This worksheet walks you through a couple of non-trivial applications of Differential Equations.

Forensic Mathematics

A detective discovers a murder victim in a hotel room at 9:00am one morning. The temperature of the body is 80.0°F. One hour later, at 10:00am, the body has cooled to 75.0°F. The room is kept at a constant temperature of 70.0°F. Assume that the victim had a normal temperature of 98.6°F at the time of death. We'll use differential equations to find the time the murder took place.

Let $u(t)$ be the temperature of the body after t hours. By Newton's Law of Cooling we have the differential equation

$$\frac{du}{dt} = k(u - 70)$$

where k is a constant (to be determined). We'll solve the differential equation and get a formula for $u(t)$.

1 Multiply both sides by dt to get a differential form of the equation. Now do some easy algebra to get the variable u on the same side as the du. (Leave the k where it is).

$$\frac{du}{u-70} = k\,dt$$

2 Integrate both sides of the equation. Integrate the right side with respect to t and the left with respect to u. You can combine the integration constants into one "$+C$" on the right side.

$$\int \frac{du}{u-70} = \int k\,dt$$

$$\ln(u-70) = k\,t + C$$

We may assume $u - 70 \geq 0$ since the body will never be cooler than the room.

3 Solve for u as a function of t. Your function will involve the constants k and C.

$$u - 70 = e^{\ln(u-70)} = e^{k\,t+C} = e^C\,e^{k\,t}$$

Set $A = e^C$. (This is an arbitrary positive constant.) Then $u = 70 + A\,e^{k\,t}$.

4 Take $t = 0$ when the body was found at 9:00am. Plug in $t = 0$ and $u = 80.0$°F and solve for C. (It's easier to solve for $A = e^C$ and use this in your formula).

$80 = 70 + A\,e^{k\cdot 0} = 70 + A$ *so* $A = 10$ *and* $u = 70 + 10\,e^{k\,t}$.

5 Plug in $t = 1$ and $u = 75.0°$F and solve for k. (This'll take some log tricks).

$75 = 70 + 10\,e^{k\cdot 1}$ *so* $\frac{1}{2} = e^k$ *and* $k = \ln\left(\frac{1}{2}\right) = -\ln 2$. *Thus* $u = 70 + 10\,e^{-t\ln 2}$.

6 Set $u = 98.6°$F and solve for t. At what time did the murder take place?

$98.6 = 70 + 10\,e^{-t\ln 2}$ *so* $2.86 = e^{-t\ln 2}$ *and* $t = -\dfrac{\ln 2.86}{\ln 2} \approx -1.516$

1.516 *hours is about* 1 *hour and* 31 *minutes. The murder took place at 7:29am.*

Spread of a Rumor

The Xylocom Company has 1000 employees. On Monday a rumor began to spread among them that the CEO had suddenly moved to Brazil. It is reasonable to assume that the rate of the spread of the rumor is proportional to the number of possible encounters between employees who have heard the rumor and those who have not. Let $y = y(t)$ be number of employees who have heard the rumor after t days.

1 Explain why the number of possible meetings between employees who have heard the rumor and those who have not equals $y\,(1000 - y)$.

The number of possible meetings = (the number who've heard the rumor)× (the number who haven't)

2 Write a differential equation that describes this model of the spread of a rumor. (Remember that "is proportional to" means "is some constant k times".)

$$\frac{dy}{dt} = k\,y\,(1000 - y)$$

3 Proceed as in the cooling body problem to solve the differential equation for $y(t)$. You will need to use the method of partial fractions. Your answer should involve two constants: the proportionality constant k and a constant C from integrating. (As in part 2 of the previous problem, you can combine the integration constants into one "$+C$" on the right side.)

First separate the variables: $\dfrac{dy}{y\,(1000 - y)} = k\,dt$

Integrate both sides: $\displaystyle\int \frac{dy}{y\,(1000 - y)} = \int k\,dt = k\,t + C$

Use Partial Fractions: $\quad \dfrac{1}{y\,(1000-y)} = \dfrac{A}{y} + \dfrac{B}{1000-y} \qquad 1 = A\,(1000-y) + B\,y$

Set $y = 0$ *to get* $A = \frac{1}{1000}$. *Set* $y = 1000$ *to get* $B = \frac{1}{1000}$.

So $\displaystyle \int \frac{dy}{y\,(1000-y)} = \frac{1}{1000}\int \frac{1}{y} + \frac{1}{1000-y}\,dy = \frac{1}{1000}\,[\ln(y) - \ln(1000-y)] + C'.$

Thus we get $\quad \dfrac{1}{1000}\ln\left(\dfrac{y}{1000-y}\right) = k\,t + C. \qquad$ *(Note that* $\dfrac{y}{1000-y} \geq 0.$*)*

Solving for y*, we first get* $\quad \dfrac{y}{1000-y} = A\,e^{1000\,k\,t}$ *where* $A = e^{1000C}$

and finally $\quad y = 1000\,\dfrac{A\,e^{1000\,k\,t}}{1 + A\,e^{1000\,k\,t}}.$

4 At the very beginning, 50 people had heard the rumor (they all attended the same meeting). Compute the constant $A = e^C$.

[Note: It is easier to use $A = e^{1000C}$ *as we did in the previous part.]*

Set $t = 0$ *and* $y = 50$ *to get* $\quad 50 = 1000\,\dfrac{A}{1+A}. \qquad$ *This gives* $A = \frac{1}{19}$.

So $\quad y = 1000\,\dfrac{e^{1000\,k\,t}}{19 + e^{1000\,k\,t}}.$

5 On Tuesday morning, 100 people had heard the rumor. Compute the constant k.

On Tuesday morning, $t = 1$ *and* $y = 100$*. This gives* $\quad 100 = 1000\,\dfrac{e^{1000\,k}}{19 + e^{1000\,k}}.$

Solving, we get $\quad e^{1000\,k} = \dfrac{19}{9} \quad$ *and so* $\quad k = \dfrac{1}{1000}\ln\left(\dfrac{19}{9}\right).$

Thus $\quad y = 1000\,\dfrac{e^{t\,\ln(19/9)}}{19 + e^{t\,\ln(19/9)}}.$

6 When will 800 people have heard the rumor?

Solve $\quad 800 = 1000\,\dfrac{e^{t\,\ln(19/9)}}{19 + e^{t\,\ln(19/9)}} \quad$ *for* t *to get* $\quad t = \dfrac{\ln(76)}{\ln(19/9)} \approx 5.7958$ *days.*

This is about 5 days, 19 hours, and 6 minutes after the rumor started. If, for example, the rumor started at 9:00 AM Monday morning, this would be 4:06 AM Sunday morning.