvex domains.

8.3 Analytic Continuation of Function Elements

Definition 95. .

(a) A ***function element*** *is a pair (f, Δ) where $\Delta \subseteq \mathbb{C}$ is a disk and $f \in H(\Delta)$.*

(b) A ***chain of disks*** *is a sequence $(\Delta_0, \ldots, \Delta_n) = \mathcal{C}$ such that $\Delta_i \cap \Delta_{i+1} \neq \emptyset$.*

(c) Given $(f_0, \Delta_0$ and a chain $\mathcal{C} = \{\Delta_0, \ldots, \Delta_n\}$ if there exist $f_i \in H(\Delta_i)$ such that $f_i = f_{i-1}$ on $\Delta_{i-1} \cap \Delta_i$ then (f_n, Δ_n) is called the ***analytic continuation of*** *(f_0, Δ_0)* ***along*** *$\mathcal{C}$.*

Fact 2. *In (c), f_n is uniquely determined by f_0 and $\mathcal{C}$ if g_1 and f_1 are continuations on Δ_1 then they are the same.*

8.3.1 Example

1. Let $\Delta_0 = \mathbb{D}(1, 1/2)$ and $\Delta_1 = \mathbb{D}(0, 3/4)$. Let $f_0 =$ a branch of $\sqrt{z} = e^{\frac{1}{2}\log z}$. If $\mathcal{C} = \{\Delta_0, \Delta_1\}$ then no analytic continuation of (f_0, Δ_0) along $\mathcal{C}$ exists.

   Proof. Suppose not and such a continuation exists. Then we have $z = f_0^2(z) = f_1^2(z)$ in $\Delta_0 \cap \Delta_1$. By uniqueness we know that $f_1^2 = z$ in Δ_1. So we can write

   $$f(z) = \sum_0^\infty a_n z^n$$

   As $f_1(0) = 0$ we know that $a_0 = 0$ and $z = (\sum_1^\infty a_n z^n)^2$. Thus

   $$z = a_1^2 z^2 + O(z^3)$$

   Then we have $a_1^2 z + O(z^2) = 1$ as $z \to 0$ then $0 = 1$. □

2. Let $\Delta_0 = \mathbb{D}(1, \epsilon)$ and let f_0 be a branch of $\sqrt{z}$ with $f_0(1) = 1$. Let $\mathcal{C}^+$ be a chain covering the upper half of $\partial\mathbb{D}$ and $\mathbb{C}^-$ be a chain covering the lower half of $\partial\mathbb{D}$. Also make sure that 0 is not contained in either chain. By taking $\arg z \in (-\frac{\pi}{2}, \frac{3\pi}{2})$ we get an analytic continuation of f_0 along $\mathcal{C}^+$. And, by taking $\arg(z) \in (-\frac{3\pi}{2}, \frac{\pi}{2})$ we get an analytic continuation of f_0 along $\mathbb{C}^-$. But final elements have different values at -1. So does not work.

3. Let (f_0, Δ_0) be as in example 2, but let $\mathcal{C}_1 = (\mathcal{C}^+, -\mathcal{C}^-)$ where $-\mathcal{C}^-$ mean reverse order. Then (f_0, Δ_0) may be continued along $\mathcal{C}_1$ and at its final element we have $f(1) = -1$.

Definition 96. *Let $\gamma : [a, b] \to \mathbb{C}$ and Δ_0 be a disk containing $\gamma(a)$,*

*(a) We say (f_0, Δ_0) can be **continued along** γ if there exists a chain $\mathcal{C}$ with initial disk Δ_0 and s_i $i = 0, 1, \ldots, n+1$ so that $\gamma[s_i, s_{i+1}] \subseteq \Delta_i$ and (f_0, Δ_0) can be continued along $\mathcal{C}$.*

*(b) (f_n, Δ_n) is an **analytic continuation of** (f_0, Δ_0) **along** γ.*

Theorem 114 (Theorem 4)**.** *Let γ be a curve and Δ_0 a disk centered at $\gamma(a)$. Suppose there exists $\mathcal{C} = (\Delta_0, \ldots, \Delta_n)$ and $\mathcal{C} = (\Delta_0, \Delta_1', \ldots, \Delta_m')$ with associated special points s_i and s_j' so that $\gamma([s_i, s_{i+1}]) \subseteq \Delta_i$ and $\gamma([s_j, s_{j+1}']) \subseteq \Delta_j'$. Also we have (f_0, Δ_0) can be continuad along $\mathcal{C}$ to (f_n, Δ_n) and along $\mathcal{C}'$ to (g_m, Δ_m') then*

$$f_n = g_m, \qquad \text{in } \Delta_n \cap \Delta_m'$$

Proof. Let $I_i = [s_i, s_{i=1}]$ and $I_j' = [s_j', s_{j+1}']$. Define

$$S := \{(i, j) \in \{0, \ldots, n\} \times \{0, 1, \ldots, m\} : I_i \cap I_j' \neq \emptyset \text{ but } f_i \not\equiv g_j \text{ in } \Delta_i \cap \Delta_j'\}$$

We would like to prove that $S = \emptyset$. Suppose that $S \neq \emptyset$. And let $(i, j) \in S$ be so that $i + j$ is minimal. Certainly $i + j \geq 1$ without loss of generality suppose that $s_j' \leq s_i$. Then we must have $i \geq 1$. Now, $S_i \in I_{i-1} \cap I_i \cap I_j'$. By hypothesis we have $f_i = f_{i-1}$ in $\Delta_i \cap \Delta i - 1$ and $f_{i-1} = g_j$ in $\Delta_{i-1} \cap \Delta_j'$ by minimality. Additionally,

$$\Delta_i \cap \Delta_{i-1} \cap \Delta_j' \neq \emptyset$$

By the 3-disk Lemma we know that $f_i = g_j$ which contradicts minimality. So $S = \emptyset$ and $f_n = g_m$ in $\Delta_n \cap \Delta_m$. □

Theorem 115 (Theorem 5)**.** *Suppose $\gamma_0 : [a,b] \to \mathbb{C}$ is a curve and Δ_0 is centered at $\gamma_0(a)$. Let (f_0, Δ_0) be a function element which can be analytically continued along γ. Then there exists an $\epsilon > 0$ so that if $\gamma_1; [a,b] \to \mathbb{C}$ is another curve with the same endpoints and $\sup_{s\in[a,b]} |\gamma(s) - \gamma_1(s)| < \epsilon$ then (f_0, Δ_0) can be analytically continued along γ_1 and moreover the final elements $(f_n, \Delta_{n,0})$ and $(g_m, \Delta_{m,1})$ satisfy $g_m = f_n$ on $\Delta_{n,0} \cap \Delta_{m,1}$.*

Proof. Start with $\mathcal{C} = (\Delta_0, \dots, \Delta_p)$ with associated function elements $h_i \in H(\Delta_i)$ and points s_i for which (f_0, Δ_0) may be continued along γ_0. Set

$$\delta_i = \text{dist}\,(\gamma_0[s_i, s_{i+1}], \partial\Delta_i).$$

Then each $\delta_i > 0$. Set $\epsilon = \frac{1}{2}\min_i\{\delta_i\}$. Thus $\gamma_1([s_i, s_{i+1}]) \subseteq \Delta_i$, which means that the analytic continuation of γ_1 may also be accomplished with (h_i, Δ_i) with function element (h_p, Δ_p).

For uniqueness use Theorem 4 to show that $f_n = h_p$ in $\Delta_p \cap \Delta(n,0)$ and $g_m = h_p$ in $\Delta_p \cap \Delta(m,1)$. Then the 3-disk lemma holds, yielding $f_n \equiv g_m$ in $\Delta(m,1) \cap \Delta(n,0)$.

□

8.4 Monodromy Theorems

Theorem 116 (Theorem 6)**.** *(Monodromy I) Suppose*

(a) $\gamma_t : [a,b] \to \mathbb{C}$, $0 \le t \le 1$ is a 1-parameter continuous family of curves with fixed endpoints.

(b) Δ_0 is a disc containing $\gamma_0(a)$.

(c) (f_0, Δ_0) is a function element which can be analytically continued along γ_t for all $t \in [0,1]$.

Then the final element $(f_n, \Delta(n,0))$ and $(g_m, \Delta(m,1))$ of the analytic continuation of (f_0, Δ_0) along γ_0 and γ_1, respectively, satisfy $f_n \equiv g_m$ in $\Delta(n,0) \cap \Delta(m,1)$.

Proof. Let $S := \{t \in [0,1] :$ uniqueness conclusion holds when γ_1 is replaced with $\gamma_t\}$. Now Theorem 4 implies that $0 \in S$. Let $t_0 = \sup S$. Let $\epsilon > 0$ be associated to γ_0 as in Theorem 5. As the homotopy $H(s,t) = \gamma_t(s)$ is uniformly continuous on $[a,b] \times [0,1]$ we know there is a $\delta > 0$ so that

$$0 \le t \le \delta \implies \sup_{s\in[a,b]} |\gamma_t(s) - \gamma_0(s)| \epsilon$$

Hence $[0,\delta] \subseteq S$. This means that $t_0 \ge \delta > 0$. To show a contradiction suppose that $t_0 < 1$. Let $\epsilon > 0$ associated to γ_{t_0} by Theorem 5 be given. Take $\delta > 0$ so that if $|t - t_0| \le \delta$ we have $\sup_{s\in[a,b]} |\gamma_t(s) - \gamma_{t_0}(s)| < \epsilon$. Take $t_1 \in S$ with $t_1 \le t_0 < t_1 + \delta$. Take $t_2 \in [0,1]$ with $t_0 < t_2 < t_0 + \delta$. By the hypothesis the function element (f_0, Δ_0) may be continued along γ_{t_0} and γ_{t_2} and γ_{t_1}. Let $(g^i_{m_i}, \Delta(m_i, t_i))$ be the final elements for $i = 0,1,2$ of some analytic continuation of (f_0, Δ_0) along γ_{t_i}. Theorem 5 implies that $g^0_{m_0} = g^1_{m_1}$ in $\Delta(m_0, t_0) \cap \Delta(m, t_1)$ and $g^0_{m_0} = g^2_{m_2}$ in $\Delta(m_0, t_0) \cap \Delta(m_2, t_2)$. Also $t_1 \in S$ implies that $g^1_{m_1} = f_n$ in $\Delta(m_1, t_1) \cap \Delta(n, 0)$. when $(f_n, \Delta(n,0))$ is the final element of any analytic continuation of (f_0, Δ_0) along γ_0. Since $\Delta(m_i, t_i)$ and $\Delta(n, 0)$ all contain $\gamma_0(\underline{\ })$ two applications of the 3-disk lemma imply that

$$g^2_{m_2} \equiv f_n, \qquad \text{in } \Delta(m_2, t_2) \cap \Delta(n, 0)$$

Thus $t_2 \in S$. But $t_2 > t_0$ and t_0 is the supremum of S. A contradiction. We conclude that $t_0 = 1$. The same argument applied with $t_0 = 1$ and $t_1 > 1 - \delta$ with δ associated to an ϵ for γ_1 by Theorem 5 implies that $1 \in S$. So we are done.

□

Lemma 23. *Suppose $D \subseteq \mathbb{C}$ is simply connected and that curves $\alpha, \beta : [a,b] \to D$ with the same endpoints. Then there exists a 1-parameter family of curves γ_t with $\gamma_0 = \alpha$, $\gamma_1 = \beta$ with fixed endpoints.*

Proof. If D is convex then we use a straight line homotopy, $H(s,t) = (1-t)\alpha(s) + t\beta(s)$. If D is not convex then use the Riemann mapping theorem. $\square$

Definition 97. *The function element (f_0, Δ_0) is said to admit* ***unrestricted continuation in the domain*** *D if $\Delta_0 \subseteq D$ and (f_0, Δ_0) can be analytically continued along every curve $\gamma : [a,b] \to D$ when $\gamma(a)$ is the center of Δ_0.*

8.4.1 Example

Consider $\Delta_0 = \mathbb{D}(1, \frac{1}{2})$, $f_0 = \sqrt{z}$ and $D = \mathbb{C} \setminus \{0\}$. We have (f_0, Δ_0) admits unrestricted continuation.

Theorem 117 (Theorem 7)**.** *(Monodromy II) Let Δ_0 be a disc in a domain $\Omega \subseteq \mathbb{C}$. Suppose (f_0, Δ_0) admits unrestricted continuation in Ω. Then if Ω is simply connected f_0 has an extension $F \in H(\Omega)$.*

Proof. For each $z \in \Omega$ let $\gamma_z \subseteq \Omega$ connect z_0 (the center of Δ_0) to z. Continue (f_0, Δ_0) along γ_z to the final element (f_z, Δ_z). For $z = z_0, (f_{z_0}, \Delta_{z_0}) = (f_0, \Delta_0)$. We claim that

$$(*) \qquad \text{for } z_1, z_2 \in \Omega \text{ if } \Delta_{z_1} \cap \Delta_{z_1} \neq \emptyset \text{ then } f_{z_1} = f_{z_2} \text{ in } \Delta_{z_1} \cap \Delta_{z_2}.$$

Assuming the claim, we would define $F(z) = f_{z_1}(z)$ where $z_1 \in \Omega$ and $z \in \Delta_{z_1}$. F is well defined by (*) and $F \in H(\Omega)$. Also, $F = f_0$ in Δ_0 by (*) as well. It remains to prove (*).

Let $w \in \Delta_{z_1} \cap \Delta_{z_2}$. Let α a curve from z_0 to w by following γ_{z_1} to z_1 then $[z_1, w]$ and β a similar curve with γ_{z_2}. Reparametrize α, β so the domain is $[a,b]$. And, since $[z_1, w] \subseteq \Delta_{z_1}$ and $[z_2, w] \subseteq \Delta_{z_2}$ it follows that (f_{z_1}, Δ_{z_1}) and (f_{z_2}, Δ_{z_1}) are final elements of an analytic continuation along α and β respectively. As Ω is simply connected there exists a homotopy between α and β. By Monodromy I we conclude that $f_{z_1} = f_{z_2}$ in the intersection $\Delta_{z_1} \cap \Delta_{z_2}$.

$\square$

9 More Normal Families and Picard's Theorems

9.1 Spherical Metric and Derivative

We now wish to extend our study of normal families to Meromorphic families of functions.

Definition 98. *Let $z, w \in \mathbb{C}^*$ and let $P(z), P(w)$ be images of z, w under stereographic projection $(P : \mathbb{C}^* \to S^2)$. We define the* ***distance between*** *z* ***and*** *w to be*

$$\sigma(z,w) := \textit{length of the smaller arc of the great circle of } S^2 \textit{ joining } P(z) \textit{ and } P(w)$$

9.1.1 Facts

1. σ is invariant under rotations of the sphere. Thus $\sigma(z,w) = \sigma(1/z, 1/w)$.

2. On any bounded set in $\mathbb{C}$ we have σ is equivalent to the Euclidean metric:

$$\exists B_R : |z|, |w| < R \implies \frac{1}{B_R}|z-w| \leq \sigma(z,w) \leq B_R|z-w|$$

3. Any subset of $\mathbb{C}^*$ at positive Euclidean distance to the origin 0 has σ equivalent to Euclidean distance between inverse points. There exists a C_ϵ such that $|z|, |w| > \epsilon$ implies that

$$\frac{1}{C_\epsilon}|1/z - 1/w| \leq \sigma(z, w) \leq C_\epsilon |1/z - 1/w|$$

Definition 99. *A sequence of meromorphic functions on a domain $D \subseteq \mathbb{C}$* ***converges locally uniformly on D*** *if the sequence converges uniformly in the spherical metric on compact subsets.*

9.1.2 Note

If $f_n \to f$ locally uniformly then $1/f_n \to 1/f$ locally uniformly.

Theorem 118 (Theorem 1)**.** *If $\{f_n(z)\}$ is a sequence of meromorphic functions on a domain D and it converges locally uniformly to $f(z)$ then (i) $f(z) \in M(D)$ or $f \equiv \infty$. If in addition $f_n \in H(D)$ then (ii) $f \in H(D)$ or $f \equiv \infty$.*

Proof. Part (i)

Case I: Let $z \in D$ be such that $f(z) \neq \infty$. The limit function must be continuous with respect to the spherical metric at z. Let $\epsilon > 0$ be so that $\sigma(f(z), \infty) > \epsilon$. Then there exists $\delta > 0$ such that $|z - w| < \delta$ then $\sigma(f(z), f(w)) < \epsilon$. On the spherical ball of radius ϵ centered at $f(z)$ we know that σ is equivalent to the Euclidean metric. Hence $f_n \to f$ on this $\mathbb{D}(z, \delta)$ in the Euclidean metric, and eventually all the f_n are holomorphic on $\mathbb{D}(z, \delta)$. So f is holomorphic in a neighborhood z.

Case II: Let $z \in D$ where $f(z) = \infty$. Then $\frac{1}{f_n(z)}$ converges in σ to $\frac{1}{f(z)}$ and thus $1/f$ is holomorphic with $\frac{1}{f(z)} = 0$. Then $\frac{1}{f}$ is either a constant (in which case $f \equiv \infty$ via uniqueness) or $\frac{1}{f}$ is analytic and so f is meromorphic at z.

Part (ii)

Suppose f has no points $f(z_0) = \infty$ then via the above argument we know that f is holmorphic. Suppose now that there exists $z_0 \in D$ such that $f(z_0) = \infty$. Then $\frac{1}{f_n}$ (which converges to $\frac{1}{f}$) has

$$\frac{1}{f_n}(z_0) \to 0$$

But $\frac{1}{f_n}$ are zero free near z_0. By Hurwitz's theorem (see below) it follows that $\frac{1}{f_n} \to 0$ locally uniformly these $f_n \to \infty$ locally uniformly with respect to σ. □

Theorem 119 (Hurwitz's Theorem)**.** *Let $\{f_k(z)\} \in H(D)$ and $f_k \to f$ locally uniformly in D. Suppose also that f has a zero of order N at $z_0 \in D$. Then there exists $\rho > 0$ such that for all large k it holds that $f_k(z)$ has N zeroes in $\{|z - z_0| < \rho\}$ and all these zeros converge to z_0 as $k \to \infty$.*

Proof. Choose $\rho > 0$ so small that $f(z) \neq 0$ for $0|z - z_0| < \rho$. Choose $\delta > 0$ so that $|f(z)| \geq \delta$ on $|z - z_0| = \rho$. So, for large k, by local uniform convergence we have $|f_k(z)| > \delta/2$ on $|z - z_0| = \rho$. Also $\frac{f_k'(z)}{f_k(z)} \to \frac{f'}{f}$ uniformly on $|z - z_0| = \rho$. Hence as uniform limits we get

$$\lim_{k\to\infty} \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f_k'}{f_k} dz = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f'}{f} dz$$

By the argument principle we have must have $N_k := \#$ of zeros of f_k in $|z - z_0| < \rho$ converges to $N := \#$ of zeros for f in $|z - z_0| < \rho$.

The second part of the theorem follows by taking ρ smaller and smaller. □

Definition 100. *A family of functions $\mathcal{F} \subseteq M(D)$ is said to be* ***normal*** *if every sequence in $\mathcal{F}$ has a subsequence which converges locally uniformly with respect to σ.*

Definition 101. *Let $\gamma \subseteq \mathbb{C}^*$ be a curve. The* ***spherical length*** *of γ is defined to be the length of γ projected onto $S^2 \subseteq \mathbb{R}^3$.*

Definition 102. *Let $f \in M(D)$. The* ***spherical derivative of*** *f denoted by $f^\sharp$ is defined as*

$$f^\sharp(z) = \frac{2|f'(z)|}{1+|f(z)|^2}$$

(we will show below how to define at poles)

9.1.3 Remark

- The spherical length of the curve $f \circ \gamma$ is given by $\int_\gamma f^\sharp |dz|$.
- Spherical derivative is invariant under inversion.

Proof. $\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$. It follows that $\left(\frac{1}{f}\right)^\sharp = \frac{2\left|\frac{f'}{f}\right|}{1+\left|\frac{1}{f}\right|^2} = \frac{2|f'|}{|f|^2+1} = f^\sharp$. □

Lemma 24. *If $f_k \to f$ with the f_k meromorphic, locally uniformly with respect to σ on D, then $f_k^\sharp \to f^\sharp$ locally uniformly on D with respect to the Euclidean metric.*

Proof. If f is analytic at $z_0 \in D$ this follows immediately. If f is not analytic at $z_0 \in D$ then $\frac{1}{f}$ is analytic and we are done. □

Theorem 120 (Theorem 2)**.** *(Marty's Theorem) A family of functions $\mathcal{F} \in M(D)$ is normal in D if and only if the family $\mathcal{F}^\sharp = \{f^\sharp : f \in \mathcal{F}\}$ are locally uniformly bounded with respect to the Euclidean metric.*

Proof. ($\Leftarrow$) Suppose $\mathcal{F}^\sharp$ is locally uniformly bounded. Let $z_0 \in D$ and say $f^\sharp \leq C$ on $\mathbb{D}(z_0, r)$. Let $z \in \mathbb{D}(z_0, r)$ and γ the straight line segment from z_0 to z. We have

$$\sigma(f(z_0), f(z)) \leq \int_\gamma f^\sharp |dz| \leq C|z - z_0|$$

So, the family $\mathcal{F}$ is locally equicontinuous and $(\mathbb{C}^*, \sigma)$ is compact. Arzelá-Ascoli implies that $\mathcal{F}$ is normal.

($\Rightarrow$) Suppose that $\mathcal{F}^\sharp$ is not bounded on some compact set. In other words there exists $f_k \in \mathcal{F}$ such that $f_k^\sharp$ is not bounded on K. By lemma if f_k had a uniformly convergent subsequence $f_k^\sharp$ would also. This is a contradiction of $f_k^\sharp$ being unbounded. Hence $\mathcal{F}$ is not normal. □

9.1.4 Remark

We may define normal families for $D \subseteq \mathbb{C}^*$. we require $\mathcal{F}$ to be σ-normal on $D \setminus \{\infty\}$ and $g(w) = f(1/w)$ to form a normal meromorphic family in some disk containing 0. In this new case Marty's theorem applies.

9.2 Zalcman's Lemma

Theorem 121 (Theorem 3)**.** *(Zalcman's Lemma) Let $\mathcal{F}$ be a family of meromorphic functions on a domain which is not normal. Then there exist points $z_n \in D$ with $z_n \to z \in D$ and $\rho_n > 0$ with $\rho_n \to 0$ and $f_n \in \mathcal{F}$ such that the functions $g_n(\zeta) := f_n(z_n + \rho_n \zeta)$ converge locally uniformly (with respect to the spherical metric σ) to a non-constant $g \in M(\mathbb{C})$ satisfying*

$$g^\sharp(0) = 1, \qquad g^\sharp(\zeta) \le 1$$

for $\zeta \in \mathbb{C}$.

Proof. By Marty's Theorem there exists a sequence $\{w_n\} \subseteq K \subseteq D$ with K compact and $f_n \in \mathcal{F}$ such that

$$f_n^\sharp(w_n) \to \infty, \qquad \text{and} \quad w_n \to w$$

Without loss of generality we may assume that $w = 0$ and by scaling/translating we can assume that $\overline{\mathbb{D}} \subseteq D$. Define the number R_n as

$$R_n := \max_{|z|=1} f_n^\sharp(z)(1 - |z|)$$

As $w_n \to 0$, after finitely many of them $|w_n| \le 1$. Hence $R_n \ge f_n^\sharp(w_n)(1 - |w_n|)$. This implies that $R_n \to \infty$. Let z_n be the points where the max is obtained. We then have $R_n := f_n^\sharp(z_n)(1 - |z_n|)$. Because $(1 - |z_n|) \le 1$ we know that $f_n^\sharp(z_n) \to \infty$.

Let $\rho_n = \frac{1}{f^\sharp(z_n)}$. Then $\rho_n \to 0$. Now $\mathbb{D}(z_n, 1 - |z_n|) \subseteq \mathbb{D} \subseteq D$ and

$$(1) \qquad \mathbb{D}(z_n, 1 - |z_n|) = \mathbb{D}(z_n, \rho_n R_n).$$

Then if $\zeta \in \mathbb{D}(0, R_n)$ we have $\zeta \mapsto z_n + \rho_n \zeta$ is conformal onto $\mathbb{D}(z_n, 1 - |z_n|)$. Let

$$g_n(\zeta) := f_n(z_n + \rho \zeta), \qquad |\zeta| < R_n.$$

Since $R_n \to \infty$ we know for all $\zeta \in \mathbb{C}$ the function $g_n(\zeta)$ is defined. The spherical chain rule is

$$(f \circ h)^\sharp = f^\sharp(h(\zeta)) \cdot \left| h'(\zeta) \right|$$

So we have

$$g_n^\sharp(\zeta) = \rho_n \cdot f_n^\sharp(z_n + \rho_n \zeta), \qquad |z| < \mathbb{R}_n.$$

Let $R > 0$ and choose N such that $N \ge R_n > R$ for $n \ge N_1$. Maximality tells us that

$$f_n^\sharp(z_n + \rho_n \zeta)(1 - |z_n + \rho_n \zeta|) \le R_n.$$

It follows that

$$\begin{aligned} g_n^\sharp(\zeta) &\le \rho_n \cdot \frac{R_n}{1 - |z_n + \rho_n \zeta|} \\ &\le \rho_n \cdot \frac{R_n}{1 - |z_n| - \rho_n R} \\ &\overset{(1)}{=} \frac{\rho_n R_n}{\rho_n R_n - \rho_n R} \\ &\overset{(2)}{=} \frac{1}{1 - \frac{R}{R_n}} \end{aligned}$$

By Marty's we know that $\{g_n^\sharp\}$ is bounded locally which implies that $\{g_n\}$ is normal. Hence letting $R \nearrow \infty$ and diagonalizing we get

$$g_n n(\zeta) \to g(\zeta), \qquad \text{locally for all } \zeta \in \mathbb{C}$$

From (2) and the Lemma preceding Marty's Theorem we know that $g_n^\sharp(\zeta) \leq 1$. This implies that $g^\sharp(\zeta) \to 1$. Also, $g_n^\sharp(0) = \rho_n f_n^\sharp(z_n) = 1$. □

9.2.1 Example

Let $\mathcal{F} = \{$all conformal maps of domain D which omit $0\}$. If $\mathcal{F}$ is not normal then z_n, ρ_n, g_n satisfy the conclusion of Zalcman's lemma. We have the following chain of facts

- $g^\sharp(0) = 1$, so g is not constant.
- $g \in H(\mathbb{C})$.
- g is conformal.
- But g is zero free.

This is a contradiction of Hurwitz Theorem and thus $\mathcal{F}$ is normal.

9.3 Omitted values and Montel's Theorem II

Theorem 122 (Theorem 4)**.** *(Montel's Theorem II) Let $a, b, c \in \mathbb{C}^*$ be distinct. Let $\mathcal{F}$ be a family of meromorphic functions of a domain D whose values omit a, b and c then $\mathcal{F}$ is normal.*

Proof. Because normality is local, let $D = \mathbb{D}$. Let $T \in \mathfrak{M}$ be the möbius transformation which maps $a \mapsto 0, b \mapsto 1, c \mapsto \infty$. Then $\mathcal{F}^* = T \circ \mathcal{F}$ omits $0, 1\infty$. In particular we may take roots of the $f \in \mathcal{F}^*$. Let $\mathcal{F}_k^* := \{g : g^{2^k} \in \mathcal{F}^*\}$. Now $\mathcal{F}_k^*$ omits $0, \infty$ and all 2^k roots of 1. Also,

$$\mathcal{F}_k^* \text{ normal } \iff \mathcal{F}^* \text{ normal } \iff \mathcal{F} \text{ normal}$$

Assume for a contradiction that $\mathcal{F}^*$ is not normal. We can apply Zalcman's lemma to conclude that for each $k \geq 1$ there exists $G_k(\zeta)$ such that

- The G_k are entire.
- $G_k^\sharp(\zeta) \leq 1$.
- $G_k^\sharp(0) = 1$.
- G_k is a limit of restrictions of functions in $\mathcal{F}_k^*$ appropriately scaled.

These appropriately scaled restrictions avoid the $(2^k)^{th}$-roots of unity. By Hurwitz, G_k also avoids these values. Consider $\mathcal{G} = \{G_k L k \geq 1\}$. Let $\mathcal{G}$ be a local uniform limit of subsequence of G_k. We know that

- $G^\sharp(0) = 1$, which implies G is non-constant.
- G omits the $(2^k)^{th}$-roots of unity for all k.
- G omits 0.

- $G \in H(\mathbb{C})$ and so G is an open mapping as well and thus it omits S^1 (since the 2^k roots of unity are dense in S^1).

It follows that $|G| < 1$ or $|1/G| < 1$ (note that since G avoid 0) we know that G and $1/G$ are both entire, which contradicts Liouville. Thus the family is normal. □

Definition 103. *Let $f \in M(\{z : 0 < |z - z_0| < \delta\})$ for some $\delta > 0$. The value $w_0 \in \mathbb{C}^*$ is an **omitted value of** f **at** z_0 if there exists an $\epsilon > 0$ such that $f(z) \neq w_0$ for $0 < |z - z_0| < \epsilon$.*

9.3.1 Remark

$w + 0$ is not omitted at z_0 if and only if $\exists z_n \to z_0 \ni$ such that $f(z_n) = w_0$.

9.3.2 Example

$e^{1/z}$ at $z_0 = 0$ has 0 and ∞ are omitted.

9.4 Picard's Theorems

Theorem 123 (Theorem 5)**.** *(Big Picard) Let $f \in M(\{0 < |z - z_0| < \delta\})$, $\delta > 0$ and $z_0 \in \mathbb{C}^*$. If f omits 3 distinct points at z_0 then $f(z)$ has a meromorphic extension to $\mathbb{D}(z_0, \delta)$.*

Proof. Via Möbius transformations we may assume that $z_0 = 0$ and that f omits 0 and ∞. Then there exists a disc Δ, possibly smaller such that $f \in H(\Delta \setminus \{0\})$. Let $\{\epsilon_n\}$ be such that $\epsilon_n \searrow 0$, $\epsilon_1 < 1$. Let $g_n(z) = f(\epsilon_n z)$ for $z \in \Delta \setminus \{0\}$. Then $\mathcal{G} = \{g_n : n \geq 1\}$ omits 0, ∞ and some third value. So $\mathcal{G}$ is normal via Montel. Hence a subsequence of $g_n \to g$ locally uniformly. By Theorem 1, we know that $g \in H(\Delta \setminus \{0\})$ or $g \equiv \infty$.

Case 1: Assume $g \not\equiv \infty$. Let $o < \rho < \delta$ be such that $\rho \in \Delta \setminus \{0\}$ be given. Let

$$M_1 = \max_{|z|=\rho} |g(z)|$$

$$M_2 = \max_{|z|=\rho} |f(z)|$$

Let $\epsilon > 0$ be given for all but a finite number of n's.

$$|g_n(z)| \leq M - 1 + \epsilon, \qquad |z| = \rho$$

Hence $|f(z)| < \max\{M_1 + \epsilon, M_2\} := M$ on $\{|z| = \epsilon_n \rho\} \cup \{|z| = \rho\}$. By the maximum principle we know that

$$|f(z)| < M, \qquad z \in A(0, \epsilon_n \rho, \rho)$$

Let $n \to \infty$ to get $|f(z)| < M$ in $\Delta \setminus \{0\}$. So, f has an isolated singularity and thus a holomorphic extension.

Case 2: Assume $g \equiv \infty$. We can repeat with the function $\frac{1}{f}$. It follows that $\frac{1}{f}$ extends analytically to be 0 at $z = 0$ and thus f extends to have a pole at 0. □

Theorem 124 (Theorem 6)**.** *(Little Picard) A non-constant entire function assumes every value in $\mathbb{C}$ with at most one possible exemption.*

Proof. Such a function is holomorphic in a deleted neighborhood of ∞. It already omits one value, if it omitted two more, there would be a meromorphic extension. If it does have a meromorphic extension then $f \in M(C^*)$ and thus f must be a polynomial. But polynomials hit every point. □