#4: Asymptotes, Limit Definition of Derivative
Chapters 2.6, 2.7,2.8

Contents

1 Asymptotes

Diagram

Goal: Come to terms with the infinite, ∞...

1.1 Intro and Definition

There are two kinds of asymptotes:

1. Vertical (either come from dividing by zero or taking $\ln(0)$. We've already seen these.
2. Horizontal Asymptotes. Look like

Diagram

Definition: A function f has a horizontal asymptote at L if the values of f get arbitrarily close to L as x gets very big or very small.

WRABD: What we actually mean by x gets very big or very small is that $x \to \infty$ or $x \to -\infty$. So, we actually are taking a limit to compute a horizontal asymptote

Definition: If $\lim_{x\to\infty} f(x) = L$ then we call L a horizontal asymptote. Similarly, if $\lim_{x\to-\infty} f(x) = L_2$ then we call L_2 a horizontal asymptote.

Fact: A function can have **at most 2** horizontal asymptotes (however a function can have infinitely many vertical asymptotes... can you think of such a function?).

1.2 Toolbox

Here are the barebones limits you need to know

1. $\lim_{x\to\pm\infty} \frac{1}{x^r} = 0$, for any constant $r > 0$
2. $\lim_{x\to-\infty} e^x = 0$

3. $\lim_{x\to\infty} \arctan x = \pi/2$, and $\lim_{x\to-\infty} \arctan x = -\pi/2$

Diagram

1.3 Some Examples

Example: Functions don't always have horizontal asymptotes. Consider the functions x^2 or $\ln(x)$.

Diagram

Example: "The Divide Out Method"... Consider $\lim_{x\to\infty} \dfrac{2x^2+2x+1}{5x^2+2}$.

Computation

In general given a rational functions $f(t) = \frac{p(t)}{q(t)}$ we divide out by the $\frac{1}{x^n}$ with n the relative degree of $q(t)$.

Example: Squeeze Theorem. We can use the squeeze theorem to show that $\lim_{x\to\infty} \frac{\sin x}{x} = 0$.

Computation

Example: "You cannot treat ∞ like a number." Remember that ∞ is not a number, it is instead a symbol that represents the idea of unboundedness. So, we cannot treat ∞ and $-\infty$ like numbers. For example,

$$\lim_{x\to\infty} \sqrt{x^2+2x} - \sqrt{x^2+3x} \neq \infty - \infty = 0 \qquad \text{(Because can't use limit law in this case)}$$

Instead we need to manipulate (*get $-1/2$)

Computation

2 Derivatives!

2.1 Intro

We are now ready to start talking about the main part of the course. Recall that we spent most of Wednesday trying to make sense of the velocity of a position function at a point, and what we ended up finding was that the velocity at a is given by shrinking the interval on which we looked at average velocity.

$$v(a) = \lim_{t\to a} \mathcal{A}_{[a,t]}(p(t))$$

And we also found that this was equivalent to finding the slope of the tangent line to the curve $p(t)$ at a.

Diagram

Let's start out by ignoring any interpretations and just look at tangent lines.

2.2 Finding Tangent Lines

Definition: The tangent line to the curve $f(x)$ at the point $P = (a, f(a))$ is the line through P with slope

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

Example: Find the tangent line to $f(x) = x^2 + x$ at $x = 1$. Ans: $y - 2 = 3(x - 1)$

Computation

LINE DIGRESSION Sell Point-Slope Form. There is a better formula and I recommend that you use it exclusively. Essentially if $P = (x_1, y_1)$ and we want the equation of the line of slope m through P we have the equation is

$$y - y_1 = m(x - x_1)$$

This is equivalent to slope-intercept form but much easier for the types of problems we do.

Also, an easier formula to calculate the slope of the tangent to a function f at the point $x = a$ is given by

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

We obtained this from the first definition by substituting $x = h + a$. The above is the formula to use!

Diagram

Example: Find the tangent line to $\sqrt{x}$ at $x = 4$.

$$m = \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} = \frac{1}{4}$$

Computation

2.3 The Derivative of a Function

Since we know that velocity, $v(t)$, is the slope of the tangent line to the position function, $p(t)$ we obtain the following formula for velocity

Fact:

$$v(a) = \lim_{h \to 0} \frac{p(a+h) - p(a)}{h}$$

WRABD: If you think about it for a second, the slope of the tangent line is actually telling us exactly how fast the function is changing at that moment. Large slopes mean a great deal of change, small slopes mean small amounts of change. That is why velocity is the tangent to position.

Diagram

We will see that the slope of the tangent is an extremely important mathematical tool. It is so important we give it a name (which the name of this course is derived from)

Definition: The *derivative* of a function f at a number a, denoted by $f'(a)$ is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Just to cover everything we define the tangent line to the curve $f(x)$:

Definition: The tangent line to the curve f at a is the line through $(a, f(a))$ with slope $f'(a)$.

Example: Compute the derivative of $f(x) = \frac{1}{x+1}$

Computation

Name of the Game When computing the derivative of a function using limits most every problem will follow these steps:

1. Set up the proper limit.
2. We start out dividing by zero, so we need to use algebra to somehow extract an h from the numerator and cancel with the denominator.
3. Once we cancel the h's we can take the limit as $h \to 0$ without any peril and obtain an answer.

2.4 The Derivative as a Rate of Change

I talked about how the derivative tells us about the rate of change of a function. Let us delve deeper.

The Setup: Let $y = f(x)$. So y is a function of x.

Diagram

$$\frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x} = \text{Slope of Secant} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

And when we take the limit of this as $\Delta x \to 0$ we see that:

Fact: The derivative $f'(a)$ is the *instantaneous rate of change* of $y = f(x)$ with respect to x when $x = a$.

2.5 Keeping Track of Units

So we think of the derivative as a rate of change. Here is an example.

Example: Suppose that $f(t)$ describes the position of a particle in feet. What are the units of $f'(a)$?

Example: Suppose that $g(x)$ is a production function that says how many liters of petroleum can be made, where x represents liters of crude oil. What are the units of $g'(a)$?

Example: (2.8.45) The cost of producing x ounces of gold from a new gold mine is $C = f(x)$ dollars.

1. What is the meaning of $f'(x)$? What are the units?
2. What does $f'(800) = 17$ represent?

3 The Derivative as a Function

3.1 Intro

Hearkening back to last section notice that we only defined the derivative pointwise (i.e. our definition of the derivative is just at a point), but if we stop and think sensibly for a moment we realize that if our function $f(x)$ has a derivative at every point in its domain then the derivative is a brand new function which we will denote $f'(x)$. Replacing a by x in our old definition we obtain the following definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We call the above the derivative of $f(x)$.

Example: $f(x) = x^2$ we use limits and obtain $f'(x) = 2x$.

Computation

! Note that we treat x as a constant in the limit (since the limit only involves h)

! The derivative may not be defined everywhere that f is. Consider the function $f(x) = \sqrt{x}$ at the point $x = 0$.

Computation

! Two alternate notations for the derivative are $\frac{d}{dx}f(x)$ and $\frac{df}{dx}$ this will be important when we start doing more complicate types of differentiation.

3.2 Slope Interpretation

Since the derivative *is* the slope of our function f at each point x then in the example above we have the slope of the tangent line to f at any point x for $f(x) = x^2$ is $2x$.

Diagram

This leads us to the first stumbling block for 124 students.

3.2.1 Stumbling Block

Given a graph of a function, we can see what the slope of a function is roughly doing. So, if you are super keen about what the derivative is and have good spatial reasoning, you can sketch the derivative.

Diagram

We will look at a harder example later!

3.3 How a function can fail to be differentiable

Example: $f(x) = |x|$ is not differentiable at $x = 0$.

Computation

Technical Reason The limit doesn't exist.

WRABD: Look at graph.

Diagram

So, a function is not differentiable at a point if one of the following situations arises.

1. The point is not in the domain.
2. Sharp corner.
3. f is discontinuous at the point.
4. The tangent is vertical. (What does this mean the limit does?)

Definition: A function f is said to be differentiable on the interval (a, b) if $f'(x)$ exists for all $x \in (a, b)$.

3.4 Higher Order Derivatives

3.4.1 Second Derivative

If f is differentiable we just learned that f' is a new function. So, there is no reason that we can't consider the derivative of f' which we will denote $f''(x)$. Some alternate notations are

$$f''(x), \qquad f^{(2)}(x), \qquad \text{and} \qquad \frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2}$$

WRABD: The second derivative describes the rate of change of the first derivative. The most tangible example of this comes from looking at position functions.

Example: If $p(t)$ is a position function we showed that $p'(t) = v(t)$ =velocity, what about $p''(t)$? This would be $v'(t)$ which is the change in velocity. But the rate of change of velocity is something we know... acceleration. So, when considering position, we have the following

$$p(t) = \text{position}$$
$$p'(t) = v(t) = \text{acceleration}$$
$$p''(t) = v'(t) = a(t) = \text{acceleration}$$

3.4.2 n^{th} derivatives

If the second derivative exists then we can consider its derivative and take the third derivative $f'''(x)$. So long as the derivatives keep existing then we can keep taking them. We denote the n^{th} derivative of f by

$$f^{(n)}(x)$$

! Can you think of a function that has a first derivative defined on the same domain, but its second derivative is defined on a strictly smaller domain? *hint: Think about root functions.*

Example: If $f(x) = x^2 + x$ find $f''(x)$.

Computation

3.5 Left and Right Derivatives

Just like with limits we can define a left and right derivative. Look at tangents from just one side.

Diagram

Definition: The left derivative of a function f at the number a is given by

$$\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

The right derivative of a function f at the number a is given by

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

Fact: $f'(a) = m$ if and only the left and right derivatives at a are equal to m.

Example: Find numbers b and c so that the function below is differentiable at 1.

$$f(x) = \begin{cases} cx^2 + 1, & x \leq 1 \\ x^3 + b, & x \geq 1 \end{cases}$$

Computation