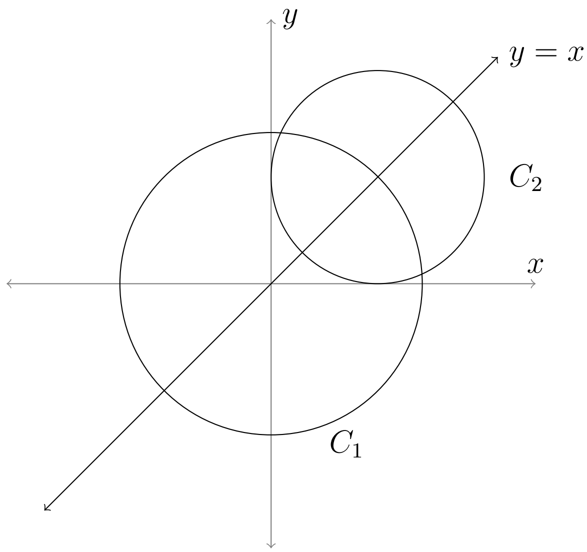
Worksheet for Week 1: Circles and lines

This worksheet is a review of circles and lines, and will give you some practice with algebra and with graphing. Also, this worksheet introduces the idea of "tangent lines" to circles. Later on in Math 124, you'll learn how to find tangent lines to many other types of curves.

1. Two circles, called C_1 and C_2, are graphed below. The center of C_1 is at the origin, and the center of C_2 is the point in the first quadrant where the line $y = x$ intersects C_1. Suppose C_1 has radius 2. C_2 touches the x and y axes each in one point. What are the equations of the two circles?



Solution: The equation for a circle with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2.$$

The circle C_1 is centered at $(0, 0)$ and has radius 2, so its equation is $x^2 + y^2 = 4$.

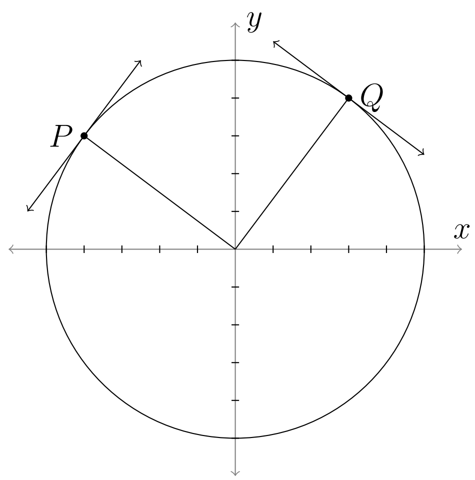
C_2's center is at the point where the line $y = x$ meets C_1. So we need to solve for x and y provided that the two equations

$$x^2 + y^2 = 4 \qquad x = y$$

are true. Substituting $x = y$ into the left equation and solving for x and y, we get two solutions: $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. Since the point we want is in the first quadrant, we know the center of C_2 must be $(\sqrt{2}, \sqrt{2})$. We know C_2 just touches the x and y axes in one point, so the radius of C_2 is $\sqrt{2}$. Finally, the equation of C_2 is

$$(x - \sqrt{2})^2 + (y - \sqrt{2})^2 = 2.$$

2. Let C be the circle of radius 5 centered at the origin. The **tangent line** to C at a point Q is the line through Q that's perpendicular to the radial line connecting Q to the center. (See picture.) Use this information to find the equations of the tangent lines at P and Q below.



Note: Later in Math 124, you'll learn how to find tangent lines to curves that are not circles!

Solution: If one line has slope m and another line has slope n, then the lines are perpendicular if $m = -\frac{1}{n}$.

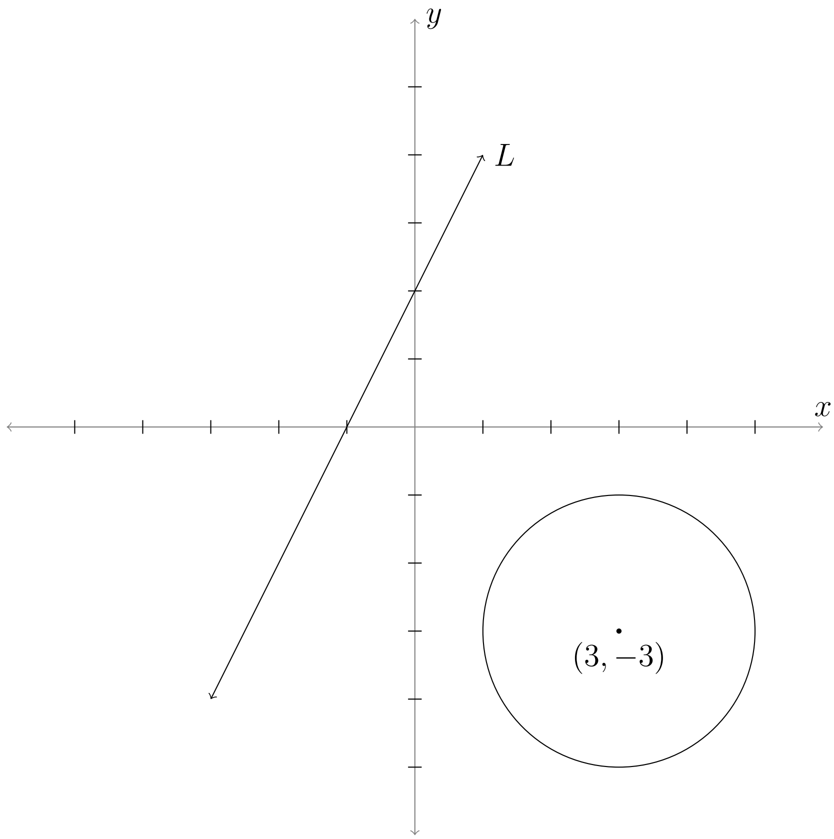
The radial line through $(0, 0)$ and $Q = (3, 4)$ has slope $\frac{4}{3}$. So the tangent line must have slope $-\frac{3}{4}$. Now use the point-slope formula to get the equation of the line through $(3, 4)$ with slope $-\frac{3}{4}$:

$$y = -\frac{3}{4}x + \frac{25}{4}.$$

Similarly, the radial line through $(0, 0)$ and $P = (-4, 3)$ has slope $-\frac{3}{4}$. So the tangent line through P has slope $\frac{4}{3}$. Its equation is

$$y = \frac{4}{3}x + \frac{25}{3}.$$

3. Sketch the circle of radius 2 centered at $(3, -3)$ and the line L with equation $y = 2x + 2$. Find the coordinates of all the points on the circle where the tangent line is perpendicular to L.



Solution: The line L has slope 2, so we want to find tangent lines to the circle with slope $-\frac{1}{2}$. We can do this by finding points P on the circle so that the radial line from $(3, -3)$ to P has slope 2. In other words, we want points (x, y) so that the following two equations are true:

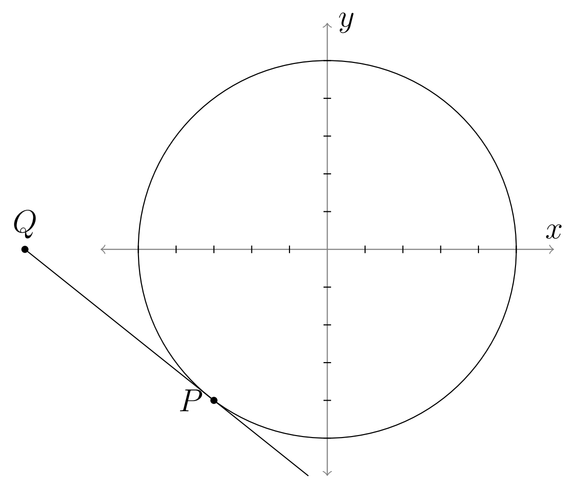
$$(x-3)^2 + (y+3)^2 = 4 \qquad \frac{y+3}{x-3} = 2.$$

Simplifying, the second equation becomes $y = 2x - 9$. Plug this in to the first equation and solve for x and y. We get two points:

$$\left(3 + \frac{2}{\sqrt{5}}, -3 + \frac{4}{\sqrt{5}}\right) \quad \text{and} \quad \left(3 - \frac{2}{\sqrt{5}}, -3 - \frac{4}{\sqrt{5}}\right).$$

4. Draw the circle with equation $x^2+y^2 = 25$ and the points $P = (-3,-4)$ and $Q = (-8,0)$. Explain why P is on the circle. Is the line through P and Q tangent to the circle? How do you know?

Solution:



This circle is centered at the origin and has radius 5. The point P is on the circle because its coordinates satisfy the circle's equation: $(-3)^2 + (-4)^2 = 25$.

Draw the line through Q and P. It looks tangent to the circle, but we need to check with the definition in Question 2. The radial line from the center of the circle to P has slope $\frac{4}{3}$, so the tangent line at P must have slope $-\frac{3}{4}$. But the line through P and Q has slope $-\frac{4}{5}$, so it actually is **not** tangent to the circle.