San Francisco State University
ECON 715

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Midterm Exam 2

October 27, 2017

Time: 1 hour, 30 minutes

Name: ______________________________

Instructions:
1. One double-sided sheet with any content is allowed.
2. Calculators are NOT allowed.
3. Show all the calculations, and explain your steps.
4. If you need more space, use the back of the page.
5. Fully label all graphs.

1. (5 points). Prove that

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$$

This is a limit of the form $\frac{0}{0}$. Using L'Hôpital's rule,

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{x \to 0} \frac{\frac{1}{1+x}}{1} = 1$$

2. (10 points). Prove that
$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$$
Remark: This is an important result in finance. When periodic interest rate is r, and the interest is compounded n times per period, the amount after one period is: $(1+r/n)^n$. For example, r is usually annual percentage rate (APR), and $n = 12$ if the interest compounds monthly, and $n = 360$ if interest compounds daily, etc. If interest compounds every instant, the amount can be approximated by e^r - continuously compounded formula.

We can rewrite the required function in a way that allows the use of substitution rule and L'Hôpital's rule:

$$\left(1+\frac{r}{n}\right)^n = \exp\left(n\ln\left(1+\frac{r}{n}\right)\right) = \exp\left(r\frac{\ln\left(1+\frac{r}{n}\right)}{r/n}\right) = \exp\left(r\frac{\ln(1+x)}{x}\right)$$

where $x = r/n$. Note that

$$\begin{aligned} \lim_{n\to\infty} x &= 0 \\ \lim_{x\to 0}\frac{\ln(1+x)}{x} &= 1 \quad \text{[We proved previously, using L'Hôpital's rule]} \end{aligned}$$

Thus, the required limit can be written as

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = \lim_{x\to 0}\exp\left(r\frac{\ln(1+x)}{x}\right) = \lim_{y\to 1}\exp(ry) = e^r$$

3. (10 points). Suppose the production function of a firm is $Y = AK^{\theta}L^{1-\theta}$, where $0 < \theta < 1$, Y is output, A is productivity parameter, K is physical capital, and L is labor. This is known as the Cobb-Douglas production function.

   (a) Calculate the elasticities of output with respect to capital and labor.

   Taking logs of the production function:

   $$\ln Y = \ln A + \theta \ln K + (1-\theta)\ln L$$

   Then the elasticities are:

   $$\begin{aligned} \eta_{Y,K} &= \frac{\partial \ln Y}{\partial \ln K} = \theta \\ \eta_{Y,L} &= \frac{\partial \ln Y}{\partial \ln L} = 1-\theta \end{aligned}$$

   (b) Suppose that $\theta = 0.35$, and the firm increases the amount of capital employed by 1%. As a result, the output will increase approximately by <u>0.35%</u>. (fill in the blank).

   (c) Suppose that $\theta = 0.35$, and the firm increases the amount of labor employed by 1%. As a result, the output will increase approximately by <u>0.65%</u>. (fill in the blank).

4. (10 points). Suppose a firm has a total cost function of $TC(Q)$, where Q is output level. Let the average total cost function be $ATC(Q) = TC(Q)/Q$, and the marginal cost function $MC(Q) = TC'(Q)$. Prove that $ATC(Q)$ is increasing in Q if and only if $MC(Q) > ATC(Q)$.

$$\frac{d}{dQ}ATC(Q) = \frac{d}{dQ}\left(\frac{TC(Q)}{Q}\right) = \frac{TC'(Q)\,Q - TC(Q)}{Q^2} = \frac{TC'(Q) - TC(Q)/Q}{Q}$$

Thus

$$\begin{aligned} \frac{d}{dQ}ATC(Q) &> 0 \\ &\iff \underbrace{TC'(Q)}_{MC(Q)} - \underbrace{TC(Q)/Q}_{ATC(Q)} > 0 \\ &\iff MC(Q) - ATC(Q) > 0 \end{aligned}$$

The last inequality is the required result: $MC(Q) > ATC(Q)$.

5. (15 points). Suppose that the demand function for some good is given by $x = Ap_x^{-1}p_y^{-0.5}I^{1.5}$, where p_x is the price of x, p_y is the price of some other good y, and I is buyers' income.

(a) Based on the given information, x is a gross $\boxed{\text{complement}}$/substitute/unrelated to good y. Circle the correct answer and provide a brief explanation.

The demand for x is decreasing in the p_y, that is, when y is more expensive, buyers buy less of both goods.

(b) Calculate the price elasticity of demand for x, the cross price elasticity of demand for x with respect to p_y, and income elasticity of demand for x. Denote these elasticities with η_{x,p_x}, η_{x,p_y} and $\eta_{x,I}$.

Taking logs of the demand function

$$\ln x = \ln A - \ln p_x - 0.5 \ln p_y + 1.5 \ln I$$

The required elasticities are:

$$\begin{aligned}
\eta_{x,p_x} &= \frac{\partial \ln x}{\partial \ln p_x} = -1 \\
\eta_{x,p_y} &= \frac{\partial \ln x}{\partial \ln p_y} = -0.5 \\
\eta_{x,I} &= \frac{\partial \ln x}{\partial \ln I} = 1.5
\end{aligned}$$

(c) Calculate the approximate % change in the demand for x as a result of a 1% simultaneous increase in p_x, p_y and I.

$$\begin{aligned}
\%\Delta Q &= \eta_{x,p_x} \cdot \%\Delta p_x + \eta_{x,p_y} \cdot \%\Delta p_y + \eta_{x,I} \cdot \%\Delta I \\
&= -1 \cdot 1\% - 0.5 \cdot 1\% + 1.5 \cdot 1\% = 0
\end{aligned}$$

Remark. This has to be the case with any demand function. When prices and income increase by the same proportion, the budget constraint does not change, and the demand does not change as well. We therefore say that demand (for any good) is always *homogeneous of degree zero.*

6. (10 points). Given the utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}, \quad \sigma > 0$$

(a) Prove that

$$\lim_{\sigma \to 1} u(c) = \ln(c)$$

Since the above limit has $\frac{0}{0}$ form (i.e. the numerator and the denominator have limit of 0 as $\sigma \to 1$), we can apply L'Hôpital's rule:

$$\lim_{\sigma \to 1} \frac{c^{1-\sigma} - 1}{1-\sigma} = \lim_{\sigma \to 1} \frac{-\ln(c)\, c^{1-\sigma}}{-1} = \ln(c)$$

Here we used the rule of derivatives $(c^x)' = \ln(c)\, c^x$, together with the chain rule.

(b) Calculate the Arrow-Pratt measure of Relative Risk Aversion (RRA).

$$RRA = -\frac{u''(c)}{u'(c)} c = -\frac{-\sigma c^{-\sigma-1}}{c^{-\sigma}} c = \sigma$$

7. (10 points). Suppose $u(x,y)$ is a utility function, and $v(x,y) = f(u(x,y))$ is another utility function generated by composition $f \circ u$, where f is differentiable function. Using the implicit function theorem, prove that the indifference curves of u and v have the same slope at every point.

Indifference curves of u and v, in the form of implicit function:

$$\begin{aligned} u(x,y) &= \bar{u} \\ v(x,y) &= f(u(x,y)) = \bar{v} \end{aligned}$$

Using the Implicit Function Theorem, the slopes of indifference curves are:

$$\begin{aligned} [IC_u] &: \frac{dy}{dx} = -\frac{u_x(x,y)}{u_y(x,y)} \\ [IC_v] &: \frac{dy}{dx} = -\frac{v_x(x,y)}{v_y(x,y)} = -\frac{f'(u(x,y))\,u_x(x,y)}{f'(u(x,y))\,u_y(x,y)} = -\frac{u_x(x,y)}{u_y(x,y)} \end{aligned}$$

The last step is valid as long as $f'(u)$ is not zero. Thus, the slope of indifference curves of u is the same as the slope of indifference curves of v, as long as $f'(u) \neq 0$.

Remark. The above is an important result in consumer choice theory. If f is also *monotone increasing* function, then consumer with utility function $v(x,y) = f(u(x,y))$ will have the same demand as consumer with utility function $u(x,y)$. This result is known as *invariance of utility functions under monotone increasing transformation*. In other words, the same preferences can be represented with infinitely many utility functions, as long as they are monotone increasing transformations of each other.

8. (5 points). Suppose a firm's profit, as a function of output q and advertising a is given by $\pi(q,a)$. Suppose that at some point (q,a) we have $\pi_1(q,a) = 0.2$ and $\pi_2(q,a) = -0.1$. Using total differential, what is the approximate change in profit if the firm increases output by 1 unit and also increases advertising by 2 units (simultaneously)?

   The total differential of π with increments $dq = 1$ and $da = 2$ is

$$\begin{aligned} d\pi &= \pi_1(q,a) \cdot dq + \pi_2(q,a) \cdot da \\ &= 0.2 \cdot 1 + (-0.1) \cdot 2 = 0 \end{aligned}$$

9. (15 points). Let the utility function be $u(x,y) = [\alpha x^\sigma + (1-\alpha) y^\sigma]^{\frac{1}{\sigma}}$, $\sigma \leq 1$.

(a) Write the equation of a generic indifference curve with utility level $\bar{u}$, and find its slope dy/dx.

Indifference curve:

$$[\alpha x^\sigma + (1-\alpha) y^\sigma]^{\frac{1}{\sigma}} = \bar{u}$$

Note, the above is implicit function. Using the implicit function theorem:

$$\begin{aligned}
\frac{dy}{dx}u(x,y) &= -\frac{u_x(x,y)}{u_y(x,y)} = -\frac{\frac{1}{\sigma}[\alpha x^\sigma + (1-\alpha) y^\sigma]^{\frac{1}{\sigma}-1}\sigma\alpha x^{\sigma-1}}{\frac{1}{\sigma}[\alpha x^\sigma + (1-\alpha) y^\sigma]^{\frac{1}{\sigma}-1}\sigma(1-\alpha) y^{\sigma-1}} \\
&= -\frac{\alpha x^{\sigma-1}}{(1-\alpha) y^{\sigma-1}} = -\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{y}{x}\right)^{1-\sigma}
\end{aligned}$$

(b) Find the marginal rate of substitution between goods x and y ($MRS_{x,y}$).

The marginal rate of substitution is the absolute value of the slope of indifference curves:

$$MRS_{x,y} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{y}{x}\right)^{1-\sigma}$$

(c) Find the elasticity of substitution between goods x and y ($ES_{x,y}$).

$$\begin{aligned}
\frac{1}{ES_{x,y}} &= \frac{\partial MRS_{x,y}}{\partial(y/x)} \cdot \frac{y/x}{MRS_{x,y}} = \left(\frac{\alpha}{1-\alpha}\right)(1-\sigma)\left(\frac{y}{x}\right)^{-\sigma} \cdot \frac{y/x}{\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{y}{x}\right)^{1-\sigma}} = 1-\sigma \\
ES_{x,y} &= \frac{1}{1-\sigma}
\end{aligned}$$

10. (10 points). Briefly explain what the following Matlab commands are doing.

(a) `z = linspace(-10,10,101)`

Creating evenly spaced grid (named `z`) of 101 points between -10 and 10.

(b) `syms x y`

Declaring symbolic variables x and y.

(c) `limit(log(1+x)/x,x,0)`

Finding the limit

$$\lim_{x\to 0} \frac{\ln(1+x)}{x}$$

(d) `diff(x^a*y^b,y)`

Calculating the partial derivative

$$\frac{\partial}{\partial y} x^a y^b$$

(e) `diff(sqrt(x),x,2)`

Calculating the second order derivative

$$\frac{d^2}{dx^2}\sqrt{x}$$