San Francisco State University
ECON 715

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Fall 2017

Midterm Exam 1

September 21, 2017

Time: 1 hour, 30 minutes

Name: ______________________________

Instructions:
1. One double-sided sheet with any content is allowed.
2. Calculators are NOT allowed.
3. Show all the calculations, and explain your steps.
4. If you need more space, use the back of the page.
5. Fully label all graphs.

1. (10 points). Consider the following matrices:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(a) Calculate the matrix product AB.

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(b) Calculate the matrix product BA.

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

2. (5 points). Consider the matrix

$$A = \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}$$

Circle **all** the correct statements about A:

(a) $\boxed{A \text{ is square matrix.}}$

(b) A is invertible.

(c) $\boxed{A \text{ is singular.}}$

(d) A is symmetric.

(e) $\boxed{A \text{ is idempotent.}}$

3. (5 points). Suppose $A_{3\times 5}$ and $B_{7\times 5}$ are two matrices. Circle **all** the correct statements:

(a) $\boxed{\text{The matrix product } AB' \text{ has dimension } 3 \times 7.}$

(b) The matrix product $A'B$ has dimension 5×7.

(c) The matrix product $B'A'$ has dimension 5×5.

(d) The matrix product $A'B'$ has dimension 3×5.

(e) $\boxed{\text{The matrix product } BA' \text{ has dimension } 7 \times 3.}$

4. (10 points). Consider the following system of equations:

$$\begin{aligned} 3x_1 - 6x_3 &= d_1 \\ x_1 + 2x_3 &= d_2 \\ 13x_1 + 7x_2 - 11x_3 &= d_3 \end{aligned}$$

(a) Write the above system in matrix form $Ax = d$.

$$\underbrace{\begin{bmatrix} 3 & 0 & -6 \\ 1 & 0 & 2 \\ 13 & 7 & -11 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}_{d}$$

(b) The above system of equations has a unique solution. $\boxed{\text{True}}$/False, circle the correct answer, and provide a proof.

A $n \times n$ system of linear equations has a unique solution if and only if the determinant of the coefficient matrix $|A| \neq 0$ (i.e. A is invertible). Using Laplace expansion along the 3rd row (easiest, because of two zeros), the determinant is

$$|A| = -7 \cdot \begin{vmatrix} 3 & -6 \\ 1 & 2 \end{vmatrix} = -7 \cdot (3 \cdot 2 - (1 \cdot -6)) = -7 \cdot 12 = -84 \neq 0$$

Therefore, the given system has a unique solution.

5. (10 points). Consider the following system of equations:

$$\begin{aligned} x_1 - x_2 + x_3 &= d_1 \\ x_2 - x_3 &= d_2 \\ -x_2 + 3x_3 &= d_3 \end{aligned}$$

(a) Using Cramer's rule, solve for x_3. Denote the solution by x_3^*.

$$x_3^* = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 1 & -1 & d_1 \\ 0 & 1 & d_2 \\ 0 & -1 & d_3 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 3 \end{vmatrix}} = \frac{d_3 + d_2}{2}$$

(b) Based on your result in the last section, find the change in x_3^* due to an increase in d_3 by 1 unit (i.e. $\Delta d_3 = 1$).

$$\Delta x_3^* = \frac{1}{2}\Delta d_3 = \frac{1}{2}$$

Thus, x_3^* will increase by half a unit.

6. (10 points). Let $A_{3\times 3}$ and $B_{3\times 3}$ be two matrices, with determinants $|A| = 3$ and $|B| = 4$. Calculate the following (show steps):

(a) $|A'|$

$$|A'| = |A| = 3$$

(b) $|AB|$

$$|AB| = |A| \cdot |B| = 3 \cdot 4 = 12$$

(c) $|A^{-1}|$

$$\left|A^{-1}\right| = \frac{1}{|A|} = \frac{1}{3}$$

(d) $|2 \cdot B|$

$$|2 \cdot B| = 2^3 \, |B| = 8 \cdot 4 = 32$$

(e) $|A^{-1}B|$

$$\left|A^{-1}B\right| = \left|A^{-1}\right| \cdot |B| = \frac{1}{3} \cdot 4 = 1\frac{1}{3}$$

7. (5 points). Let A, B, C be matrices such that the product ABC is well defined. Prove that $(ABC)' = C'B'A'$. You are allowed to use the result proved in class, that $(AB)' = B'A'$ (transpose of a product of two matrices is equal to the product of transposes, in reverse order).

We simply apply the property $(AB)' = B'A'$ twice:

$$\begin{aligned} (ABC)' &= (BC)'\,A' \\ &= C'B'A' \end{aligned}$$

8. (5 points). Let $A_{n\times n}$ be a square matrix, such that $AA = A$. Moreover, suppose that A^{-1} exists. Prove that $A = I$, i.e. A must be identity matrix.

The given $AA = A$ means that A is idempotent matrix. Premultiply $AA = A$ by A^{-1}, gives:

$$\begin{aligned} A^{-1}AA &= A^{-1}A \\ IA &= I \\ A &= I \end{aligned}$$

Thus, the above proves that idempotent matrices are not invertible, with the only exception of identity matrix.

9. (10 points). Consider the following matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

Calculate the inverse of the matrix if it exists, or prove that A is not invertible.

We proved in class that the determinant of a triangular matrix is equal to the product of the elements on its diagonal. Thus,

$$|A| = a_{11} \cdot a_{22} \cdot 0 \cdot a_{44} = 0$$

Thus, A is not invertible. Recall, that if A is invertible, the inverse is given by:

$$A^{-1} = \frac{1}{|A|} adj\,(A)$$

10. (20 points). Suppose that the input-output matrix for some economy is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(a) Find the technology matrix (Leontief matrix) T.

$$T = I - A = \begin{bmatrix} 1 - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & 1 - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & 1 - a_{33} \end{bmatrix}$$

(b) Suppose the consumer demand vector is $d = \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix}'$. The social planner wants to find the output levels in each industry, $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}'$, which will satisfy simultaneously the consumer demand and the inter-industry demand. Write the social planner's problem in matrix form.

$$\underbrace{\begin{bmatrix} 1 - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & 1 - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & 1 - a_{33} \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}_{d}$$

$$Tx = d$$

(c) Write the solution to $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}'$, obtained via matrix inversion. No need to calculate the output; just write the formula.

$$x^* = T^{-1}d$$

(d) Consumers' demand in some country changed from $d = \begin{bmatrix} 5 & 10 & 10 \end{bmatrix}'$ to $\tilde{d} = \begin{bmatrix} 20 & 10 & 10 \end{bmatrix}'$. This can reflect an increase in demand for health care as population ages. As a result, the social planner's production plan changed from $x = \begin{bmatrix} 80 & 85 & 85 \end{bmatrix}'$ to $x = \begin{bmatrix} 140 & 130 & 130 \end{bmatrix}'$. Explain intuitively, why the production of all goods can increase even though the demand increased only for good 1.

The production of all goods requires inputs from all industries, potentially. Thus, higher demand for good 1 by the consumers causes higher demand by industry 1 for inputs produced by *all* industries.

11. (10 points). Consider the following Matlab program:

```
1
2    clear %Clearing memory
3    close all
4
5    x = linspace(-10,10,100);
6    y = x.^3;
7
8    figure(1)
9    plot(x,y, 'LineWidth',2)
10   title('y = x^3', 'FontSize',14)
11   xlabel('x', 'FontSize',14)
12   ylabel('y', 'FontSize',14, 'Rotation', 0) %Making the label vertical
```

(a) What is the purpose of the above program?

Plotting a graph of a function $y = x^3$.

(b) What is the purpose of the statements that follow after the % symbol?

These are comments that explain how the program works. Comments are useful for other readers of the code, as well as for the programmer himself.

(c) What does the command in line 5 do?

Creates a grid of 100 equally spaced points between -10 and 10.