1. Recall that a subset S of $\mathbb{R}^n$ is a subspace if i) $\vec{0} \in S$, ii) $\vec{u} + \vec{v} \in S$ for all $\vec{u}, \vec{v} \in S$, and iii) $c\vec{u} \in S$ for all $c \in \mathbb{R}, \vec{u} \in S$.

   Determine (by checking the properties, drawing a picture and/or using some theorem) whether or not the following sets describe subspaces:

   - the subset of $\mathbb{R}^2$ consisting of vectors of the form $\begin{bmatrix} a \\ b \end{bmatrix}$, where $-a = 1 + 3b$.

   - the subset of $\mathbb{R}^3$ consisting of vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where $bc = 0$

   - the subset of $\mathbb{R}^4$ consisting of vectors of the form $\begin{bmatrix} 5a \\ 2a + b \\ 3a + b - c \\ 2b - c \end{bmatrix}$

   - the subset of $\mathbb{R}^3$ consisting of vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where $4a = 3b - c$.

   - the subset of $\mathbb{R}^2$ consisting of vectors of the form $\begin{bmatrix} a \\ b \end{bmatrix}$ where $a + b \geq 0$

2. Let $A = \begin{bmatrix} 3 & 6 & 0 & 1 \\ 2 & 4 & 3 & -4 \\ -1 & -2 & -6 & 9 \end{bmatrix}$. Note that A has reduced echelon form:

$$B = \begin{pmatrix} 1 & 2 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{14}{9} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and A^T has reduced echelon form

$$C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(a) Find a basis for the row space of A.

(b) Find a different basis for the row space of A such that at least one vector in this new basis is **not** a multiple of a vector in your answer to (a).

(c) Find a basis for the column space of A.

(d) Find a different basis for the column space of A such that at least one vector in this new basis is **not** a multiple of a vector in your answer to (c).

(e) Is $\begin{bmatrix} 3 \\ 3 \\ 5 \\ -7 \end{bmatrix}$ in the row space? Is it in the column space?

(f) Is the linear transformation $T(\vec{x}) = A\vec{x}$ one to one, onto or neither? Why?

(g) Without doing any further calculations, find the nullity of A^T.

3. Violet is a chemical engineer for Starbucks and is attempting to design a new drink using nonfat milk, sugar, chemical X and compound Y. These have macro-nutrient profiles

$$\begin{bmatrix} \text{Calories} \\ \text{Fat} \\ \text{Carbs} \\ \text{Protein} \end{bmatrix} \in \mathbb{R}^4.$$

per 100g as follows:

$$\begin{bmatrix} M & S & X & Y \\ 32 & 400 & 895 & 575 \\ 0 & 0 & 99 & 99 \\ 5 & 100 & 0 & -50 \\ 3 & 0 & 1 & -29 \end{bmatrix}$$

This has reduced echelon form:

$$\begin{pmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Also, when augmenting an arbitrary vector and row reducing, Violet finds that:

$$\left(\begin{array}{cccc|c} 1 & \frac{25}{2} & \frac{895}{32} & \frac{575}{32} & \frac{1}{32}K \\ 0 & 1 & -\frac{179}{48} & -\frac{179}{48} & \frac{2}{75}C - \frac{1}{240}K \\ 0 & 0 & 1 & 1 & \frac{1}{99}F \\ 0 & 0 & 0 & 0 & C + \frac{9}{4}F - \frac{1}{4}K + P \end{array}\right)$$

(a) Describe the set, S, of possible drinks that Violet could make. (We will assume that negative amounts of an ingredient are possible. Say by telling customers to do very specific exercises in combination with the drinks.)

(b) Show that S is a subspace by confirming each of the the three properties.

(c) List all collections of ingredients that could form a basis for the subspace.

(d) What is the dimension of the column space?

(e) Write a basis for the nullspace of the matrix containing the 4 column vectors.

(f) What is the nullity of the matrix containing the 4 column vectors?

(g) Violet decides to make a smoothie with profile, $\begin{bmatrix} 337 \\ 4 \\ 75 \\ 1 \end{bmatrix}$. Using one of the bases from the previous question describe the recipe to make this. You do not need to simplify anything.

(h) Is there any other recipe that makes the same nutrition outcome *and* uses the same ingredients as the previous part? Why?