San Francisco State University — Michael Bar
ECON 715 — Fall 2018

Midterm Exam 1

September 25, 2018

Time: 1 hour, 30 minutes

Name: ______________________________

Instructions:
1. One double-sided sheet with any content is allowed.
2. Calculators are NOT allowed.
3. Show all the calculations, and explain your steps.
4. If you need more space, use the back of the page.
5. Fully label all graphs.

1. (20 points). Let X be $n \times k$ matrix.

   (a) Prove that the matrix products $X'X$ and XX' are always well defined.

   $X'_{k\times n}X_{n\times k}$ is well defined because #(columns of X') = #(rows of X) = n. Similarly, $X_{n\times k}X'_{k\times n}$ is well defined because #(columns of X) = #(rows of X') = k.

   (b) What are the dimensions of $X'X$ and XX'?

   $X'_{k\times n}X_{n\times k}$ is $k \times k$
   $X_{n\times k}X'_{k\times n}$ is $n \times n$

(c) Prove that $X'X$ and XX' are symmetric.

A symmetric matrix is equal to its transpose. Thus,

$$\begin{aligned} (X'X)' &= X'(X')' && \text{trans. of product = product of trans. in reverse order} \\ &= X'X && \text{transpose of transpose is the original matrix} \end{aligned}$$

Similarly,

$$(XX')' = (X')'X' = XX'$$

(d) Suppose that the inverse $(X'X)^{-1}$ exists. Prove that $X(X'X)^{-1}X'$ is idempotent matrix.

A matrix B is idempotent if $BB = B$. Thus

$$\begin{aligned} X(X'X)^{-1}X' \cdot X(X'X)^{-1}X' &= X\underbrace{(X'X)^{-1}(X'X)}_{I}(X'X)^{-1}X' \\ &= X(X'X)^{-1}X' \end{aligned}$$

2. (5 points). Let B be idempotent matrix. Prove that $I - B$ is also idempotent matrix.

A square matrix B is idempotent if $BB = B$. We need to prove $(I - B)(I - B) = I - B$.

$$\begin{aligned}(I - B)(I - B) &= I \cdot I - IB + BI + BB \\ &= I - B - B + B \\ &= I - B\end{aligned}$$

3. (5 points). Let A, B, C be square matrices, and suppose that the inverse $(ABC)^{-1}$ exists. Prove that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$. You are allowed to use the result proved in the notes that $(AB)^{-1} = B^{-1}A^{-1}$ (inverse of a product of two matrices is equal to the product of inverses, in reverse order).

$$\begin{aligned}(ABC)^{-1} &= ((AB)C)^{-1} \\ &= C^{-1}(AB)^{-1} \\ &= C^{-1}B^{-1}A^{-1}\end{aligned}$$

We applied the property $(AB)^{-1} = B^{-1}A^{-1}$ twice.

4. (10 points). Consider the following system of equations:

$$\begin{aligned} a_{11}x_1 &= d_1 \\ a_{21}x_1 + a_{22}x_2 &= d_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= d_3 \end{aligned}$$

(a) Write the above system in matrix form $Ax = d$.

$$\underbrace{\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}_{d}$$

(b) Suppose that $a_{11}, a_{22}, a_{33} \neq 0$. Then, the above system of equations has a unique solution. $\boxed{\text{True}}$/False, circle the correct answer, and provide a proof.

A $n \times n$ system of linear equations has a unique solution if and only if the determinant of the coefficient matrix $|A| \neq 0$ (i.e. A is invertible). Using Laplace expansion along the 1st row (easiest, because of two zeros), the determinant is

$$|A| = a_{11} \cdot \begin{vmatrix} a_{22} & 0 \\ a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot a_{22} \cdot a_{33} \neq 0$$

Therefore, the given system has a unique solution.

5. (10 points). Consider the following system of equations:

$$\begin{aligned} x_1 &= d_1 \\ x_1 + x_2 &= d_2 \\ x_1 + x_2 + 3x_3 &= d_3 \end{aligned}$$

(a) Using Cramer's rule, solve for x_2. Denote the solution by x_2^*.

$$x_2^* = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & d_1 & 0 \\ 1 & d_2 & 0 \\ 1 & d_3 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 3 \end{vmatrix}} = \frac{3(d_2 - d_1)}{3} = d_2 - d_1$$

(b) Based on your result in the last section, find the change in x_2^* due to an increase in d_1 by 1 unit (i.e. $\Delta d_1 = 1$).

$$\Delta x_2^* = -\Delta d_1 = -1$$

Thus, x_2^* will decrease by 1 unit.

6. (10 points). Let $A_{3\times 3}$ and $B_{3\times 3}$ be two matrices, with determinants $|A| = 5$ and $|B| = 2$. Calculate the following (show steps):

(a) $|A'|$

$$|A'| = |A| = 5$$

(b) $|AB|$

$$|AB| = |A| \cdot |B| = 5 \cdot 2 = 10$$

(c) $|A^{-1}|$

$$\left|A^{-1}\right| = \frac{1}{|A|} = \frac{1}{5}$$

(d) $|2 \cdot B|$

$$|2 \cdot B| = 2^3 \, |B| = 8 \cdot 2 = 16$$

(e) $|B \cdot A^{-1}|$

$$\left|B \cdot A^{-1}\right| = |B| \cdot \left|A^{-1}\right| = 2 \cdot \frac{1}{5} = \frac{2}{5}$$

7. (20 points). Suppose that the input-output matrix for some economy, consisting of 3 sectors, is given by:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(a) Find the technology (Leontief) matrix T.

$$T = I - A = \begin{bmatrix} 1 - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & 1 - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & 1 - a_{33} \end{bmatrix}$$

(b) Suppose the consumers' demand vector is $d = \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix}'$. The social planner wants to find the output levels in each industry, $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}'$, which will satisfy simultaneously the inter-industry demand and the consumers' demand. Write the social planner's problem in matrix form.

$$\underbrace{\begin{bmatrix} 1 - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & 1 - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & 1 - a_{33} \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}_{d}$$

$$Tx = d$$

(c) Write the solution to $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}'$, obtained via matrix inversion. No need to calculate the output; just write the formula.

$$x^* = T^{-1}d$$

(d) Consumers' demand in some country changed from $d = \begin{bmatrix} 5 & 10 & 10 \end{bmatrix}'$ to $\tilde{d} = \begin{bmatrix} 20 & 10 & 10 \end{bmatrix}'$. This can reflect an increase in demand for military production as the nation prepares for war. As a result, the social planner's production plan changed from $x = \begin{bmatrix} 80 & 85 & 85 \end{bmatrix}'$ to $\tilde{x} = \begin{bmatrix} 140 & 130 & 130 \end{bmatrix}'$. Explain intuitively, why the production of all goods can increase even though the demand increased only for good 1.

The production of all goods requires inputs from all industries, potentially. Thus, higher demand for good 1 by the consumers causes higher demand by industry 1 for inputs produced by *all* industries.

8. (10 points). Consider the multiple regression model

$$Y_i = \beta_1 + \beta_2 X_{2,i} + \beta_3 X_{3,i} + ... + \beta_k X_{k,i} + u_i, \; i = 1, ..., n$$

(a) Write the above model in matrix notation.

$$Y = X\beta + u$$

(b) Let $b = [b_1, ..., b_k]'$ denote the OLS estimator of $\beta = [b_1, ..., b_k]'$. Write the OLS problem (in matrix notation) which b solves.

$$\min_b RSS = (Y - Xb)' (Y - Xb)$$

Here the residuals are given by $e = Y - Xb$.

9. (3 points). Suppose you want to compute $b = (X'X)^{-1}X'Y$, where $X_{n\times k}$ and $Y_{n\times 1}$ are matrices. Which Matlab command will perform this calculation? (circle the correct answer)

   (a) `b = adjoint(X'*X)*X'*Y;`
   (b) `b = inv(X'*X)*X'*Y;`
   (c) `b = det(X'*X)*X'*Y;`
   (d) `b = rank(X'*X)*X'*Y;`

10. (4 points). Consider the following Matlab code.

```
3   A = [4 3;
4        2 5];
5   d = [28 42]';
6
7   x = zeros(size(d));
8   d_A = det(A);
9   for j = 1:size(A,2)
10      Aj = A; Aj(:,j) = d;
11      x(j,1) = det(Aj)/d_A;
12  end
13  disp(x)
```

   What does the command `Aj(:,j) = d` in line 10 do? (Circle the correct answer)

   (a) Replacing the content of column `j` in matrix `Aj` with the determinant of matrix `A`.
   (b) Replacing the content of row `j` in matrix `Aj` with the determinant of matrix `A`.
   (c) Replacing the content of column `j` in matrix `Aj` with `d`.
   (d) Replacing the content of row `j` in matrix `Aj` with `d`.

11. (3 points). What does the command `x = linspace(-10,10,100);` do?

   (a) Creating a grids of points from -10 to 100, in steps of 10.
   (b) Plotting a linear function with intercept of -10, slope of 10, using 100 grid points.
   (c) Creating a grid of 100 equally spaced points between -10 and 10, and storing in variable `x`.
   (d) Storing the vector $[-10, 10, 100]$ in variable `x`.