Shape-from-operators: recovering shapes from intrinsic differential operators

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ICS retreat,
Disentis, Switzerland
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Introduction

real world:
3D **shape**

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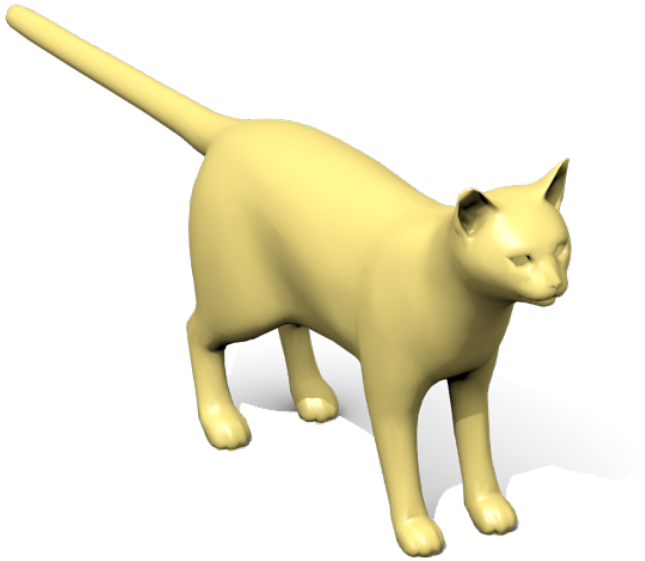
3D **shape**

mathematical model:
simply-connected, smooth, compact,
2-dim **Riemannian manifold** X,
without boundary



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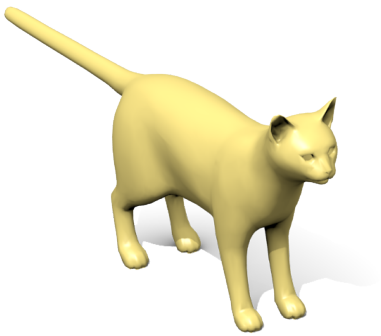
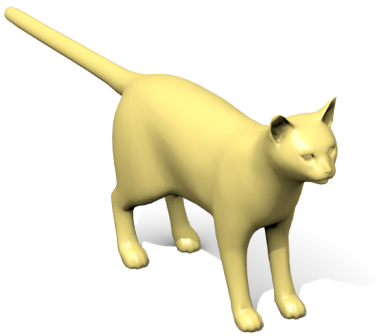


$\longrightarrow \quad \Delta \quad \longrightarrow$ properties



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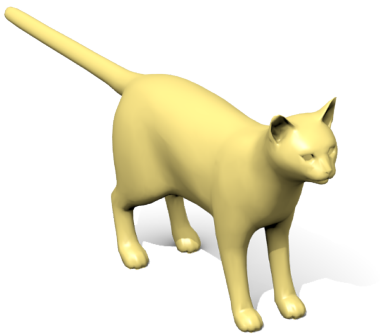
Synthesize shapes from intrinsic operators

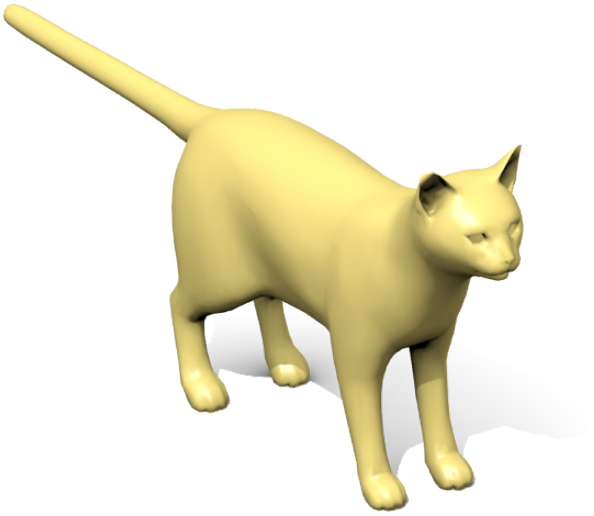


$\longleftarrow \quad \Delta \quad \longleftarrow$ properties

Synthesize shapes from intrinsic operators

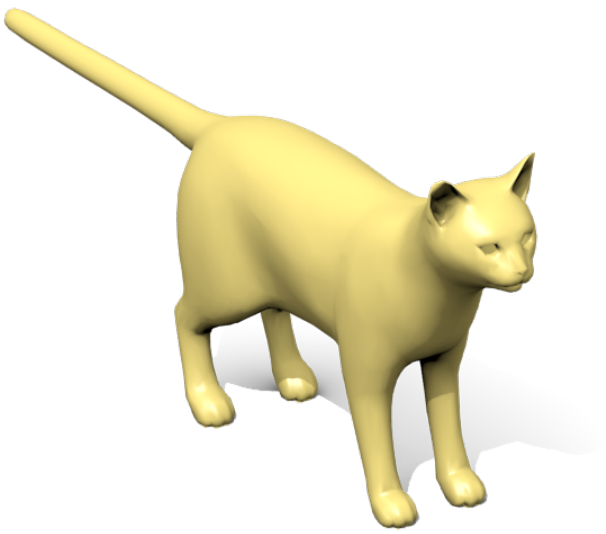
depend only on the **metric**

From shapes to operators



$$\longrightarrow \quad \Delta$$

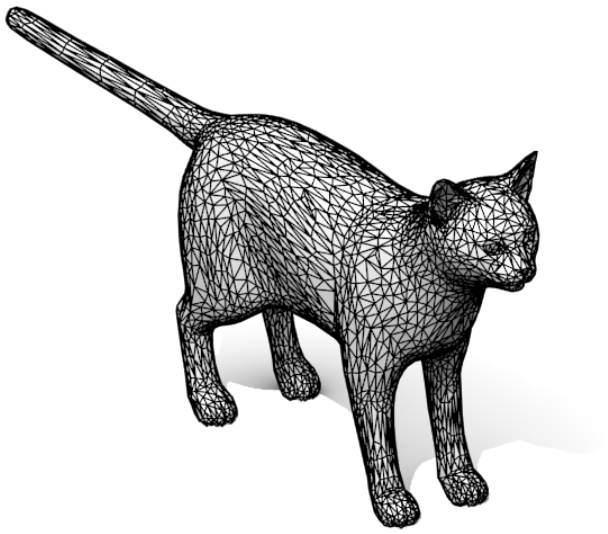
Shape discretization



- triangular mesh (X, E, F)
- discrete metric = set of edge lengths

$$\boldsymbol{\ell} = (\ell_{ij},\ (i,j) \in E)$$

Shape discretization



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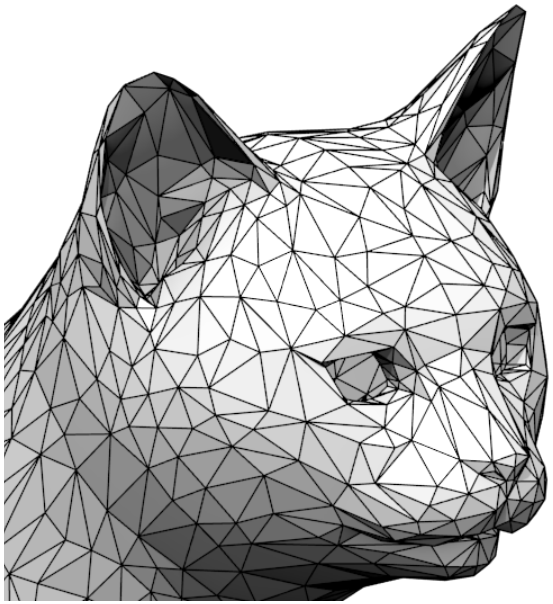

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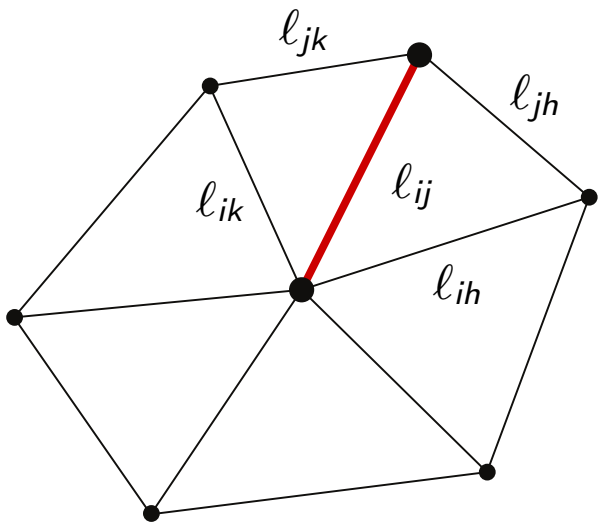
Laplace-Beltrami discretization (intrinsic)

- **intrinsic** edge weights

$$w_{ij} = \frac{-\ell_{ij}^2 + \ell_{jk}^2 + \ell_{ik}^2}{8A_{ijk}} + \frac{-\ell_{ij}^2 + \ell_{jh}^2 + \ell_{ih}^2}{8A_{ijh}}$$



Jacobson and Sorkine, 2012

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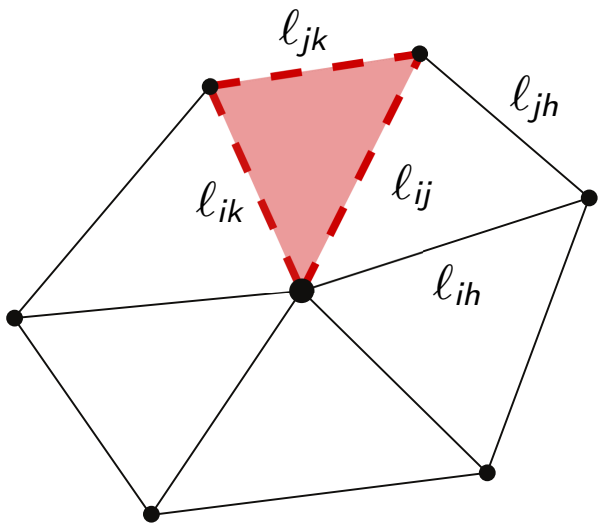
- Heron's formula

$$A_{ijk} = \sqrt{s(s-\ell_{ij})(s-\ell_{jk})(s-\ell_{ik})},$$

where s is the semi-perimeter



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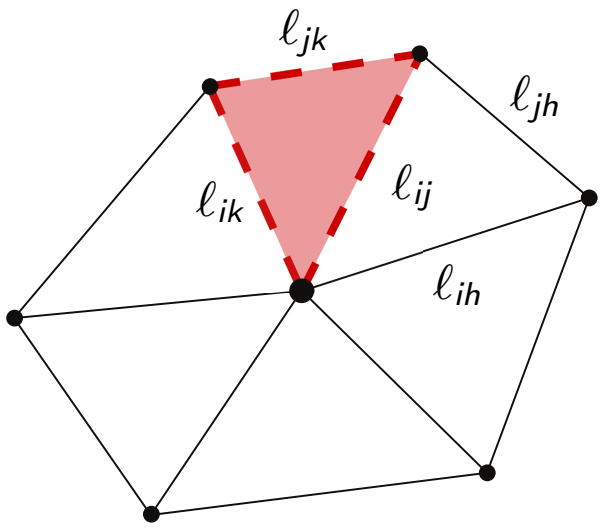

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- Laplace-Beltrami operator is the $|X| \times |X|$ matrix

$$\mathbf{L} = \mathbf{D} - \mathbf{W},$$

where $\mathbf{D} = \mathrm{diag}(\sum_{i \neq j} w_{ij})$

Jacobson and Sorkine, 2012

Laplace-Beltrami discretization (extrinsic)

- an embedding of the mesh in $\mathbb{R}^3$ specifies the coordinates $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$

- the coordinates $\mathbf{X}$ induce the metric

$$\boldsymbol{\ell} = (\|\mathbf{x}_i - \mathbf{x}_j\|, (i,j) \in E)$$

Pinkall and Polthier, 1993

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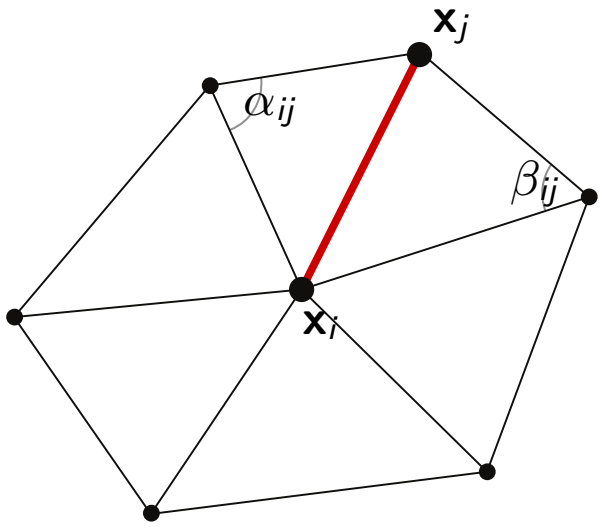
$$\boldsymbol{\ell} = (\|\mathbf{x}_i - \mathbf{x}_j\|, (i,j) \in E)$$

- cotangent edge weights

$$w_{ij} = \begin{cases} \dfrac{\cot(\alpha_{ij}) + \cot(\beta_{ij})}{2}, & (i,j) \in E \\ 0, & \text{otherwise} \end{cases}$$



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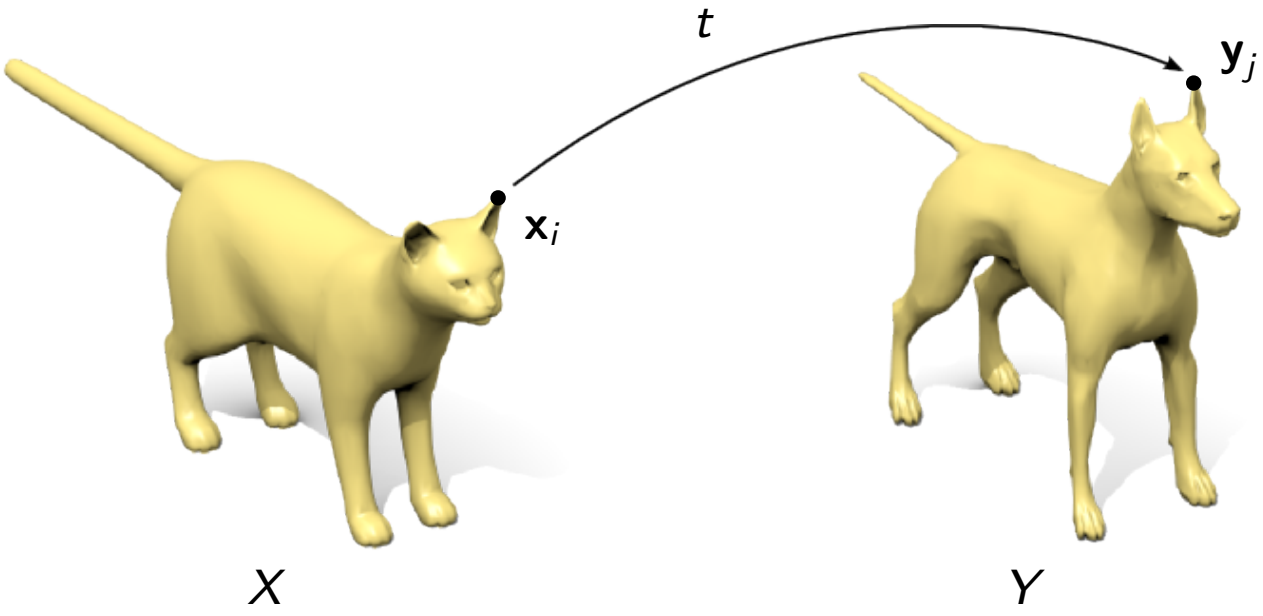
$$\Delta \longleftarrow \text{properties}$$

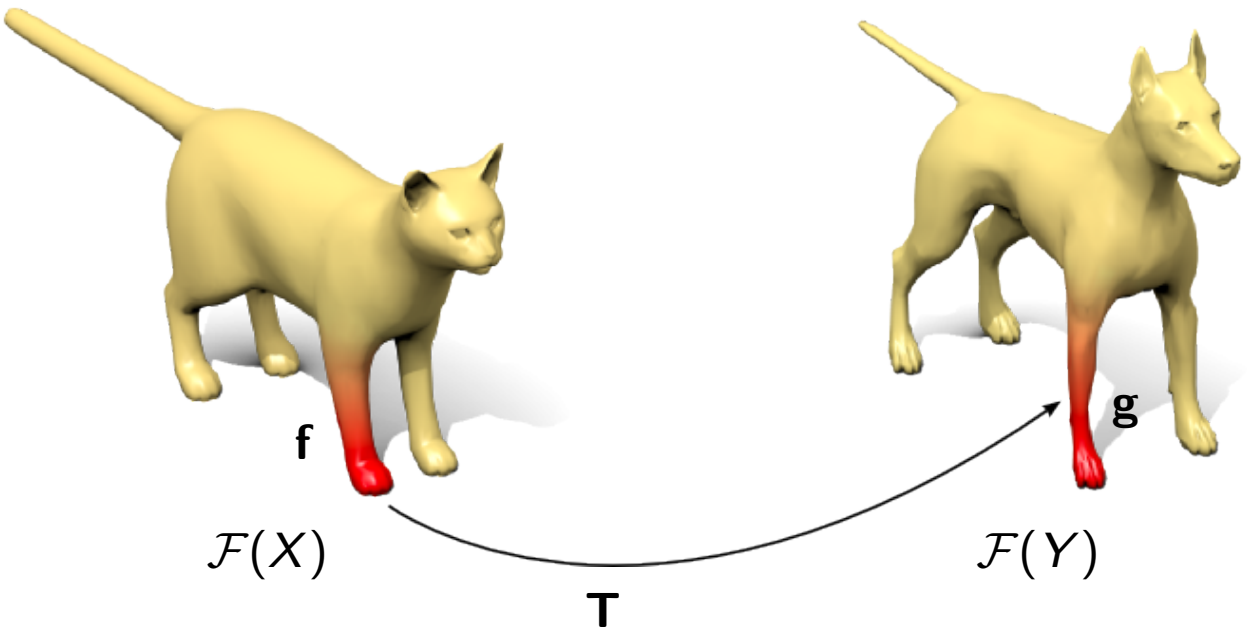
Functional maps



Point-wise maps $t\colon X \to Y$

Ovsjanikov et al., 2012

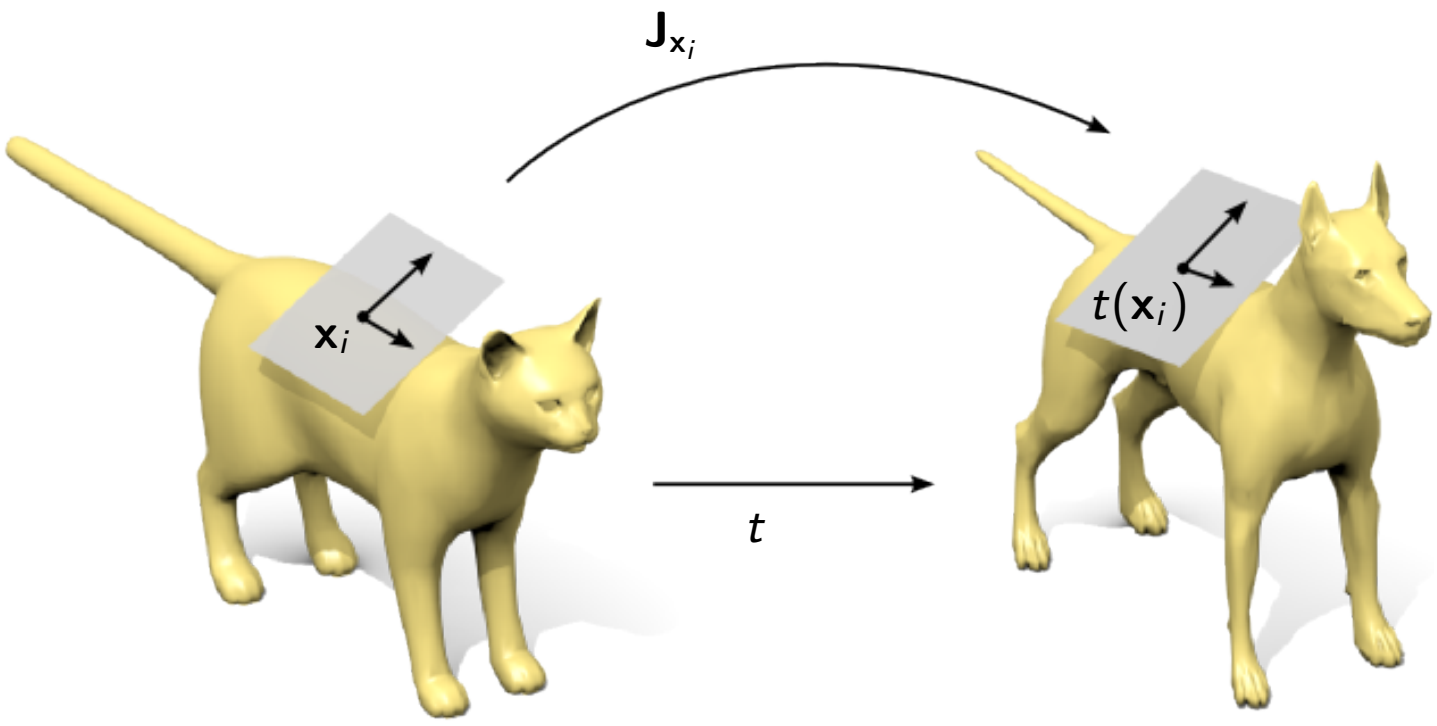
Functional maps



Functional maps $\mathbf{T}\colon \mathcal{F}(X) \to \mathcal{F}(Y)$

Ovsjanikov et al., 2012

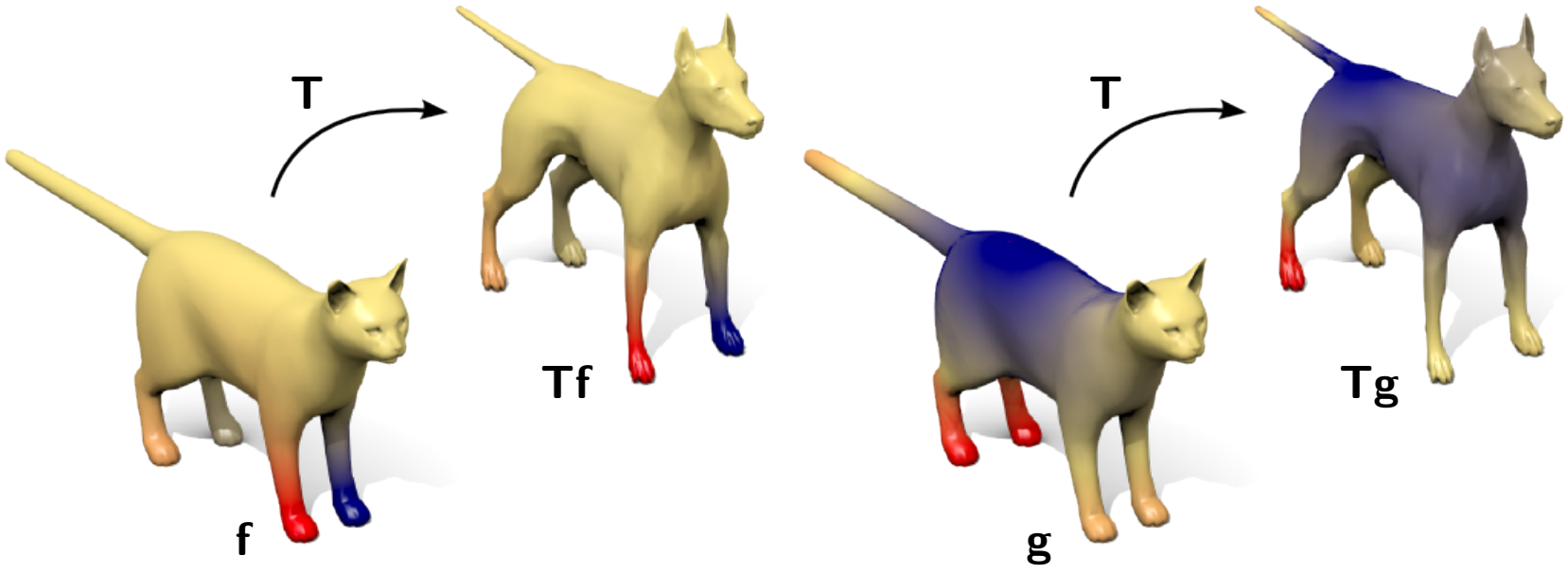
Shape difference operators



Distortions induced by a map = change of inner products of **vectors**

Rustamov et al., 2013

Shape difference operators



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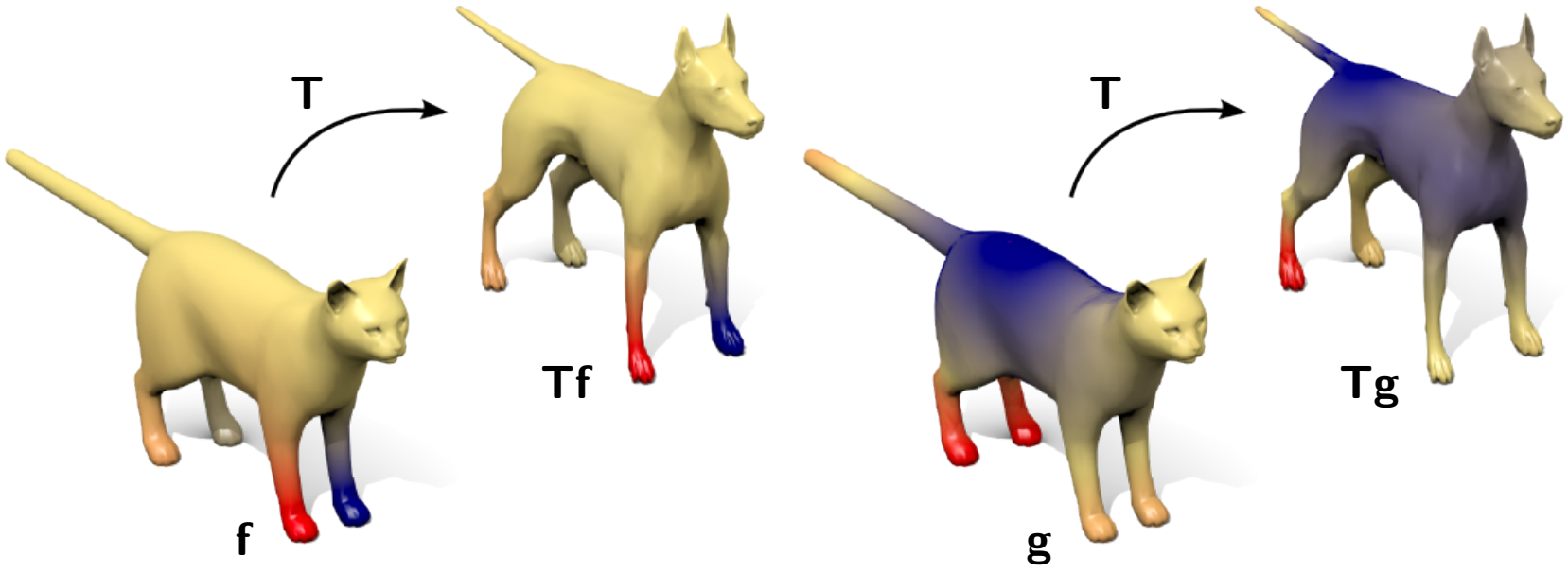
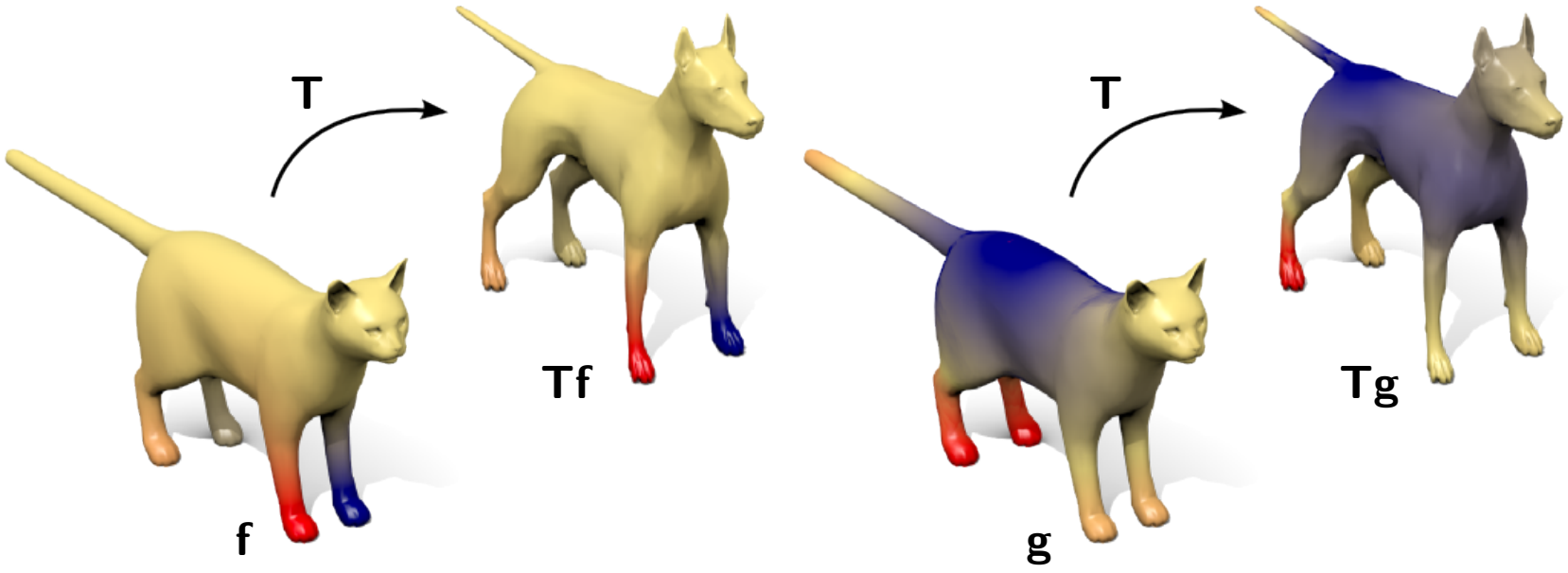


$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{F}(X)} \neq \langle \mathbf{Tf}, \mathbf{Tg} \rangle_{\mathcal{F}(Y)}$$

Rustamov et al., 2013

Shape difference operators



$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{F}(X)} \neq \langle \mathbf{Tf}, \mathbf{Tg} \rangle_{\mathcal{F}(Y)}$$

Riesz theorem: there exist a unique **self-adjoint** linear operator

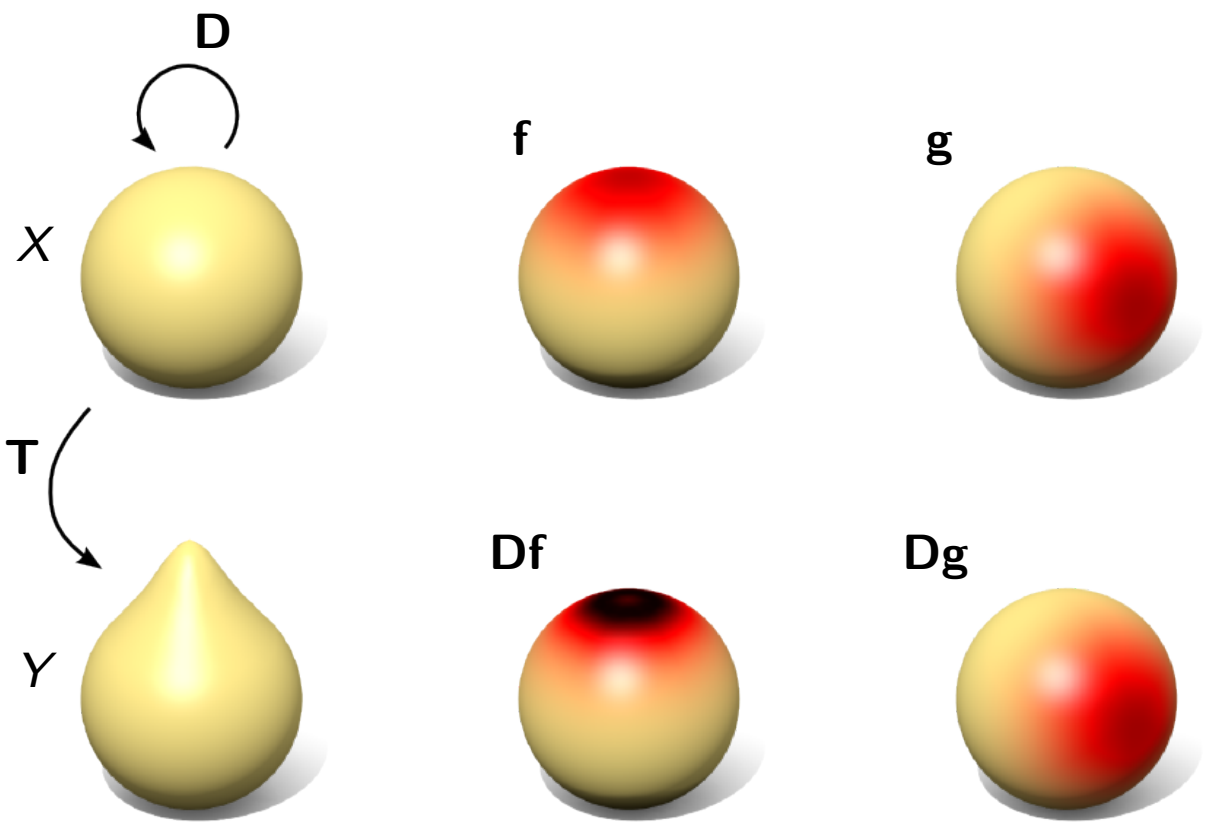
$$\mathbf{D} \colon \mathcal{F}(X) \to \mathcal{F}(X)$$

such that

$$\langle \mathbf{Tf}, \mathbf{Tg} \rangle_{\mathcal{F}(Y)} = \langle \mathbf{f}, \mathbf{Dg} \rangle_{\mathcal{F}(X)}$$

Rustamov et al., 2013

Shape difference operators



- Captures the **difference** in the geometry of the two shapes
- **Depends** on choice of inner product

Rustamov et al., 2013

Shape difference discretization

- **area**-based,

$$\langle f, g \rangle_{L^2(X)} = \int_X f(x) g(x) d\mu(x)$$

$$\mathbf{D} = \mathbf{V}_{X,Y} = \mathbf{A}_X^{-1} \mathbf{T}^\top \mathbf{A}_\mathbf{Y} \mathbf{T}$$

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- **conformal**-based,

$$\langle f, g \rangle_{H^1(X)} = \int_X \nabla f(x) \nabla g(x) d\mu(x)$$

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- if $\mathbf{V} = \mathbf{I}$, the map preserves the areas

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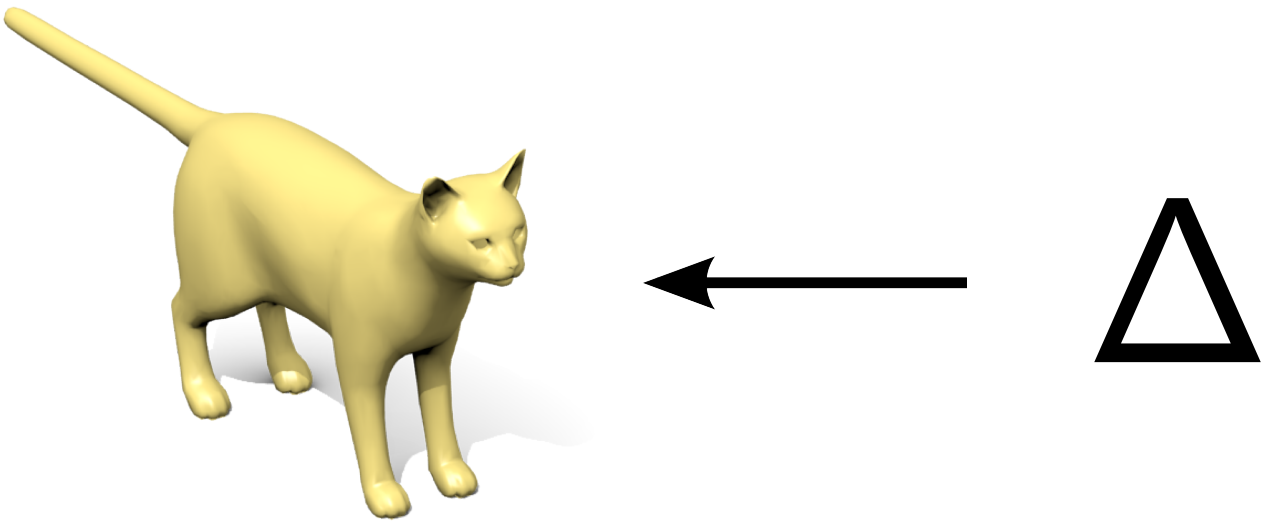
- if $\mathbf{R} = \mathbf{I}$, the map preserves the angles

- if $\mathbf{V} = \mathbf{R} = \mathbf{I}$, the map is an isometry

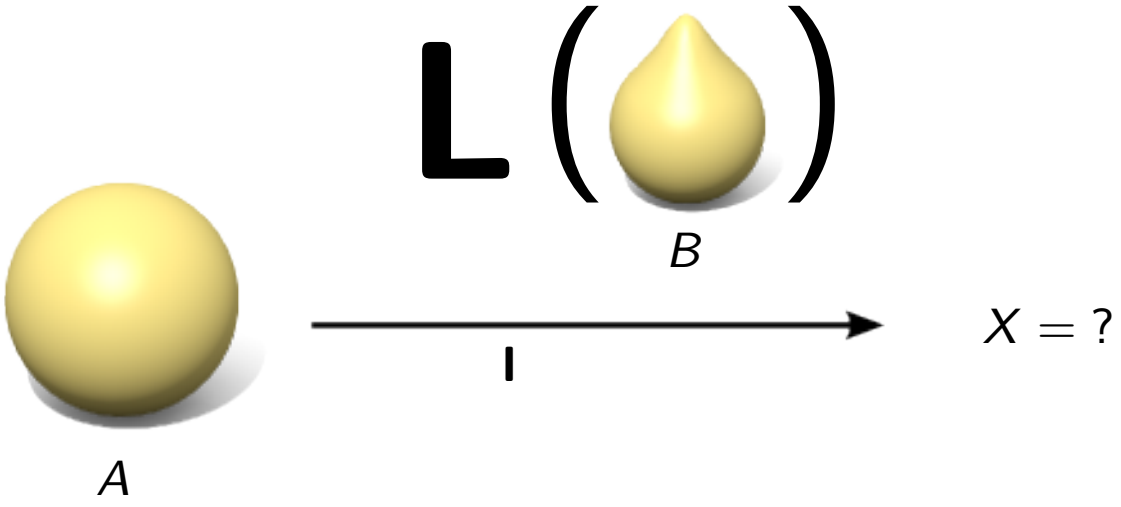
Rustamov et al., 2013

From operators to shapes



Synthesize shapes from intrinsic operators:

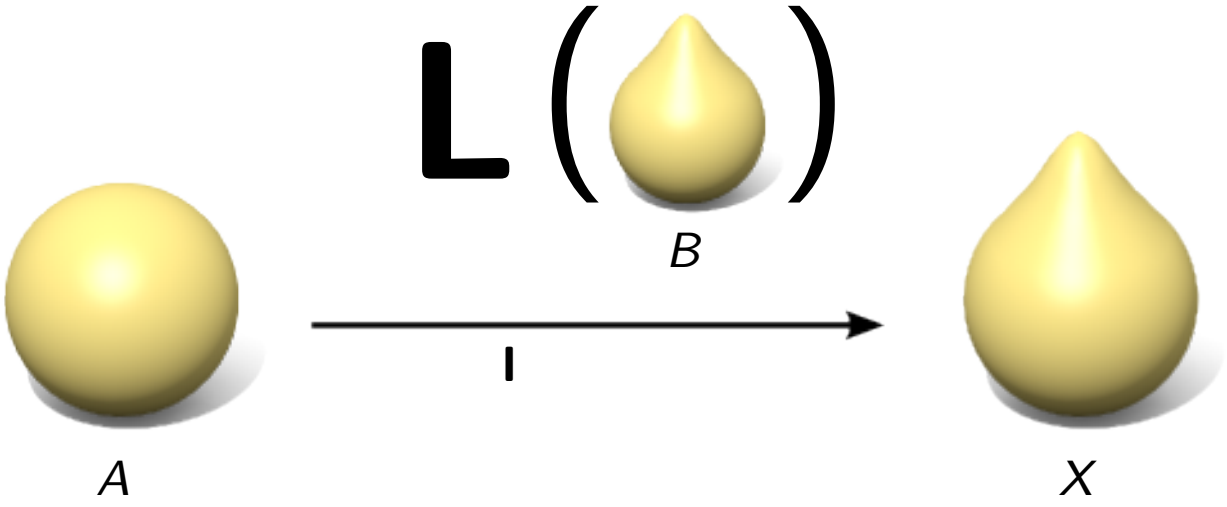
$$\min_{\mathbf{X}} \mathcal{E}\big(\mathbf{O}(\ell(\mathbf{X})), \mathbf{O}_0\big)$$



Shape-from-Laplacian problem: starting from A, find shape X s.t.

$$\mathbf{L}_X \approx \mathbf{L}_B$$

Shape-from-Laplacian



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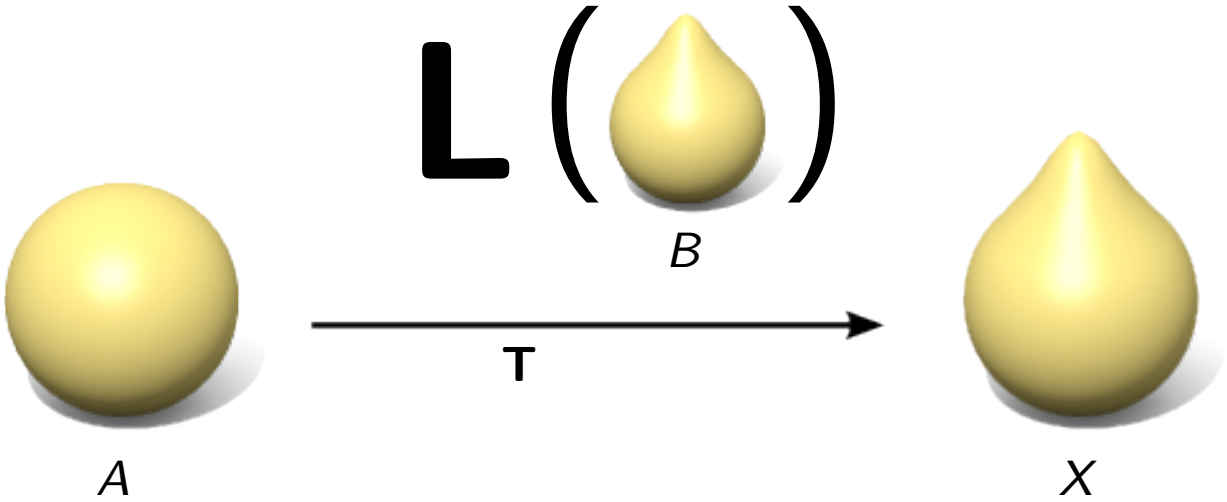
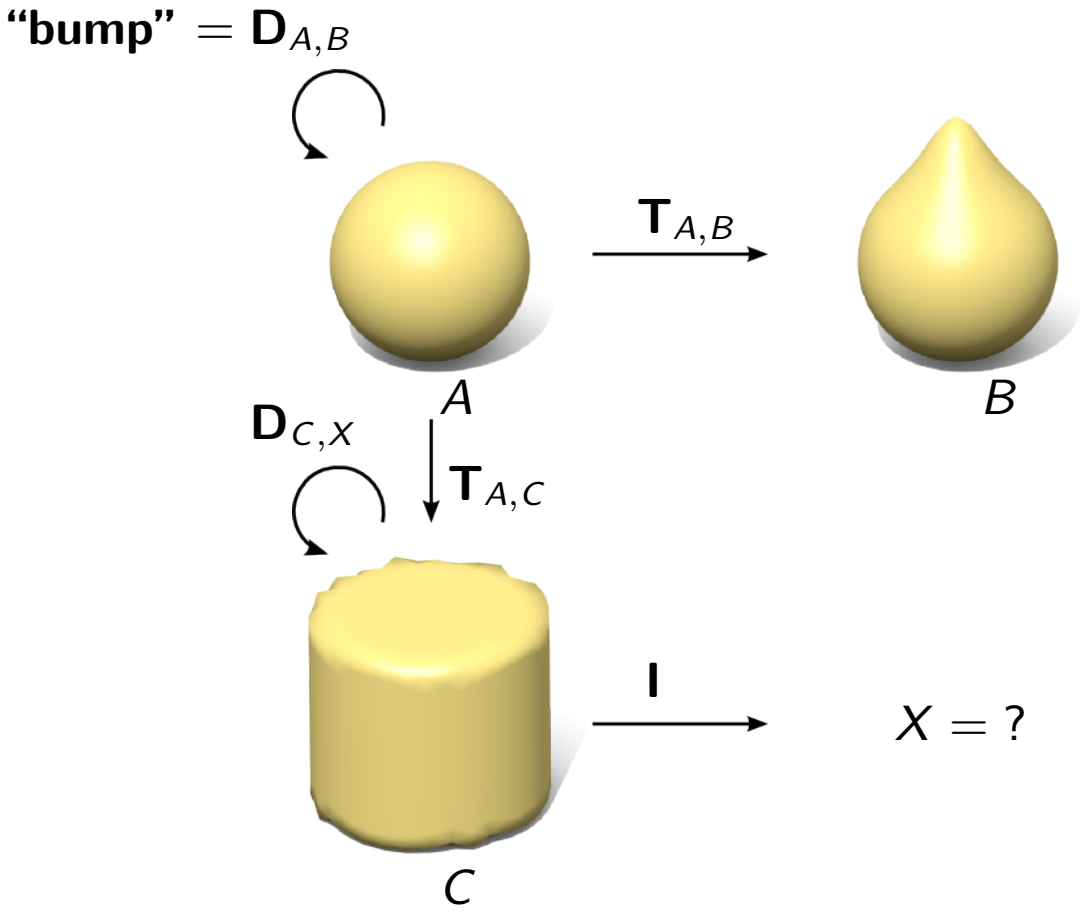


Shape-from-Laplacian problem: starting from A, find shape X s.t.

$$\mathbf{L}_X \mathbf{T} \approx \mathbf{T} \mathbf{L}_B$$

Shape analogies



Shape analogy synthesis problem: find shape X different from C same way as B is different from A, i.e.

$$\mathbf{D}_{C,X}\mathbf{T}_{A,C} \approx \mathbf{T}_{A,C}\mathbf{D}_{A,B}$$

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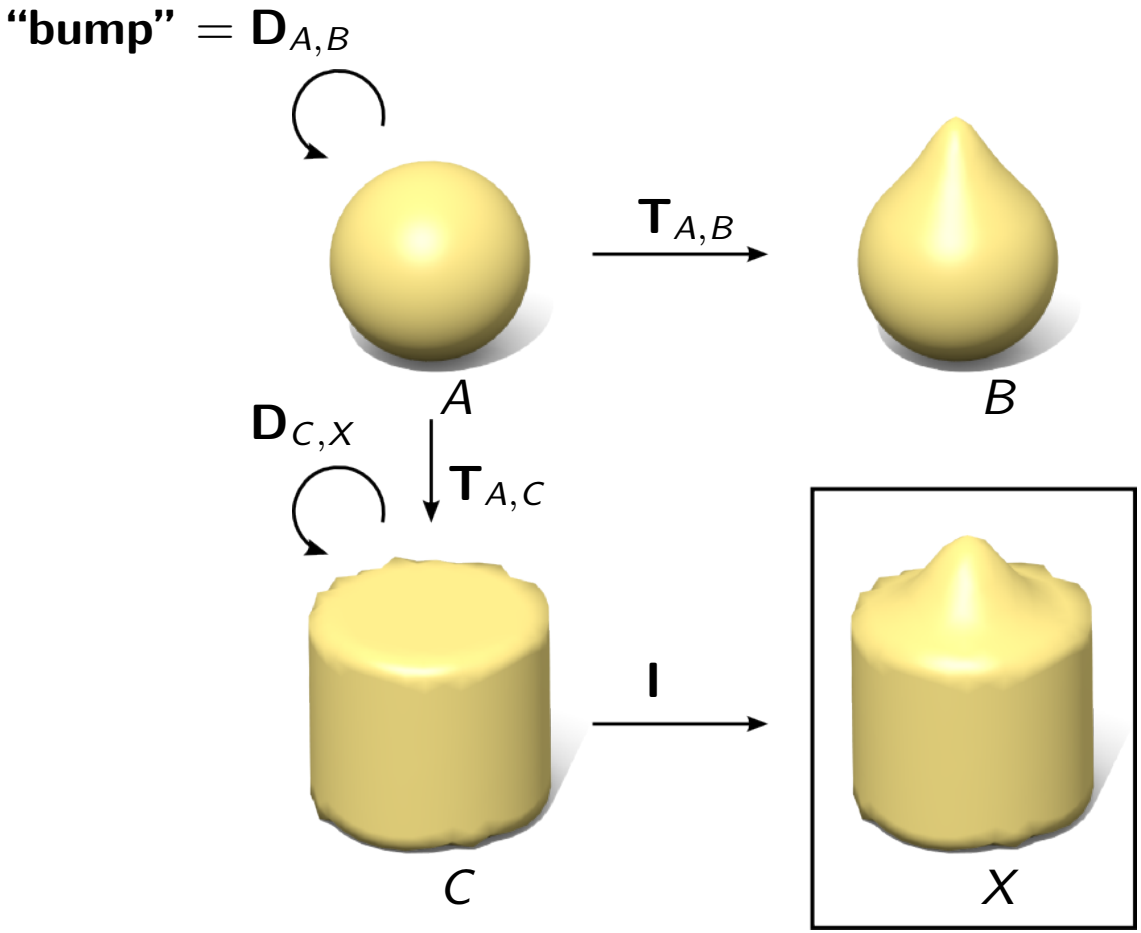

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Our approach

Original problem:

$$\min_{\mathbf{X}} \mathcal{E}\big(\mathbf{O}(\ell(\mathbf{X})), \mathbf{O}_0\big)$$

Split into two subproblems:

- metric-from-operator, $\mathbf{O}(\ell) \to \ell$

$$\min_{\ell} \ \mathcal{E}(\mathbf{O}(\ell), \mathbf{O}_0) \quad \text{s.t.} \quad \text{triangle inequality}$$

- embedding-from-metric, $\ell \to \mathbf{X}$

$$\min_{\mathbf{X}} \sum_{i,j=1}^{n} v_{ij}\big(\|\mathbf{x}_i - \mathbf{x}_j\| - \ell_{ij}\big)^2,$$

Embedding-from-metric

Special setting of **MDS**: given a metric ℓ, find its Euclidean realization by minimizing the stress

$$\min_{\mathbf{X}} \sum_{i,j=1}^{n} v_{ij}(\|\mathbf{x}_i - \mathbf{x}_j\| - \ell_{ij})^2,$$

where

$$v_{ij} = \begin{cases} 1 & \text{if } ij \in E, \\ 0 & \text{otherwise} \end{cases}$$

Leeuw et al., 1977

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SMACOF algorithm: fixed point iteration of the form

$$\mathbf{X} \leftarrow \mathbf{Z}^{\dagger}\mathbf{B}(\mathbf{X})\mathbf{X}$$

where

$$\mathbf{Z} = \begin{cases} -v_{ij} & \text{if } i \neq j, \\ \sum_{i \neq j} v_{ij} & \text{if } i = j \end{cases} \qquad \mathbf{B}(\mathbf{X}) = \begin{cases} -\dfrac{v_{ij}\ell_{ij}}{\|\mathbf{x}_i - \mathbf{x}_j\|} & \text{if } i \neq j \text{ and } \mathbf{x}_i \neq \mathbf{x}_j, \\ 0 & \text{if } i \neq j \text{ and } \mathbf{x}_i = \mathbf{x}_j, \\ \sum_{i \neq j} b_{ij} & \text{if } i = j \end{cases}$$

Leeuw et al., 1977

Numerical optimization

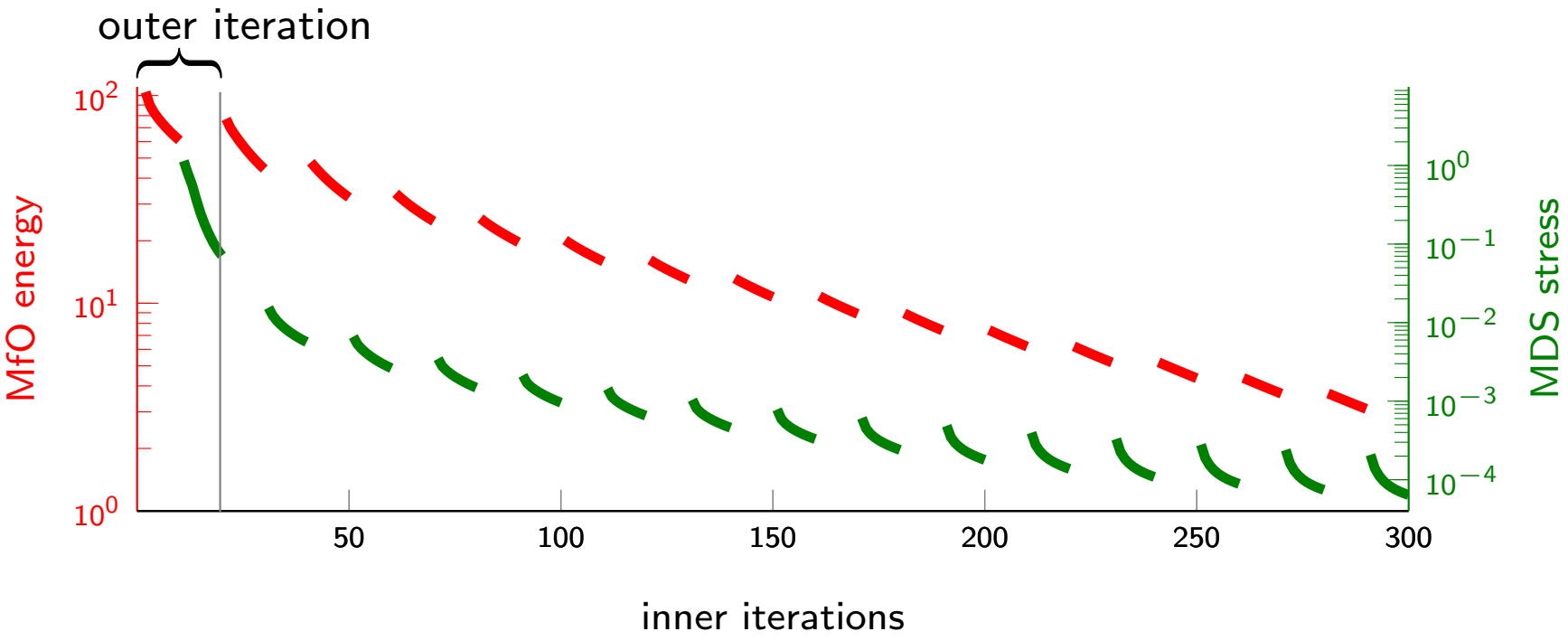
Alternating scheme:

- unconstrained minimization of the energy

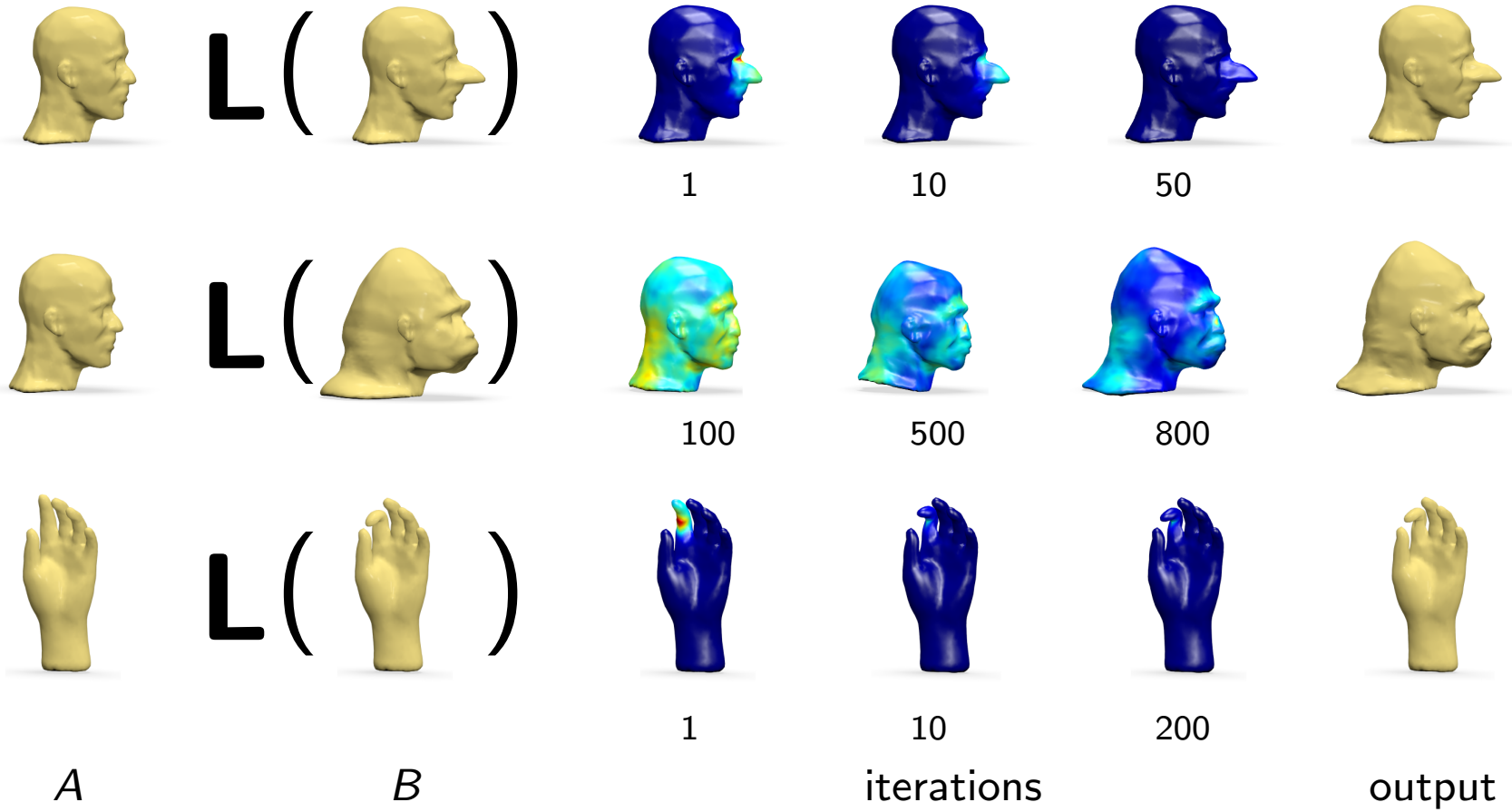
$$\min_{\ell} \ \mathcal{E}(\mathbf{O}(\ell), \mathbf{O}_0)$$

- metric embedding to enforce **triangle inequality** constraints

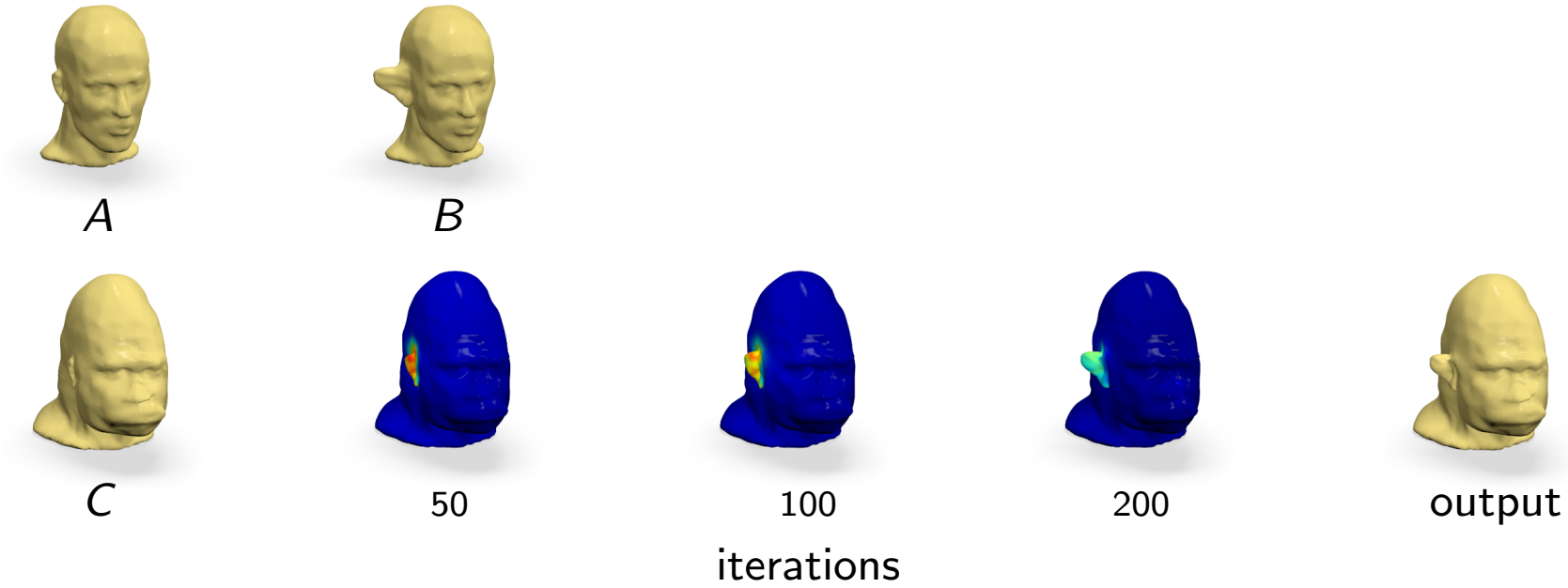
$$\ell \to \mathbf{X} \to \text{feasible } \tilde{\ell}$$



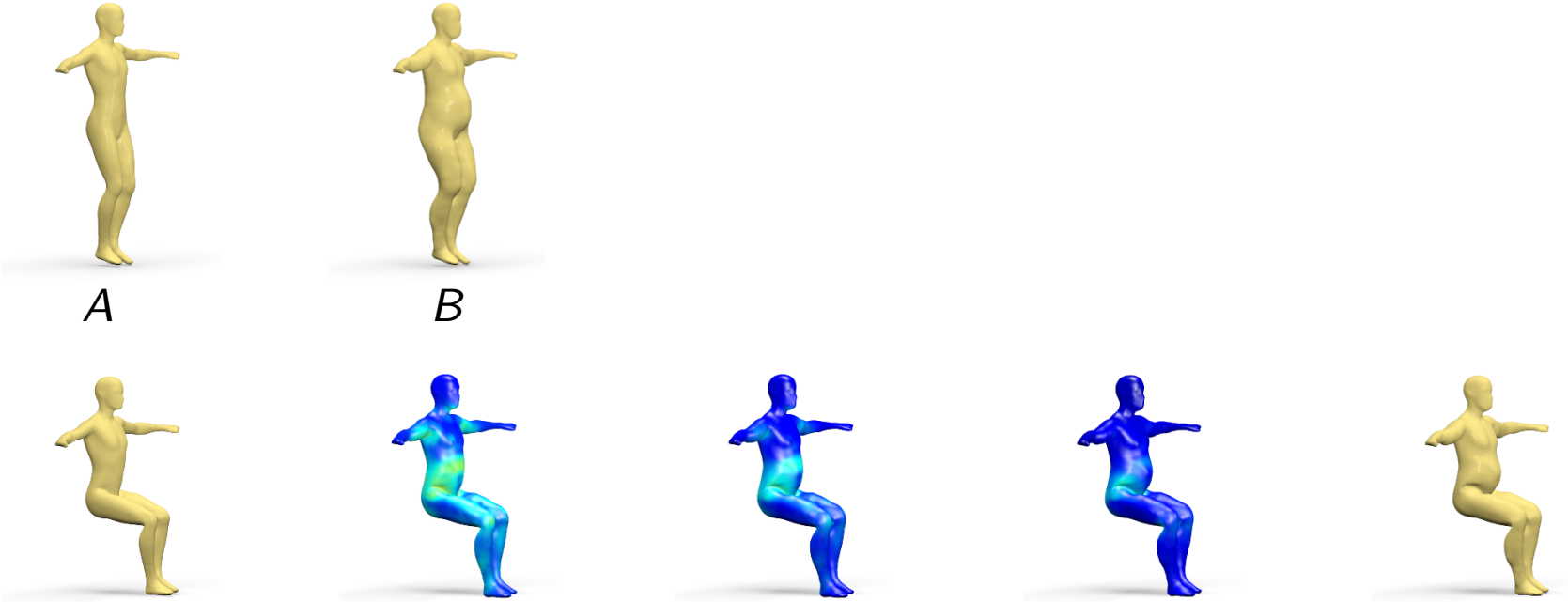
Results: shapes-from-Laplacian



Results: shapes-from-shape differences operators

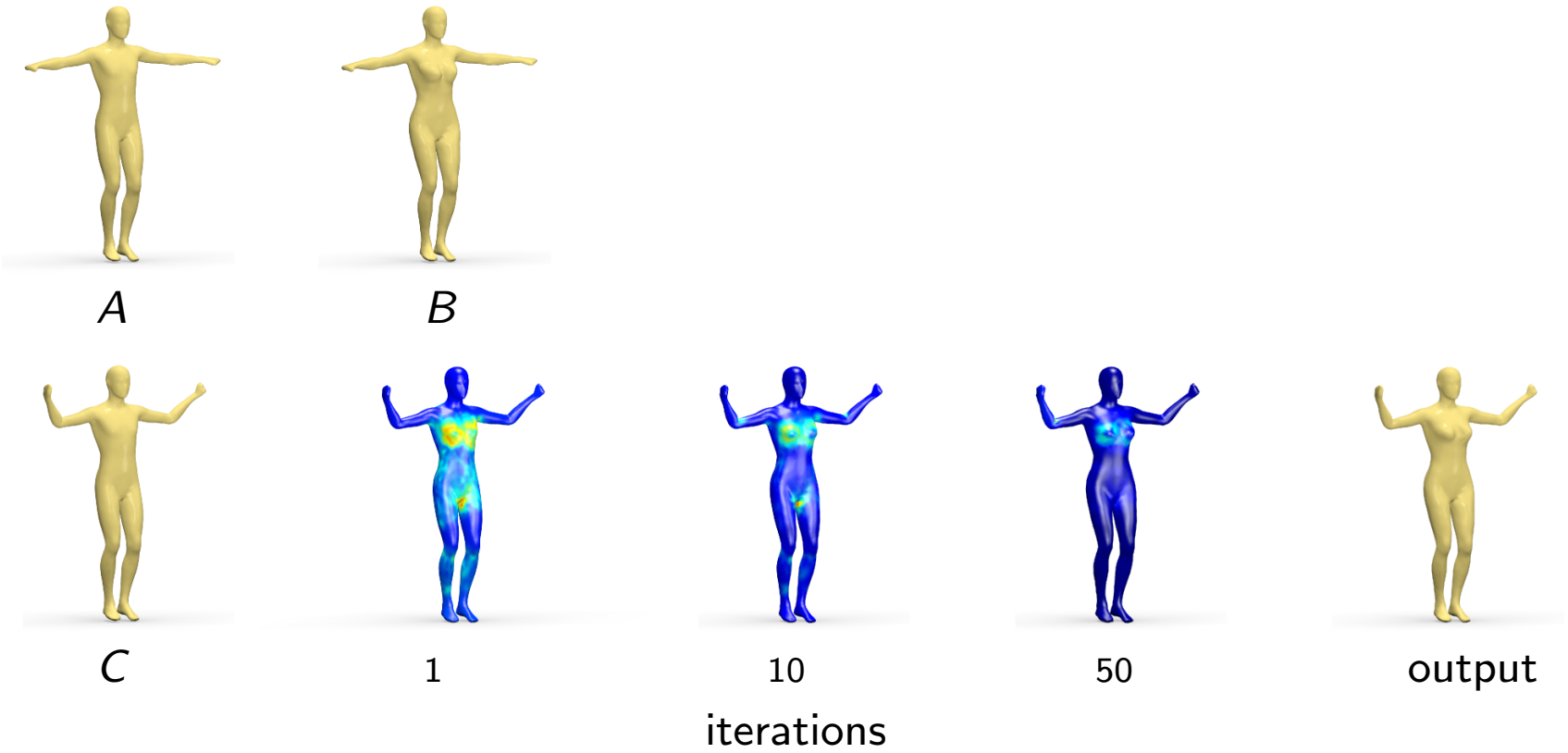


Results: shapes-from-shape differences operators

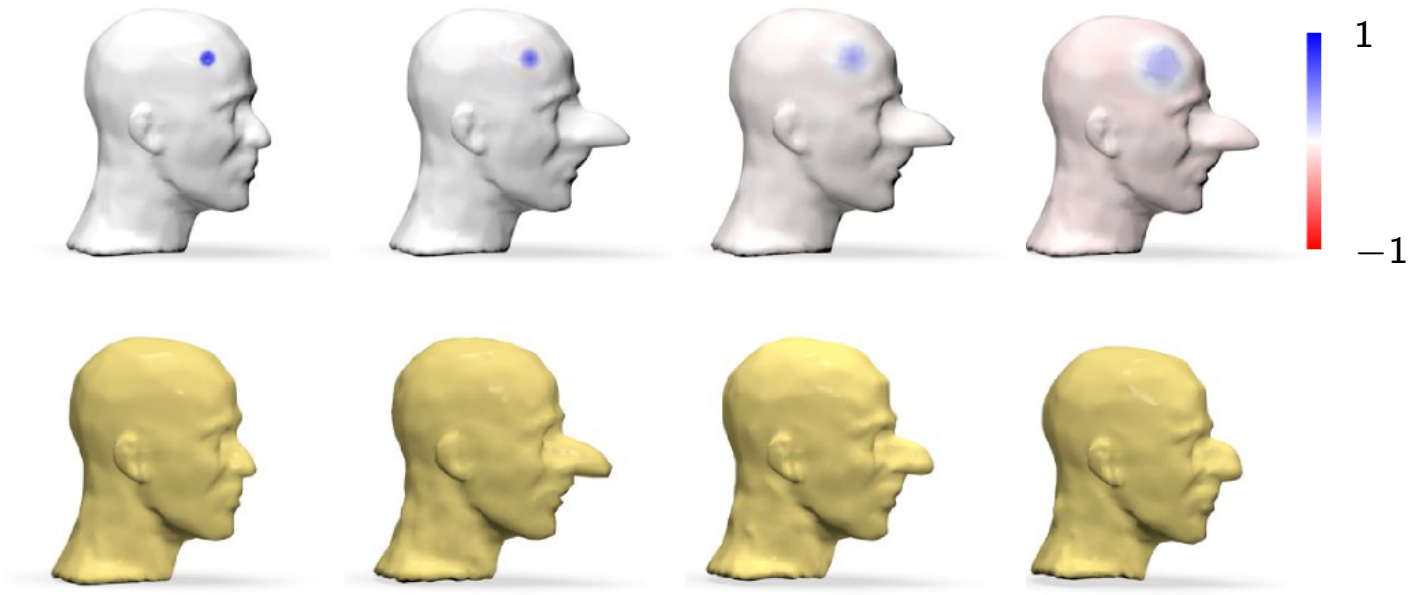


iterations

Results: shapes-from-shape differences operators



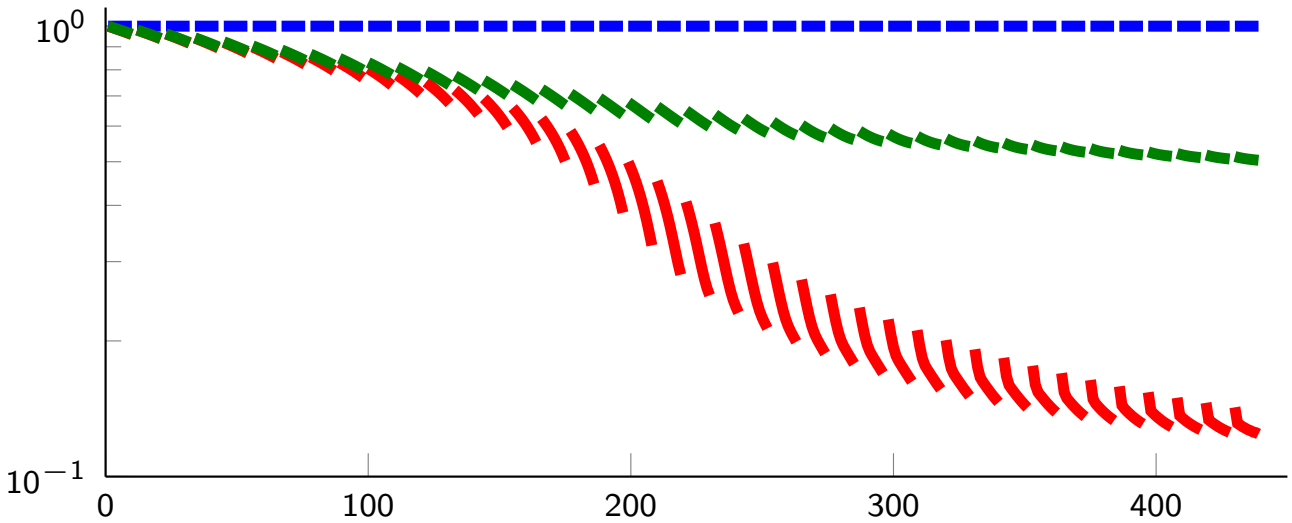
Sensitivity to functional map quality

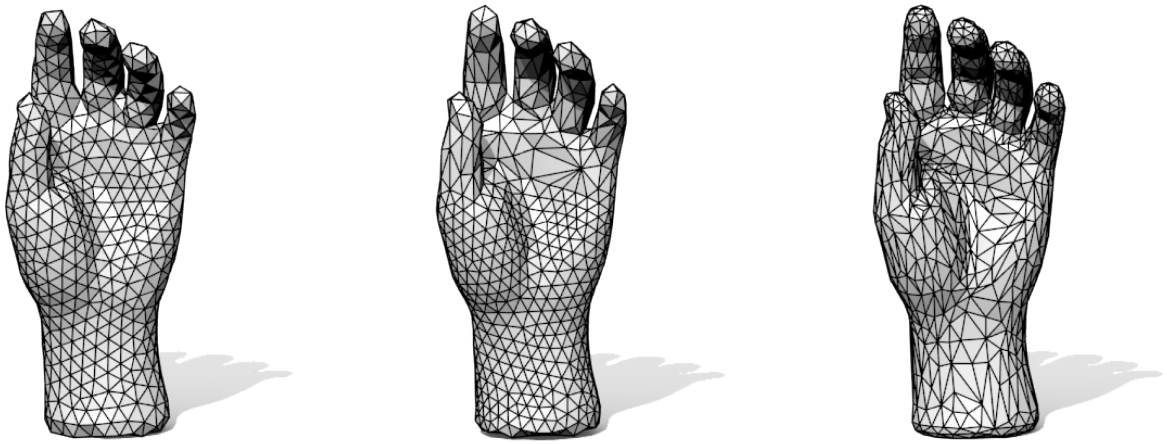


$k = 1000$ 200 100

$$\mathbf{T} \approx \bar{\boldsymbol{\Psi}}\bar{\boldsymbol{\Psi}}^\top \bar{\boldsymbol{\Phi}}\bar{\boldsymbol{\Phi}}^\top, \text{ where } \bar{\boldsymbol{\Phi}} = (\phi_1, \ldots, \phi_k)$$

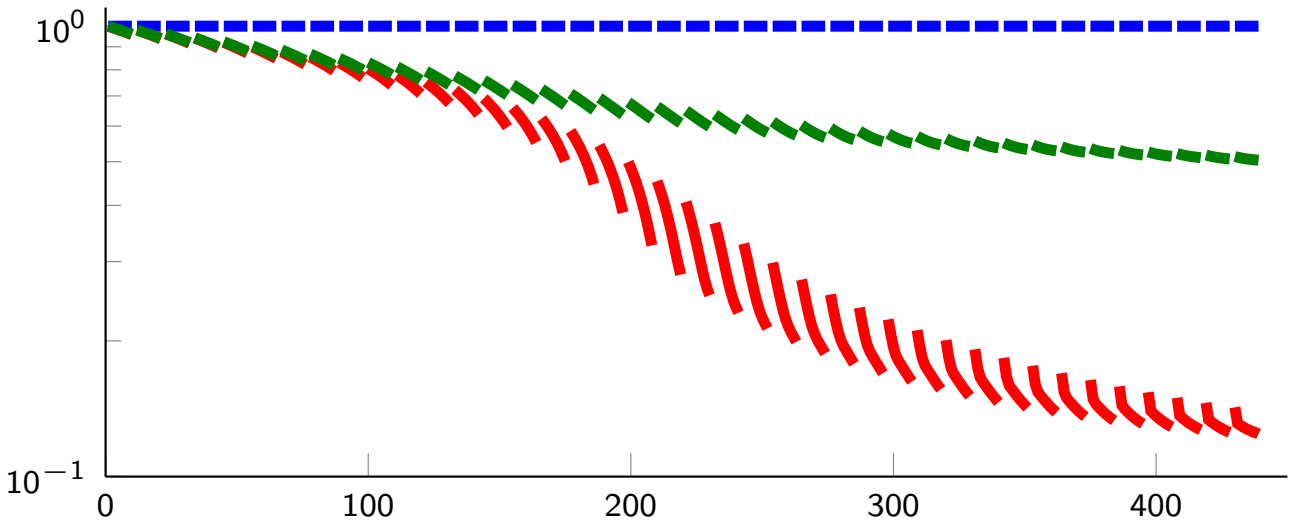
Sensitivity to mesh quality

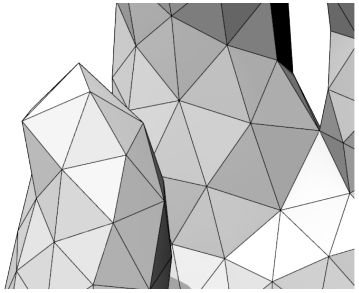




"good" "bad" "ugly"

Sensitivity to mesh quality





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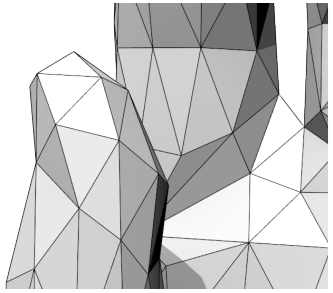

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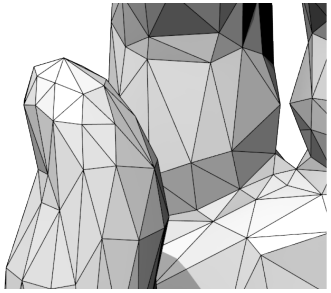

"ugly"

Future directions

- **change of variables** to avoid triangle inequality constraints (with Drosos Kourounis)

- shape-from-**eigenvectors**