Math 230, Exam 3 Name: ______________________________

March 27, 2017

Instructions.

- You are allowed one side of handwritten notes
- No calculators.
- Leave your answers unsimplified. For example, it is preferable for you to write $13C_{4,4}C_{48,1}$ rather than 624.
- There are 6 problems on 5 pages. Make sure your exam is complete.

Page	Points	Score
1	15	
2	13	
3	6	
4	8	
5	8	
Total:	50	

1. Three cards are drawn without replacement from a standard deck of cards. Let A_i be the event that the ith card is red.

[2 points] (a) What is $P(A_1)$?

Solution: 1/2

[3 points] (b) Compute $P(A_2)$ by conditioning on whether A_1 occurred or not.

Solution: $P(A_2) = P(A_2 \mid A_1)P(A_1) + P(A_2 \mid A_1^c)P(A_1^c) = \frac{1}{2}(\frac{25}{51} + \frac{26}{51}) = \frac{1}{2}$.

[7 points] (c) Let $B_1 = A_1 \cap A_2$, $B_2 = A_1^c \cap A_2$, $B_3 = A_1 \cap A_2^c$ and $B_4 = A_1^c \cap A_2^c$. Compute $P(A_3)$ by conditioning on the events $B_1, \ldots, B_4$.

Solution: We have $P(A_3 \mid B_1) = \frac{24}{50}$, $P(A_3 \mid B_2) = \frac{25}{50} = P(A_3 \mid B_3)$, and $P(A_3 \mid B_4) = \frac{26}{50}$. Also $P(B_1) = \frac{26}{52}\frac{25}{51} = P(B_4)$, $P(B_2) = P(B_3) = \frac{26}{52}\frac{26}{51}$. Use these in the formula $P(A_3) = \sum_{i=1}^{4} P(A_3 \mid B_i)P(B_i)$.

[3 points] (d) Without simplifying the sum in the previous part explain what the simplified answer must be, and *why*.

Solution: $\frac{1}{2}$. The third card drawn is equally random as the first card drawn because, without any extra information, it is not more likely to be red or black.

2. Consider the following joint distribution:

X	Y=1	2	3
1	.2	.15	.05
2	.10	0	.10
3	.05	.15	.20

.

[1 point] (a) Find $P(X = 2, Y = 3)$.

Solution: .10

[2 points] (b) Find $P(X = 3 \mid Y = 1)$.

Solution: $.05/(.2 + .1 + .05)$.

[3 points] (c) Are X and Y independent? Give evidence for why or why not.

Solution: No. $P(X = 2, Y = 2) = 0 \neq (.1 + .1)(.15 + .15) = P(X = 2)P(Y = 2)$.

[7 points] 3. Two balls are drawn without replacement from an urn containing 3 red, 2 blue and 5 green balls. Let B be the event that the drawn balls are different colors. Let A be the event that they are red and blue. Find $P(A \mid B)$.

Solution:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

We have $P(A \cap B) = \frac{C_{3,1}C_{2,1}}{C_{10,2}}$ and $P(B) = \frac{1}{C_{10,2}}(3 \cdot 7 + 2 \cdot 8 + 5 \cdot 5)$.

[6 points] 4. Ash, Lib, and Jess are taking turns trying to light a candle in a windy environment. Ash has probability 1/3 of succeeding, Lib has probability 1/2, and Jess has probability 1/3. Suppose Ash tries, then Lib, then Jess. If no one succeeds they keep trying in this order. What is the probability Lib is the one to light the candle?

Solution: Let p be the probability in question. We have

$$p = (2/3)(1/2) + (2/3)(1/2)(2/3)p.$$

Thus, $p = \frac{1}{3} + \frac{4}{18}p$ and $\frac{14}{18}p = \frac{1}{3}$ So, $p = \frac{18}{14*3} = \frac{3}{7}$.

[8 points] 5. Al has a coin and Betty has a coin. One lands heads with probability 1/4, the other is fair. Al and Betty are equally likely to have the biased coin. Let B be the event that, after each flipping their coin once, Al flips heads and Betty flips tails. Conditional on B what is the expected number of flips for Al to get his next head? Recall $E[\text{Geo(p)}] = 1/p$.

Solution: Let A be the event that Al has the biased coin. Let X be the number of flips needed conditional on A. We have

$$EX = 4P(A \mid B) + 2P(A^c \mid B).$$

Using Bayes formula we get

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^c)P(A^c)} = \frac{\frac{1}{4}\frac{1}{2}\frac{1}{2}}{\frac{1}{4}\frac{1}{2}\frac{1}{2} + \frac{3}{4}\frac{1}{2}\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{3}{4}} = \frac{1}{4}.$$

Notice that $P(A^c \mid B) = 1 - \frac{1}{4} = \frac{3}{4}$. So the answer is

$$EX = 4\frac{1}{4} + 2\frac{3}{4} = 1 + \frac{3}{2} = \frac{5}{2}.$$

6. The probability that a patient has HIV is 1/1000 and the diagnostic test for HIV can detect the virus with a probability of 98/100. The chance of a false positive is 5/100. Let A be the event a person tests positive, and H be the event they do not have HIV.

[6 points] (a) Find $P(H \mid A)$. Do not simplify!

Solution: Bayes formula gives $P(H \mid A) = \frac{P(A|H)P(H)}{P(A|H)P(H)+P(A|H^c)P(H^c)}$. $P(A \mid H)P(H) = (5/100)(999/1000)$, $P(A \mid H^c)P(H^c) = (98/100)(1/1000)$. This gives

$$P(H \mid A) = \frac{\frac{5\cdot 999}{1000}}{\frac{5\cdot 999}{1000} + \frac{98\cdot 1}{1000}} = \frac{5 \cdot 999}{5 \cdot 999 + 98} = .981.$$

[2 points] (b) If you simplified your answer to the previous part and did the problem correctly you would get $\approx .98$. Explain what this means and why this is so near 1.

Solution: Most people are healthy, so the majority of positive tests must be false positives.