1. Let $A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

   (a) Explain why the vector $A_\theta \begin{bmatrix} x \\ y \end{bmatrix}$ is a rotation of $\begin{bmatrix} x \\ y \end{bmatrix}$ by θ. Justify this with trigonometry. Hint, one way goes by converting to polar coordinates then converting back. Two useful identities are the ones for $\cos(u+v)$ and $\sin(u+v)$.

   (b) For any $\theta, \psi \in \mathbb{R}$, show that $A_\theta A_\psi = A_{\theta+\psi}$ and explain what this means geometrically.

   (c) What is the inverse matrix of A_θ? Explain why.

2. Ada is learning about moments of inertia in her physics class and sees an opportunity to put her linear algebra skills to the test. Given a vector $\vec{b} \in \mathbb{R}^3$, she wants to construct the inertia matrix B that when multiplied by any vector $\vec{x}$ yields the cross product $\vec{b} \times \vec{x}$.

   (a) If

   $$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

   then what is $\vec{b} \times \vec{x}$?

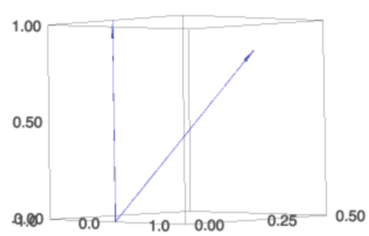
(b) Use your answer in part (a) to deduce B. The entries should only involve b_i.

(c) Is the linear transformation defined by multiplication by B a bijection? Make sense of your answer based on what you know about cross products.

3. Erika, a computer animator, wants to create a scene where a character waves gracefully back and forth in three dimensions. She knows that the rotation matrices about the x,y and z axes are given by:

$$A_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad A_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \qquad A_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) She wants the vector $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ to rotate with respect to the x-axis to make an angle of $\pi/3$ radians with the y-axis. Write down the matrix, A, that causes this transformation. (Be careful!)



(b) What is $A\vec{e}_3$?

(c) Describe in words what A^6 does to a vector.

(d) She also wants the hand to rotate a little around the y axis. She would like $A\vec{e}_3$ to rotate by $\pi/4$ with respect to the y-axis. Write the matrix, B, that corresponds to this transformation.

(e) What is $B(A\vec{e}_3)$?

(f) Explain why $B(A\vec{e}_3) = (BA)\vec{e}_3$. Do this without doing any calculations.

(g) Now she wants to animate this smoothly. She calculates that

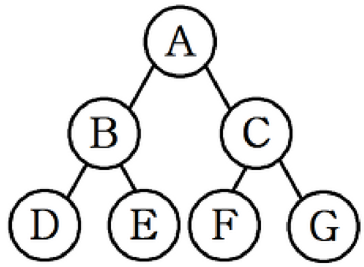
$$BA = \begin{pmatrix} \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} \\ \frac{1}{4}\sqrt{2} & \frac{1}{2}\sqrt{3} & \frac{1}{4}\sqrt{2} \\ \frac{1}{4}\sqrt{3}\sqrt{2} & -\frac{1}{2} & \frac{1}{4}\sqrt{3}\sqrt{2} \end{pmatrix}.$$

Suppose she would like the wave to take 1 second and appear smooth. Erika knows that the function $f(t) = (1-t)a + bt$ is such that $f(0) = a$ and $f(1) = b$ and it smoothly interpolates between a and b as t ranges from 0 to 1.

Using this, write a collection of matrices $C(t)$ such that $C(0) = \mathbf{I}$ and $C(1) = BA$.

(h) What would the collection $C(t)^{-1}$ animate?

4. Label the binary tree with 7 nodes as follows:



(a) Write the adjacency matrix, Q, for a binary tree with 7 nodes.

(b) Now suppose that a random walker moved randomly from node to node with equal probability. Write the transition probability matrix, P.

(c) What does the entry in Row A and Column F of Q^{10} represent?

(d) Here is P^{10}.

$$P^{10} = \begin{pmatrix} A & B & D & E & C & F & G \\ \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{275}{486} & 0 & 0 & \frac{211}{486} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{97}{486} & \frac{97}{486} & 0 & \frac{65}{486} & \frac{65}{486} \\ \frac{1}{3} & 0 & \frac{97}{486} & \frac{97}{486} & 0 & \frac{65}{486} & \frac{65}{486} \\ 0 & \frac{211}{486} & 0 & 0 & \frac{275}{486} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{65}{486} & \frac{65}{486} & 0 & \frac{97}{486} & \frac{97}{486} \\ \frac{1}{3} & 0 & \frac{65}{486} & \frac{65}{486} & 0 & \frac{97}{486} & \frac{97}{486} \end{pmatrix}.$$

Where is a random walk started at C most likely to be after 10 steps? Explain why.