Measuring An Angle

Today we will focus on a topic from elementary geometry namely the concept of *measurement of angles*. The idea of an angle is a simple one in the sense that it is made by two rays emanating from the same point. But the measurement of angles is not that simple as it appears. Many theorems in elementary geometry deal with ideas which involve the concept of measurement of angles but they assume the understanding of this measurement in an implicit fashion.

Measuring Angles

For example we have the properties of a transversal cutting two parallel lines and the corresponding as well as alternate angles being equal. What do we mean when we say that the two corresponding angles (in the last statement) are equal? Well luckily the concept of equality (or congruence) of geometrical shapes is much easier to define and comprehend. Two shapes are equal if they can be exactly superimposed on one another. Thus for two angles to be equal we need to note that when we superimpose one angle on the other the vertices as well as the rays of both the angles must coincide exactly.

In the same manner by superimposing angles such that the vertices and at least one of the rays of both angles coincide we can compare two angles and determine which of the angles is greater or smaller. This is similar to comparing two line segments by the use of a divider. However it turns out that practical applications of geometry not only require us to compare line segments but rather also measure them in a precise manner. Similar is the case with angles. A lot of practical applications of geometry (and more properly trigonometry) require us to measure angles in a precise manner.

Naturally measuring anything requires two fundamental ideas:

- Defining a standard unit for measurement
- An additive law defining how any magnitude of the "entity being measured" can be expressed in terms of the standard unit.

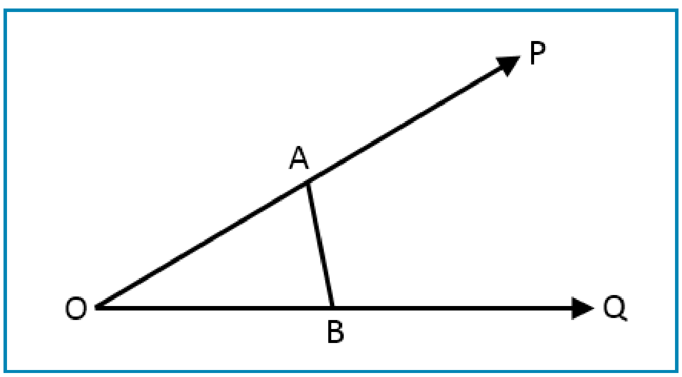
For example in case of line segments we can take any specific line segment OA and define its measure (or *length*) to be 1 and then the length of any other line segment PQ must be seen in comparison with the length of standard line segment OA whose length we have defined to be 1. More technically if line segment PQ has a point R in between P and Q such that the line segment OA can be superimposed exactly with both the line segments PR and RQ then we say that the line segment PQ has length 2. The extension to rational lengths is made possible by a very standard construction of dividing the unit line segment OA into a finite number of line segments of equal length. Finally the extension to irrational lengths is an assumption which links the branches of elementary geometry and real analysis.

The point of discussion in previous paragraph is to emphasize the fact that the concept of length of a line segment is a very fundamental one in terms of measuring magnitudes of geometrical shapes. Once we have the grasp of length it is possible to define the concept of areas of geometrical shapes (at least those shapes which are made up of line segments only). Extension to lengths and areas of shapes consisting of general curves is made possible via real analysis (via the concept of a definite integral). For the present discussion we will assume this extension of lengths and areas for any geometrical shape and proceed to measure angles.

Measuring Angles: Step 1

Intuitively the way we visualize an angle and think about comparison of angles, it is almost obvious that a greater angle somehow signifies that its rays are farther apart from one another compared to those of a smaller angle. Thus the measurement of an angle must be related somehow to the spacing between its rays. But again it is obvious that as we move away from the vertex of an angle the rays look diverging apart from each other more and more. Hence to concretely measure the spacing between the rays it is important to fix a distance from the vertex at which we want to measure the spacing between two rays.

So let POQ be an angle which we wish to measure and let A, B be points on rays OP, OQ respectively such that the length of line segments OA and OB is 1. An intuitive measure of $\angle POQ$ must be linked with a measure of separation of the points A and B. What could be simpler than just measuring the length of line segment AB? So let us define the measure of $\angle POQ$ to be the length of line segment AB where $OA = OB = 1$.



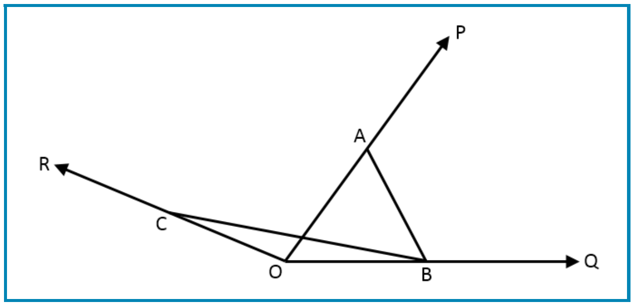
Measuring an Angle: Step 1

An equally simple measure of $\angle POQ$ could be the area of ΔAOB. In fact using area would really give a measure of space between the rays OP and OQ and the line segment AB.

However if we observe carefully both the definitions (either using length of AB or area of ΔAOB) suffer from two defects:

- The comparison of angles fails miserably, and
- The additive property also fails in a very obvious manner

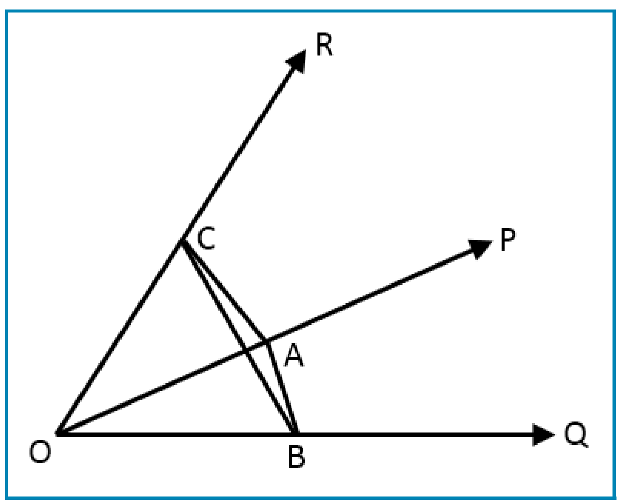
The failure of comparison of angles (using the area of ΔAOB as a measure of $\angle POQ$) is shown via the following figure:



Failure of Comparison of Angles

In the above figure we have $OA = OB = OC = 1$ and it is obvious that the $\angle ROQ$ is significantly larger than the $\angle POQ$, but the area of ΔCOB is much less than the area of ΔAOB. However note that had we used the length AB as measure of $\angle POQ$ then the comparison would have remained valid in the above figure because length of BC is obviously greater than that of AB. In order to see the failure of length AB as a measure of $\angle POQ$ we need to understand that associated with a configuration of two rays of OP and OQ emanating from same point O there are two angles possible: one the interior angle (smaller one) between OP and OQ and another the exterior angle which is the portion of plane outside the rays OP and OQ and which is significantly larger than the interior angle. When we consider this view it is obvious that both these angles must be measured by same length AB. And it might turn out that a significantly larger angle has a comparatively smaller measure.

The additive property of measurement of angles clearly does not hold with either of these definition based on length (of AB) and area (of ΔAOB) as evident from the figure below:



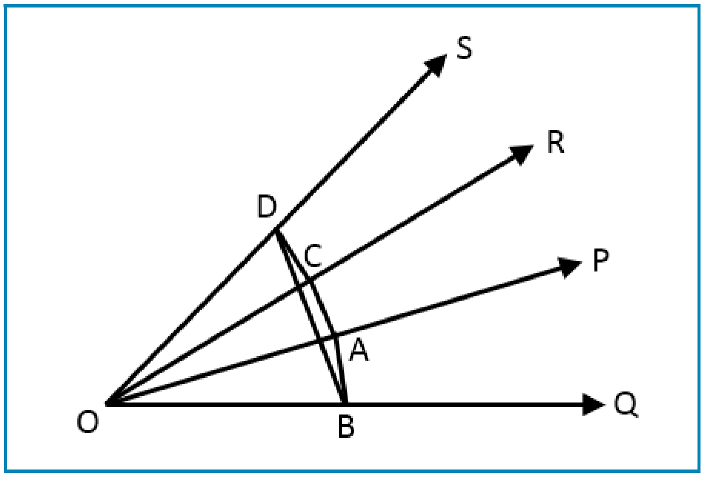
Failure of Additivity

The length BC is clearly not equal to the sum of lengths of AB and AC, rather it is slightly less than this sum. However from the same figure above it is obvious that the measure of $\angle ROQ$ must be equal to the sum of measures of $\angle ROP$ and $\angle POQ$. The same problem is seen with the areas of the triangles also as the area of ΔCOB is slightly less than the sum of areas of ΔCOA and ΔAOB.

Measuring Angles: Step 2

What went wrong with the previous definition of measure of angles? What else we need to associate with the gap between two rays emanating from the same point? Well there is no other suitable choice than the previous two approaches based on length of a line segment and area of a triangle. But they suffer from obvious defects as seen in previous paragraphs. Is it possible to fix these defects? Let us give it a try and focus on the defects more closely and perhaps we might be able to fix them.

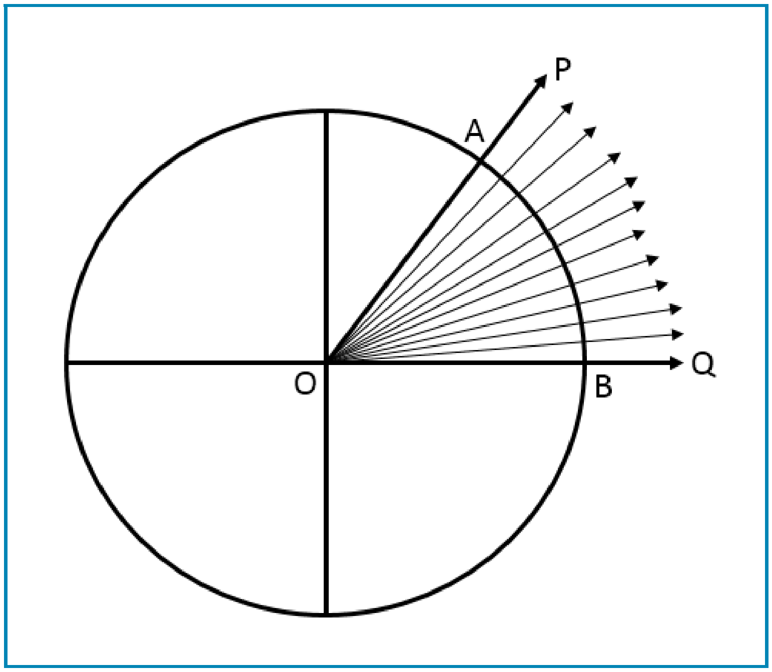
Let's take the issue of additivity. We find that the additivity fails by a very small margin. Consider the following figure:



Fixing the issue of additivity

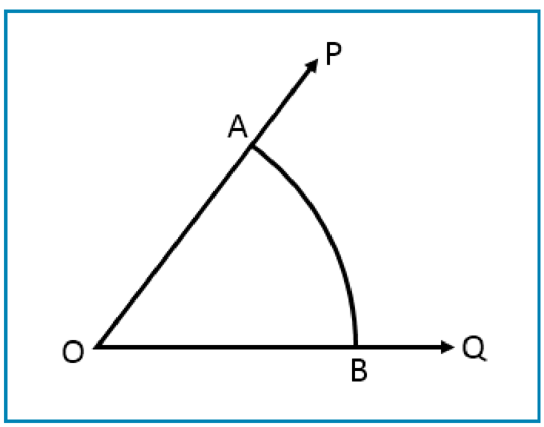
If we compare this figure with the previous figure then we note that the additivity fails here by an even smaller margin. To be explicit the difference between the length BD and the sum of lengths of AB, AC and CD is very small. Again the difference between area of ΔDOB and sum of areas of $\Delta DOC, \Delta COA$ and ΔAOB is very small. What is so different about this figure compared to the last figure? Here one angle has been expressed as the sum of three smaller angles whereas in previous case an angle was expressed as sum of two smaller angles. Intuitively it is obvious that if we express an angle as a sum of 4 smaller angles then the margin of failure of additivity in the measurement of angles will be even less. Thus to fix the issue of additivity we need to express an angle as a sum of as many smaller angles as possible.

Leaving aside the practical problem of drawing an angle as a sum of say 100 smaller angles, we can see that this is the way forward. Thus from a theoretical point of view the best option would be to let the number of smaller angles tend to infinity. When the number of smaller angles tends to infinity then we see that the polygonal line $BACD$ turns into an arc of a circle of radius 1 and centre O. And the region $OBACD$ becomes the corresponding sector associated with the arc. The situation is illustrated in the figure below:



Measuring an Angle as Sum of Many Many Angles

Starting from an intuitive approach we have finally reached the modern definition of the measure of an angle. If POQ is an angle then *its measure is defined to be the length of arc* AB *of a circle centered at* O *and radius* 1 *such that* A *lies on ray* OP *and* B *lies on ray* OQ*. The measure of* $\angle POQ$ *can also be defined as twice the area of sector* AOB*.* The unit of measurement of angles as defined above is called a *radian* and there is no specific symbol used for this unit of measurement.



Measure of an Angle

The factor of 2 comes because of a specific relation between the circumference of a unit circle (2π) and its area (π) which is based on a fact of integral calculus namely

$$\int_{0}^{1}\frac{dx}{\sqrt{1-x^{2}}}=2\cdot\int_{0}^{1}\sqrt{1-x^{2}}\,dx=\frac{\pi}{2}$$

where the integral on the left represents the length of a quadrant of a unit circle and the integral on the right represents the area of corresponding sector. We thus see that measuring an angle is intimately connected to measuring stuff related to a circle. No wonder we used semi-circular protractors in school to measure angles in degrees!

By Paramanand Singh

Sunday, November 2, 2014

Labels: Geometry