San Francisco State University
ECON 715

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Fall 2016

Midterm Exam 1

September 22, 2016

Time: 1 hour, 30 minutes

Name: ______________________________

Instructions:
1. One double-sided sheet with any content is allowed.
2. Calculators are NOT allowed.
3. Show all the calculations, and explain your steps.
4. If you need more space, use the back of the page.
5. Fully label all graphs.

1. (10 points). Consider the following matrices:

$$A = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(a) Calculate the matrix product AA and BB.

$$AA = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}$$

$$BB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) Circle **all** the correct statements about the matrices A and B in this question:

i. $\boxed{A \text{ and } B \text{ are both square matrices.}}$

ii. A and B are both invertible.

iii. A and B are both singular.

iv. A and B are both symmetric.

v. $\boxed{A \text{ and } B \text{ are both idempotent.}}$

2. (5 points). Suppose $A_{5\times 1}$ and $B_{1\times 6}$ are two matrices. Circle **all** the correct statements:

(a) The matrix product AB is not defined.

(b) $\boxed{\text{The matrix product } BA \text{ is not defined.}}$

(c) $\boxed{\text{The matrix product } AB \text{ has dimension } 5 \times 6.}$

(d) The matrix product BA has dimension 6×5.

(e) The matrix product AB has dimension 1×1.

3. (10 points). Consider the following system of equations:

$$\begin{aligned} 3x_1 - 6x_2 &= d_1 \\ x_1 - 2x_2 &= d_2 \\ 7x_1 + 5x_2 + 4x_3 &= d_3 \end{aligned}$$

(a) Write the above system in matrix form $Ax = d$.

$$\underbrace{\begin{bmatrix} 3 & -6 & 0 \\ 1 & -2 & 0 \\ 7 & 5 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}_{d}$$

(b) The above system of equations has a unique solution. True/False, circle the correct answer, and provide a proof.

A $n \times n$ system of linear equations has a unique solution if and only if the determinant of the coefficient matrix $|A| \neq 0$ (i.e. A is invertible). Using Laplace expansion along the 3rd column (easiest, because of two zeros), the determinant is

$$|A| = 4 \cdot \begin{vmatrix} 3 & -6 \\ 1 & -2 \end{vmatrix} = 4 \cdot (3 \cdot -2 - (1 \cdot -6)) = 0$$

Therefore, the given system cannot have a unique solution. It can either have no solution, or infinitely many solutions.

4. (10 points). Consider the following system of equations:

$$\begin{aligned} x_1 - x_2 + x_3 &= d_1 \\ x_2 - x_3 &= d_2 \\ -x_2 + 3x_3 &= d_3 \end{aligned}$$

(a) Using Cramer's rule, solve for x_3. Denote the solution by x_3^*.

$$x_3^* = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 1 & -1 & d_1 \\ 0 & 1 & d_2 \\ 0 & -1 & d_3 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 3 \end{vmatrix}} = \frac{d_3 + d_2}{3 - 1} = \frac{d_3 + d_2}{2}$$

(b) Based on your result in the last section, find the change in x_3^* due to an increase in d_2 by 1 unit (i.e. $\Delta d_2 = 1$).

$$\Delta x_3^* = \frac{\Delta d_2}{2} = \frac{1}{2}$$

Thus, x_3^* will increase by $\frac{1}{2}$ a unit.

5. (10 points). Consider the matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Let $|C_{ij}|$ be the cofactor of the element a_{ij}. Find the cofactors of the elements a_{31} and a_{32}.

$$|C_{31}| = (-1)^{3+1} \begin{vmatrix} \square & a_{12} & a_{13} \\ \square & a_{22} & a_{23} \\ \square & \square & \square \end{vmatrix} = a_{12}a_{23} - a_{22}a_{13}$$

$$|C_{32}| = (-1)^{3+2} \begin{vmatrix} a_{11} & \square & a_{13} \\ a_{21} & \square & a_{23} \\ \square & \square & \square \end{vmatrix} = -\left(a_{11}a_{23} - a_{21}a_{13}\right)$$

6. (10 points). Let $A_{3\times3}$ and $B_{3\times3}$ be two matrices, with determinants $|A| = 5$ and $|B| = 10$. Calculate the following (show steps):

(a) $|A'|$

$$|A'| = |A| = 5$$

(b) $|AB|$

$$|AB| = |A| \cdot |B| = 5 \cdot 10 = 50$$

(c) $|A^{-1}|$

$$\left|A^{-1}\right| = \frac{1}{|A|} = \frac{1}{5}$$

(d) $|2 \cdot B|$

$$|2 \cdot B| = 2^3 \, |B| = 8 \cdot 10 = 80$$

(e) $|A^{-1}B|$

$$\left|A^{-1}B\right| = \left|A^{-1}\right| \cdot |B| = \frac{1}{5} \cdot 10 = 2$$

7. (5 points). Let A, B, C be matrices such that the product ABC is well defined. Prove that $(ABC)' = C'B'A'$. You are allowed to use the result proved in class, that $(AB)' = B'A'$ (transpose of a product of two matrices is equal to the product of transposes, in reverse order).

We simply apply the property $(AB)' = B'A'$ twice:

$$\begin{aligned}(ABC)' &= (BC)'\,A' \\ &= C'B'A'\end{aligned}$$

8. (5 points). Let $A_{n\times m}$ and $B_{h\times k}$ be two matrices. Circle the correct answer:

   (a) The dimension of AB is $m \times h$.

   (b) The dimension of Hadamard product $A \circ B$ is $n \times k$.

   (c) The dimension of $A + B$ is $n \times k$.

   (d) Hadamard product $A \circ B$ is defined only if $n = h$ and $m = k$.

9. (5 points). Consider the following matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Calculate the inverse of the matrix if it exists, or prove that A is not invertible.

Calculating the determinant via Laplace expansion along the 3rd row:

$$|A| = 0 \cdot |C_{31}| + 0 \cdot |C_{32}| + 0 \cdot |C_{33}| + 0 \cdot |C_{34}| = 0$$

Thus, A is not invertible. Recall, that if A is invertible,

$$A^{-1} = \frac{1}{|A|} adj\,(A)$$

10. (20 points). Suppose that the input-output matrix for some economy is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(a) Find the technology matrix (Leontief matrix) T.

$$T = I - A = \begin{bmatrix} 1 - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & 1 - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & 1 - a_{33} \end{bmatrix}$$

(b) Suppose the consumer demand vector is $d = \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix}'$. The social planner wants to find the output levels in each industry, $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}'$, which will satisfy simultaneously the consumer demand and the inter-industry demand. Write the social planner's problem in matrix form.

$$\begin{bmatrix} 1 - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & 1 - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & 1 - a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

(c) Write the solution to $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}'$, obtained via matrix inversion. No need to calculate the output; just write the formula.

$$x^* = T^{-1}d$$

(d) Write the Cramer's rule formula for calculating the required output in industry 3, x_3. No need to calculate the output; just write the formula.

$$x_3^* = \frac{|T_3|}{|T|}$$

where

$$|T| = \begin{vmatrix} 1-a_{11} & -a_{12} & -a_{13} \\ -a_{21} & 1-a_{22} & -a_{23} \\ -a_{31} & -a_{32} & 1-a_{33} \end{vmatrix}, \quad |T_3| = \begin{vmatrix} 1-a_{11} & -a_{12} & d_1 \\ -a_{21} & 1-a_{22} & d_2 \\ -a_{31} & -a_{32} & d_3 \end{vmatrix}$$

11. (10 points). Consider the following Matlab program:

```
1      % My program
2      n = 3;
3      A = sym('a',[n,n]);
4      d = sym('d',[n,1]);
5
6      % Section 1
7      x1 = inv(A)*d;
8
9      % Section 2
10     d_A = det(A);
11     for j = 1:size(A,2)
12         Aj = A; Aj(:,j) = d;
13         x2(j,1) = det(Aj)/d_A;
14     end
15
16     % Section 3
17     x3 = A\d;
```

(a) What is the purpose of the above program?
Solve systems of linear equations symbolically.

(b) What is the purpose of the command in section 1 (line 7)?
Solving the linear system via matrix inversion.

(c) What is the purpose of section 2?
Solving the system via Cramer's rule?

(d) What is the purpose of section 3?
Solving the system via Gaussian elimination.

(e) The computer debugger issued a warning for line 7. What does the warning announcement say?
Matrix inversion is slower and less accurate than $A\backslash B$, which solves the system using Gaussian elimination.