Math 230, Exam 5
November 17, 2018

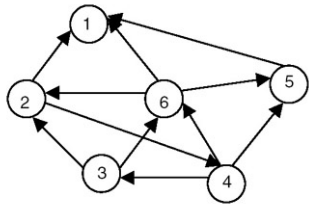
Name: ______________________________

Instructions.

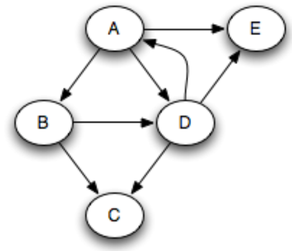
- You are allowed one side of handwritten notes
- No calculators.
- There are 6 problems on 5 pages. Make sure your exam is complete.

Run LaTeX again to produce the table

1. Two websites have the following link structures:



Website One



Website Two

[3 points] (a) Suppose a person randomly clicks links. Write the transition matrix p for Website One.

Solution:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

[2 points] (b) Given that $p^3 = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.666 & 0.139 & 0.0 & 0.055 & 0.055 & 0.083 \\ 0.5 & 0.0 & 0.139 & 0.083 & 0.139 & 0.139 \\ 0.583 & 0.055 & 0.037 & 0.194 & 0.092 & 0.037 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.61 & 0.092 & 0.055 & 0.037 & 0.092 & 0.111 \end{pmatrix}$.

What is the probability a site-surfer starting at page 6 is at page 4 after three clicks?

Solution: $p^3(6,4) = .037$.

[2 points] (c) Without doing any calculations find the stationary distribution of p.

Hint: Look at the link structure. What must happen in the long term?

Solution: $\pi = [1\ 0\ 0\ 0\ 0\ 0]$ since Page 1 is an absorbing state.

[2 points] (d) For Website Two $\pi = [0\ 0\ \frac{1}{2}\ 0\ \frac{1}{2}]$ has $\pi p = \pi$. Explain what the entries of π represent.

2. A car rental company has rental has lots at both Kennedy and LaGuardia airports. Assume that a car rented at one airport must be returned to one of the two airports. If the car was rented at LaGuardia the probability it will be returned there is $\frac{3}{4}$; for Kennedy the probability is $\frac{1}{2}$. Suppose that each week all of the cars are rented once.

[2 points] (a) Write the transition matrix, p, for this Markov chain. Let state 1 be a car is at La Guardia.

Solution:

$$p = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

[2 points] (b) Suppose that we start with 1/3 of the cars at LaGuardia. Write the row vector, q, for the initial distribution of cars.

Solution: $q = [\frac{1}{3} \quad \frac{2}{3}]$.

[2 points] (c) What is the fraction of cars at LaGuardia at the end of the first week?

Solution:
$qp = [\frac{1}{4} + \frac{1}{3} \quad \frac{1}{12} + \frac{1}{3}] = [\frac{7}{12} \quad \frac{5}{12}]$. So the answer is $\frac{7}{12}$.

[2 points] (d) In the long run what fraction of cars are at Kennedy?

Solution: The stationary distribution is $\pi = [\frac{1/2}{3/4} \quad \frac{1/4}{3/4}] = [\frac{2}{3} \quad \frac{1}{3}]$. So in the long run $\frac{1}{3}$ of the cars will be at Kennedy.

3. Gary is gambling and starts with **\$2**. He is flipping a fair coin and receives \$1 if he flips heads and loses \$1 if he flips tails. He stops playing when he has no money. Let X_n be the amount of flips for which he has n dollars.

[4 points] (a) Explain why X_2 is a geometric random variable. Find its parameter, p.

Solution: After each flip at \$2 Gary will either go broke or return to having \$2. If he returns the probability of going broke is the same. We have p_2 is the probability he flips tails twice in a row. This is $\frac{1}{4}$.

[2 points] (b) For $n \geq 2$, what is $P(X_n > 0)$? (Hint: Gambler's ruin.)

Solution: $\frac{2}{n}$.

[3 points] (c) Suppose Gary has reached n dollars. What is the probability, q, he goes broke before visiting n again?

Solution: He has to flip tails, then not return to n. Again this is gambler's ruin but this time the picture is reversed. So we have $\frac{1}{2}\frac{1}{n}$. The 1/2 is for flipping the first tails.

[3 points] (d) You have just shown that when Gary reaches $n \geq 2$ he visits it a $Y =$ geometric(q) number of times. Use (b) and (c) to compute EX_n.

Solution:

$$EX_n = E[X_n \mid X_n > 0]P(X_n > 0) = E[X_n \mid X_n > 0]\frac{2}{n} = 2\frac{EY}{n} = 4.$$

(e) One point extra credit. What is EX_n if instead Gary starts with $k \leq n$ dollars?

Solution: Now we have $P(X_n > 0) = \frac{k}{n}$, so we multiple by a factor of k. Thus, $EX_n = 2k$.

4. Let $f(x) = 1 - cx^2$ for $0 \leq x \leq 1$ and 0 for all other values of x.

[2 points] (a) Find c so that $f(x)$ is a density function.

Solution: $\int_0^1 1 - cx^2 dx = x - \frac{1}{3}x^3\Big|_0^1 = 1 - \frac{c}{3} = 1$.So $c = 0$.

[2 points] (b) Suppose that X is a continuous random variable with density function f (using the value of c you found in the previous part). What is EX?

Solution: $EX = \int_0^1 x \cdot 1 dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$.

[3 points] (c) What is var(X)? You can leave your answer as a sum of unsimplified fractions.

Solution: $EX^2 = \int x^2 \cdot 1 dx = \frac{x^3}{3}|_0^1 = \frac{1}{3}$. So var$(X) = \frac{1}{3} - \frac{1}{2} = \frac{1}{12}$.

5. Let X be a uniform(0,1) random variable and Y an independent exponential(1) random variable.

[2 points] (a) If $W = \max(X, Y)$, find the distribution function $F(w) = P(W \leq w)$.

Solution: $P(W \leq w) = P(X \leq w)P(Y \leq w) = w(1 - e^{-w})$.

[4 points] (b) If $T = \min(X, Y)$, find the distribution function $F(t) = P(T \leq t)$.

Solution: $P(T \leq t) = 1 - P(T > t) = 1 - P(X > t)P(Y > t) = 1 - (1 - t)(e^{-t})$.

[8 points] 6. Suppose that X and Y are independent uniform(0,1) random variables. Let A be the event that the line segment joining X and Y has an overlap with the interval $[\frac{1}{3}, \frac{2}{3}]$ (i.e. they have at least one shared point). What is $P(A)$?

Hint: There are two ways to do this. One is to compute $P(A^c)$. The other is to condition on the three events $B_1 = \{X < 1/3\}$, $B_2 = \{1/3 \leq X \leq 2/3\}$, and $B_3 = \{X > 2/3\}$, then use $P(A) = \sum_{i=1}^{3} P(A \mid B_i)P(B_i)$. You can use either approach.

Solution:

1. The complement of A is to have X and Y both are in $[0, 1/3)$ or both are in $(2/3, 1]$. These are disjoint and occur with probability $(1/3)^2 + (1/3)^2 = \frac{2}{9}$. So $1 - \frac{2}{9} = \frac{7}{9}$ is the answer.

2. If we don't work with the complement, we can condition on the value of X.

$$
\begin{aligned}
P(A) &= P(A \mid X < \frac{1}{3})P(X < \frac{1}{3}) + P(A \mid \frac{1}{3} \leq X \leq \frac{2}{3})P(\frac{1}{3} \leq X \leq \frac{2}{3}) + P(A \mid X > \frac{2}{3})P(X > \frac{2}{3}) \\
&= P(Y > 1/3)\frac{1}{3} + 1\frac{1}{3} + P(Y < 2/3)\frac{1}{3} \\
&= \frac{2}{3}\frac{1}{3} + \frac{1}{3} + \frac{2}{3}\frac{1}{3} \\
&= \frac{7}{9}
\end{aligned}
$$