ECON 715, Visualizing Quasiconcave Functions

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Sum of Quasiconcave Functions May not be Quasiconcave

We proved in the lecture notes that the sum of concave functions must be concave. The same is not true however for quasiconcave functions - a sum of quasiconcave functions **does not have to be** quasiconcave. In these note we provide an example of two quasiconcave functions that add up to a function which is not quasiconcave.

Consider two quasiconcave functions:

$$f(x) = x^3, \quad \text{and} \quad g(x) = -4x.$$

You should be able to verify that these functions are quasiconcave. First, we can express $f(x)$ as a monotone increasing transformation of a concave function:

$$f(x) = (\sqrt{x})^6,$$

where $\sqrt{x}$ is concave (strictly) and the 6-th power is monotone increasing function on R_+. Recall that any monotone increasing transformation of a concave function is quasiconcave.

The function $g(x) = -4x$ is linear, and therefore both concave and convex, and concave functions must be quasiconcave.

Another useful result for recognizing quasiconcave functions is that **any monotone function is both quasiconcave and quasiconvex**, and therefore both f and g given above are quasiconcave (f is monotone increasing, and g is monotone decreasing).

Yet another way to recognize that both f and g are quasiconcave is by plotting them and idenitifying their upper contour sets. This way of recognizing quasiconcave functions is not practical for functions of several variables or complicated functions.

The next figure plots the function $f(x) = x^3$, and highlights the upper contour set for level $\alpha = 2$ (bold red segment on the x-axis).

```
syms x %Declaring symbolic variable
f(x) = x^3; %Declaring symbolic function
a = 2; %Level for Upper Contour Sets
fplot(f, [-2,2], "LineWidth",2, "Color", [0 0.4470 0.7410]) %Plot of x^3
hold on
p = fplot(0, [a^(1/3), 2], "LineWidth", 3, "Color", "red");

xline(a^(1/3), 'k--', 'LineWidth',1) %Plots the y-coordinate
yline(a, 'k--', 'LineWidth',1) %Level of upper contour sets
yline(0, 'k--', 'LineWidth',1) %Plots the x-coordinate
```

```
    legend(p, ...
        ['Upper Contour Set at level \alpha = ', num2str(a)], ...
        'Location','northwest')
    title('f(x) = x^3')
    xlabel('x')
    ylabel('y', "Rotation", 0)
    grid on
    grid minor
    hold off
```
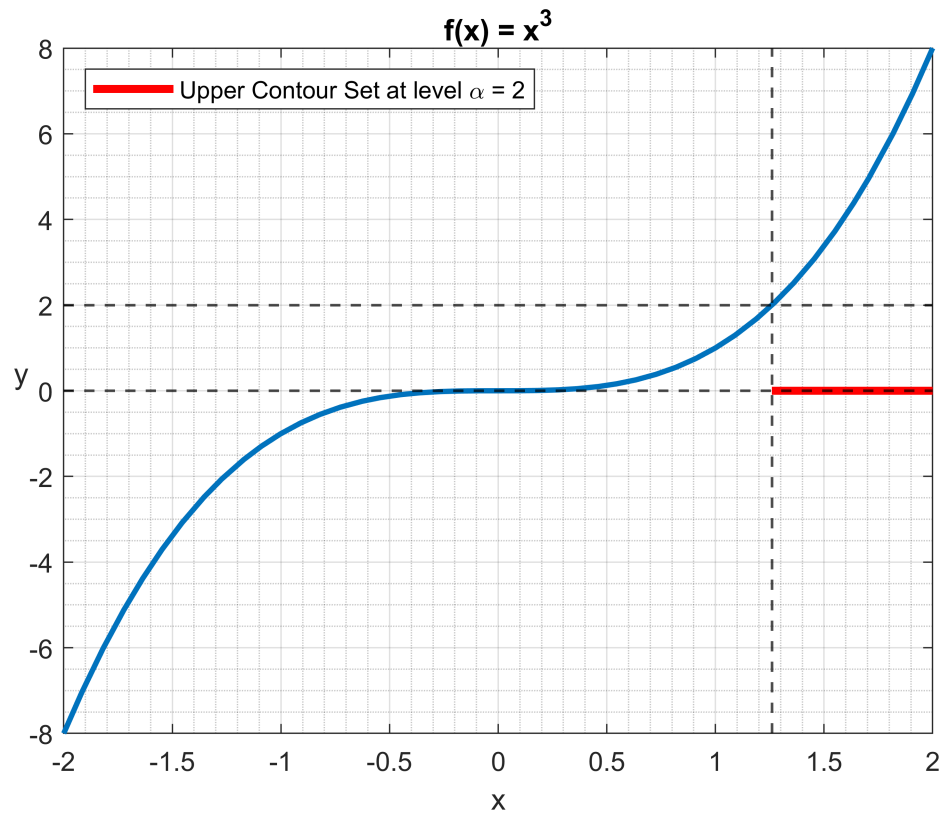

Mathematically, the above upper contour set is described as:

$$U(\alpha = 2; f(x) = x^3) = \{x | x^3 \geq 2\}$$

In words, this is the set of all x values such that the value of the given function is above 2.

We can find the upper contour set:

$$x^3 \geq 2$$

$$x \geq 2^{\frac{1}{3}} \approx 1.2599$$

This is exactly the line segment highlighted in red on the x-axis in the above graph. We can see from the above graph that the upper contour set is convex set, and this is the case for any level α .

Similarly, we can graph the function $g(x) = -4x$, and identify its upper contour sets. In the next figure, the upper contour set for level $\alpha = 2$ is highlighted on the x-axis in bold red.

```
    g(x) = -4*x; %Declaring symbolic function
```

```
fplot(g, [-2,2], "LineWidth",2, "Color", [0.8500 0.3250 0.0980]) %Plot of x^3
hold on
p = fplot(0, [-2, -1/2], "LineWidth", 3, "Color", "red");
xline(-1/2, 'k--', 'LineWidth',1) %Plots the y-coordinate
yline(0, 'k--', 'LineWidth',1) %Plots the x-coordinate
yline(a, 'k--', 'LineWidth',1) %Level of upper contour sets
legend(p, ...
    ['Upper Contour Set at level \alpha = ', num2str(a)], ...
    'Location','northeast')
title('g(x) = -4x')
xlabel('x')
ylabel('y', "Rotation", 0)
grid on
grid minor
hold off
```
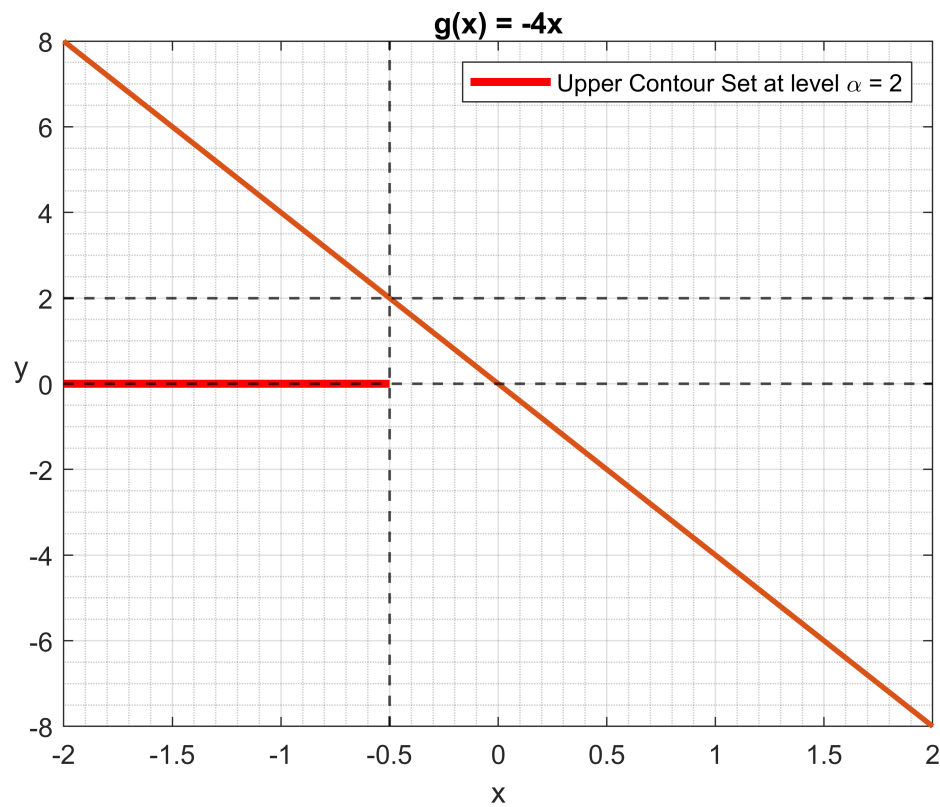

Mathematically, the above upper contour set is described as:

$$U(\alpha = 2; g(x) = -4x) = \{x \mid -4x \geq 2\}$$

We can find the upper contour set:

$$-4x \geq 2$$

$$x \leq -\frac{1}{2}$$

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Notice that again that the upper contour set is convex.

Next, we add the two quaisiconcave functions,

$$h(x) = f(x) + g(x) = x^3 - 4x$$

and observe that the sum is not quasiconcave. In the next graph, notice that the upper contour set consists of two separate segments (bold red segments on the x-axis. Thus, the upper contour is not a convex set, and therefore the function $h(x) = f(x) + g(x)$ is not quasiconcave.

```
h(x) = f(x) + g(x); %Sum of two quasiconcave functions
S = double(vpasolve(f(x) + g(x) == a)) %Solving for upper contour set
```

```
S = 3×1
   -1.6751
   -0.5392
    2.2143
```

```
fplot(h, [-3, 5], "LineWidth",2, "Color", [0.4940 0.1840 0.5560]) %Plot of x^3
ylim([-4, 5])
hold on
p = fplot(0, [S(1), S(2)], "LineWidth", 3, "Color", "red");
fplot(0, [S(3), 5], "LineWidth", 3, "Color", "red");
xline(S(1), 'k--', 'LineWidth',1) %Plots the y-coordinate
xline(S(2), 'k--', 'LineWidth',1) %Plots the y-coordinate
xline(S(3), 'k--', 'LineWidth',1) %Plots the y-coordinate

yline(0, 'k--', 'LineWidth',1) %Plots the x-coordinate
yline(a, 'k--', 'LineWidth',1) %Level of upper contour sets
legend(p, ...
    ['Upper Contour Set at level \alpha = ', num2str(a)], ...
    'Location','northwest')
title('h(x) = x^3 - 4x')
xlabel('x')
ylabel('y', "Rotation", 0)
grid on
grid minor
hold off
```
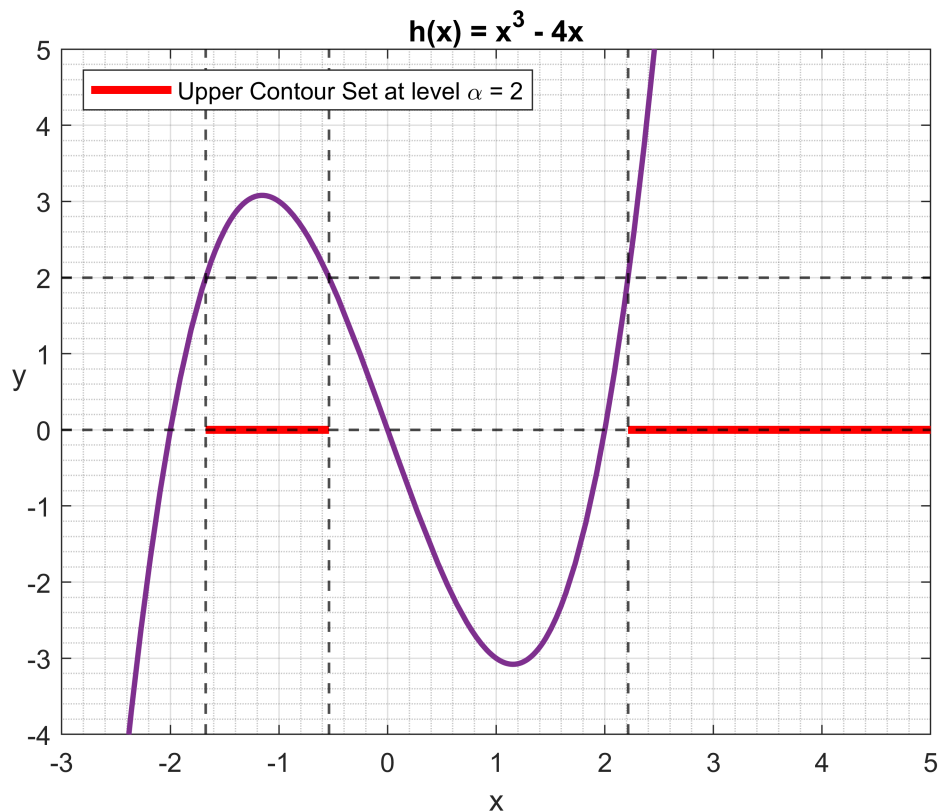
Remark. With this function $x^3 - 4x$, finding the upper contor set analytically is hard, and I used Matlab's numerical solver for that.

Summary of Main Results About Quasiconcavity

1. Any monotone increasing transformation of a concave function must be quasiconcave.
2. Any quasiconcave function must arise as a monotone increasing transformation of some concave function.
3. Any monotone (incresing or decresing) function is quasiconcave.
4. Not every quasiconcave function is monotone. Remember the bell-shaped pdf of $N(\mu, \sigma^2)$.
5. Weighted sum of quasiconcave functions does not have to be quasiconcave.

The last property was illustrated in this note.