Dynamic risk sharing with moral hazard

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Abstract

I characterize the optimal risk sharing contract in dynamic economies with moral hazard. In a full information environment, an optimal contractual arrangement prescribes that agents pool their income and divide it according to a constant sharing rule. When moral hazard is present, the sharing rule changes through time in order to reward effort. As a consequence, consumption inequality is very persistent. If agents have access to unmonitorable asset markets, then they can use their assets to smooth consumption and reduce effort. An optimal contract would avoid that, by imposing an additional cost (a wedge) on savings. As a result, trading in the asset market is restricted: the planner prevents both excessive aggregate savings and excessive aggregate borrowing.

1 Introduction

One of the main causes of the financial crisis of 2008 was excessive risk taking of financial institutions. Many banks were lending to unworthy customers, since it was

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easy to securitize loans and sell them in the derivatives market, hence transferring risk to someone else. Therefore, policymakers all over the world put strong emphasis in reforming derivatives markets and regulating them more effectively. In particular, many seem to believe that having derivatives instruments traded in centralized markets with clearinghouses would reduce excessive risk taking. A crucial question is thus how such a market must be regulated and monitored by financial authorities. This paper tries to answer this question in a general framework with moral hazard and access to insurance markets. It characterizes the optimal dynamic risk sharing contract between a pool of agents that exert hidden effort and can trade anonymously a risk free bond.

In a full information environment, an optimal contractual arrangement prescribes that agents pool their income and use a constant sharing rule. This is not optimal with moral hazard, since incentives to exert effort would be minimal. An optimal arrangement must then consider the trade-off between providing insurance and incentives. However, in many situations agents have anonymous access to markets that can provide some partial insurance. It is therefore possible that incentives are affected by these markets: by buying insurance, agents can exert less effort (and therefore generate less output on average) and still enjoy the same level of consumption. Therefore, an optimal risk sharing scheme must prevent agents from buying or selling excessive insurance in these unmonitorable markets. In order to obtain this result, the planner puts a wedge between marginal utility of consumption today and expected marginal utility of consumption tomorrow. In particular, the planner would like to make savings not profitable for people that have a good sequences of shocks, while it must make borrowing not profitable for people that have bad sequences of shocks. However, it cannot charge individual wedges since markets are anonymous. Hence, the wedge can be either positive or negative: if a majority of agents want to sell bonds, then the wedge will be negative, and viceversa[1].

The implications for the regulation of derivatives markets are then straightforward. If the moral hazard problem is in the possibly excessive risk taking of financial institutions, the derivative market can always give some partial insurance to agents,

[1] This is different from Golosov and Tsyvinski (2007), where wedges were always positive.

who then would be willing to take more risk. A benevolent planner can improve on the competitive market allocation by setting state contingent wedges, and in order to do that it does not need detailed information about the balance sheet of each financial intermediaries: it only needs information about aggregate supply and demand in the derivatives markets.

As a benchmark, I first characterize the optimal contract in a simple endowment economy with no access to asset markets. Each agent receives a stochastic income, the distribution of which depends on his own privately observed effort. The characterization of the optimal allocation can be obtained by means of endogenously evolving Pareto-Negishi weights. The contracting problem can be represented as a dynamic social welfare maximization in which the planner observes the realization of shocks and increases (respectively, decreases) the Pareto weight of each agent if the realization of the shock is "good" ("bad"). By optimally choosing these weights for each history, the planner makes sure to provide enough incentives to each agent. I show that the problem has a recursive structure: Pareto weights are new state variables that keep track of the history of shocks' realizations. Therefore, the optimal contract can be obtained by solving a standard dynamic program. In the optimum, the ratio of inverse marginal utilities of consumption is equal to the ratio of Pareto weight. It can be shown that the latter evolves as a submartingale, therefore implying persistent and increasing consumption inequality.

The setup is extended to allow agents to buy and sell a risk free bond in a unmonitorable market. Each agent can trade in a risk-free bond at market price, and these trades are private information[2]. In this case, marginal rates of substitution between consumption today and tomorrow must be equalized across agents. This implies that the previous submartingale result for the ratio of marginal utility of consumption is not valid anymore. Indeed, the ratio of Pareto weights is still a submartingale, but this ratio is not equal to the consumption marginal utilities ratio

[2]In a model with private information over labor productivity shocks, Golosov and Tsivinski (2007) show, with numerical simulations, that private insurance markets can provide almost efficient levels of insurance without public intervention. A quantitative exploration of the efficiency of private insurance markets in this framework with hidden effort is left for future research

anymore. Instead, the planner wants to put a wedge between the ratio of Pareto weights and the consumption marginal utilities in order to distort the consumption/saving decisions of the agents. In other words, the optimal contract restricts trade in the asset market.

Zhao (2007) analyzes the case of an endowment economy with two-sided moral hazard, and shows how to solve the model with recursive techniques based on the work of Abreu, Pearce and Stacchetti (1990). This approach is not easy to apply to production economies or even to endowment economies where the financial market is not monitorable by the planner. Mainly this is due to the complexities discussed in Mele (2011): as the number of state variables and agents increases, the APS approach becomes extremely burdensome for numerical simulations. Therefore, I will use the Lagrangean approach to study these models. My work extends and integrate the work by Zhao (2007), by allowing agents to have access to anonymous markets.

Friedman (1998) studies the same model in Zhao (2007), and shows that the planner problem has a recursive structure in the space of Pareto weights, also if he does not directly apply the Lagrangean approach. His work is mainly theoretical, though, and it does not provide any numerical example. Moreover, it is focused on endowment economies only.

The paper is organized as follows: Section 2 presents a model of risk sharing with repeated moral hazard in an endowment economy. Section 3 extends the basic setup to allow for hidden trades in the financial market. Section 5 presents few numerical examples of simulated 2-agents economies, and Section 6 concludes.

2 An endowment economy

As a benchmark, we first look at an endowment economy with multi-sided moral hazard wit no assets. There are N agents indexed by $i = 1, ..., N$. Each agent receive a stochastic endowment, governed by an observable Markov state process $\{s_{it}\}_{t=0}^{\infty}$, where $s_{it} \in S_i$. I assume that s_{i0} is known, and the process is common knowledge. I will denote with subscripts the single realizations, and with superscripts the whole histories of states:

$$s_i^t \equiv \{s_{i0}, ..., s_{it}\}$$

I also assume that the processes are independent across agents. Let $s_t \equiv \{s_{1t}, ..., s_{Nt}\}$ be the state of nature in the economy, let $s^t \equiv \{s_0, ..., s_t\}$ be the history of their realizations.

The agent i exerts a costly action $a_{it}(s^t)$, which is unobservable to other players. This action affects next period distribution of states of nature: let $\pi(s_{i,t+1} \mid s_{it}, a_{it}(s^t))$ be the probability that state is $s_{i,t+1}$ conditional on past state and effort exerted by the government in period t. Therefore, since the processes are independent across agents, we can define $\Pi(s^{t+1} \mid s_0, a^t(s^t)) = \prod_{i=1}^N \prod_{j=0}^t \pi(s_{i,j+1} \mid s_{ij}, a_{ij}(s^j))$ to be the cumulated probability of an history s^{t+1} given the whole history of unobserved actions $a^t(s^t) \equiv (a_0(s^0), a_1(s^1), ..., a_t(s^t))$. I assume $\pi(s_{i,t+1} \mid s_{it}, a_{it}(s^t))$ is differentiable in $a_{it}(s^t)$ as many time as necessary, and I denote its derivative with respect to $a_{it}(s^t)$ as $\pi_{a_i}(s_{i,t+1} \mid s_{it}, a_{it}(s^t))$.

The utility of the agent is

$$u\left(c_{it}\left(s^t\right)\right) - \upsilon\left(a_{it}\left(s^t\right)\right)$$

for which we assume as usual $u_c > 0$, $u_{cc} < 0$, $u_l > 0$, $u_{ll} > 0$ and Inada conditions, and $\upsilon' > 0$, $\upsilon'' \geq 0$. Aggregate resource constraint is:

$$\sum_{i=1}^N c_{it}\left(s^t\right) \leq \sum_{i=1}^N y_{it}\left(s^t\right) \tag{1}$$

where $y_i(s_t)$ is the endowment of agent i in period t. A contract is a pair of sequences $\{c_{it}(s^t), a_{it}(s^t)\}_{t=0}^\infty$ for each agent. I assume there is perfect commitment from all parts when they enter in the contract. Let us start with some definitions:

Definition 1 *A contract* $\{c_{it}(s^t), a_{it}(s^t)\}_{t=0}^\infty$ *is incentive compatible if* $\forall i$

$$a_i^\infty \in \arg\max \sum_{t=0}^\infty \sum_{s^t} \beta^t \left[u\left(c_{it}\left(s^t\right)\right) - \upsilon\left(a_{it}\left(s^t\right)\right)\right] \Pi\left(s^{t+1} \mid s^t, a^{t-1}\left(s^{t-1}\right)\right)$$

It is obviously difficult to deal with the incentive compatible constraint as defined above. In order to have a simpler problem, I apply a first order approach: I use the first order condition with respect to $a_{it}(s^t)$ of the government problem to characterize the optimal contract:

Definition 2 *A contract* $\{c_{it}(s^t), a_{it}(s^t)\}_{t=0}^{\infty}$ *is first order incentive compatible if*

$$\upsilon'\left(a_{it}\left(s^t\right)\right) = \sum_{j=1}^{\infty}\sum_{s^{t+j}|s^t}\beta^j \frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}\left[u\left(c_{i,t+j}\left(s^{t+j}\right)\right) - \right. \\ \left. -\upsilon\left(a_{i,t+j}\left(s^{t+j}\right)\right)\right]\Pi\left(s^{t+j} \mid s^t, a^{t+j-1}\left(s^{t+j-1}\right)\right) \quad (2)$$

In order to be sure that that first-order incentive compatibility is equivalent to incentive compatibility, we assume that Rogerson (1985) conditions are satisfied.

Let $\{\omega_i\}_{i=1}^{N}$ be a given vector of initial Pareto weights. Therefore the constrained efficient allocation is the solution of the following maximization problem:

$$P(s_0) = \max_{\left\{\{c_{it}(s^t), a_{it}(s^t)\}_{i=1}^N\right\}_{t=0}^{\infty}} \left\{\sum_{i=1}^{N}\omega_i\sum_{t=0}^{\infty}\beta^t\sum_{s^t}\left[u\left(c_{it}\left(s^t\right)\right) - \right.\right. \\ \left.\left. -\upsilon\left(a_{it}\left(s^t\right)\right)\right]\Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right)\right\}$$

$$s.t. \quad \upsilon'\left(a_{it}\left(s^t\right)\right) = \sum_{j=1}^{\infty}\sum_{s^{t+j}|s^t}\beta^j \frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}\left[u\left(c_{i,t+j}\left(s^{t+j}\right)\right) - \right. \\ \left. -\upsilon\left(a_{i,t+j}\left(s^{t+j}\right)\right)\right]\Pi\left(s^{t+j} \mid s^t, a^{t+j-1}\left(s^{t+j-1}\right)\right) \\ \forall i = 1, ..., N$$

$$\sum_{i=1}^{N} c_{it}\left(s^t\right) = \sum_{i=1}^{N} y_{it}\left(s^t\right)$$

We can now write down the Lagrangean of this problem. Let $\lambda_{it}(s^t)$ be the Lagrange multiplier of the (first-order) incentive compatibility constraint for agent i, and let $\phi \equiv \{\phi_i\}_{i=1}^N$, $\varsigma \equiv \{c_i, a_i\}_{i=1}^N$, $\nu \equiv \{\lambda_i\}_{i=1}^N$. By applying the methodology of Mele (2011), we get:

$$L\left(\varsigma^{\infty}, \nu^{\infty}, \phi^{\infty}\right) = \\ = \sum_{i=1}^{N}\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\left\{\phi_{it}\left(s^t\right)\left[u\left(c_{it}\left(s^t\right)\right) - \upsilon\left(a_{it}\left(s^t\right)\right)\right] - \right. \\ \left. -\lambda_{it}\left(s^t\right)\upsilon'\left(a_{it}\left(s^t\right)\right)\right\}\Pi\left(s^t \mid h_0, a^{t-1}\left(s^{t-1}\right)\right)$$

where

$$\phi_{i,t+1}\left(s^t, s_{t+1}\right) = \phi_{it}\left(s^t\right) + \lambda_{it}\left(s^t\right)\frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)} \qquad i = 1, ..., N$$

$$\phi_{i0}\left(s_0\right) = \omega_i \qquad i = 1, ..., N$$

The new variable $\phi_{it}(s^t)$ is an endogenously evolving Pareto-Negishi weight which keeps track of the incentive compatibility constraint: $\lambda_{it}(s^t) > 0$ and then $\phi_{i,t+1}(s^t, s_{t+1}) < \phi_{it}(s^t)$ for "good" states of nature, and $\phi_{i,t+1}(s^t, s_{t+1}) > \phi_{it}(s^t)$ for "bad" states of nature.

The following Proposition shows that this problem is recursive:

Proposition 1 *The constrained efficient allocation solves the following functional equation*

$$\begin{aligned} W(s,\phi) &= \min_{\nu}\max_{\varsigma}\left\{ r(\varsigma,\nu,s,\phi) + \beta\sum_{s'}\pi(s' \mid s,a)\, W(s',\phi'(s'))\right\} \\ s.t. & \qquad \phi_i'(s') = \phi_i + \lambda_i \frac{\pi_{a_i}(s' \mid s,a)}{\pi(s' \mid s,a)} \qquad i = 1,...,N \end{aligned}$$

where

$$r(\varsigma,\nu,s,\phi) \equiv \sum_{i=1}^{N}\left\{\phi_i\left[u(c_i) - \upsilon(a_i)\right] - \lambda_i \upsilon'(a_i)\right\}$$

Moreover, the operator

$$\begin{aligned} (TW)(s,\phi) &= \min_{\nu}\max_{\varsigma}\left\{ r(\varsigma,\nu,s,\phi) + \beta\sum_{s'}\pi(s' \mid s,a)\, W(s',\phi'(s'))\right\} \\ s.t. & \qquad \phi_i'(s') = \phi_i + \lambda_i \frac{\pi_{a_i}(s' \mid s,a)}{\pi(s' \mid s,a)} \qquad i = 1,...,N \end{aligned}$$

is a contraction, and $W(\cdot,\phi)$ is homogenous of degree 1, while the policy correspondences $\varsigma(\cdot,\phi)$ and $\nu(\cdot,\phi)$ are homogeneous of degree zero.

Proof. *It's a direct application of Proposition 2 in Mele (2011).* ■

It is also possible to simplify the problem, in the case with two agents. In that case, we can show that the problem is equivalent to an auxiliary problem with only one endogenous state variable that corresponds to the ratio of the costate variables. In fact, $\frac{1}{\phi_1}W(s,\phi_1,\phi_2) = W\left(s,1,\frac{\phi_2}{\phi_1}\right) \equiv \widetilde{W}\left(s,\frac{\phi_2}{\phi_1}\right)$ for any s by the homogeneity property of the value function, and clearly $\varsigma(s,\phi_1,\phi_2) = \varsigma\left(s,1,\frac{\phi_2}{\phi_1}\right) \equiv \widetilde{\varsigma}\left(s,\frac{\phi_2}{\phi_1}\right)$ for any s and $\nu(s,\phi_1,\phi_2) = \nu\left(s,1,\frac{\phi_2}{\phi_1}\right) \equiv \widetilde{\nu}\left(s,\frac{\phi_2}{\phi_1}\right)$ by the homogeneity properties of the policy correspondences.

2.1 Characterization of the contract

The following proposition is related to the result in Rogerson (1985a). It states that the optimal allocation satisfies an inverted Euler equation. Moreover, the ratio of marginal utilities depends on the ratio of the endogenous Pareto weights.

Proposition 2 *A constrained-efficient allocation* $\left\{c_{it}\left(s^t\right), a_{it}\left(s^t\right)_{i=1}^N\right\}_{t=0}^{\infty}$ *satisfies the following conditions:*

$$\frac{\phi_{jt}\left(s^t\right)}{\phi_{it}\left(s^t\right)} = \frac{u_c\left(c_{it}\left(s^t\right)\right)}{u_c\left(c_{jt}\left(s^t\right)\right)} \tag{3}$$

$$E_{i,t}\left\{\left[u_c\left(c_{i,t+1}\left(s^t, s_{t+1}\right)\right)\right]^{-1}\right\} = \frac{1}{u_c\left(c_{it}\left(s^t\right)\right)} \qquad \forall t \geq 0 \tag{4}$$

where $E_{i,t}$ *is the expectation operator under the probabilities defined by the optimal action of agent* i*.*

Proof. *First order condition for consumption of the Lagrangean imply directly first equation. For (4), rewrite first order condition for consumption at time* $t+1$

$$\phi_{i,t+1}\left(s^t, s_{t+1}\right) = \frac{1}{u_c\left(c_{i,t+1}\left(s^t, s_{t+1}\right)\right)}$$

Multiply it by $\pi\left(s_{i,t+1} \mid s_{i,t}, a_{i,t}\left(s^t\right)\right)$ *and sum over* $s_{i,t+1}$ *to get*

$$\sum_{s_{i,t+1}} \phi_{i,t+1}\left(s^t, s_{t+1}\right) \pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right) =$$
$$= \sum_{s_{i,t+1}} \frac{1}{u_c\left(c_{i,t+1}\left(s^t, s_{t+1}\right)\right)} \pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)$$

Now use the definition of $\phi_{i,t+1}\left(s^t, s_{t+1}\right)$ *to substitute for it*

$$\sum_{s_{i,t+1}} \left(\phi_{it}\left(s^t\right) + \lambda_{it}\left(s^t\right) \frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}\right) \pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right) =$$
$$= \sum_{s_{i,t+1}} \frac{1}{u_c\left(c_{i,t+1}\left(s^t, s_{t+1}\right)\right)} \pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)$$

and the first order condition for consumption at time t *to obtain:*

$$\frac{1}{u_c\left(c_{it}\left(s^t\right)\right)} + \sum_{s_{i,t+1}} \left(\lambda_{it}\left(s^t\right) \frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}\right) \pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right) =$$
$$= \sum_{s_{i,t+1}} \frac{1}{u_c\left(c_{i,t+1}\left(s^t, s_{t+1}\right)\right)} \pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)$$

It is now obvious that the second term in the LHS is zero, and we get (4) $\forall t \geq 0$. ■

Notice that by equation (4), $\frac{1}{u_c(c_{it}(s^t))}$ is a martingale with respect to the agent-i probability distribution generated by his own optimal effort. The following Proposition shows that the ratio of Pareto weights is a subbmartingale.

Proposition 3 $\Phi_t^{jk} \equiv \frac{\phi_{jt}(s^t)}{\phi_{kt}(s^t)}$ *is a submartingale.*

Proof. *Notice that we can always write:*

$$\begin{aligned}\frac{\phi_{jt}(s^t)}{\phi_{it}(s^t)} &= \frac{\sum\limits_{s_{j,t+1}} \phi_{j,t+1}(s^t)\,\pi(s_{j,t+1} \mid s_{jt}, a_{jt}(s^t))}{\sum\limits_{s_{k,t+1}} \phi_{k,t+1}(s^t)\,\pi(s_{k,t+1} \mid s_{kt}, a_{kt}(s^t))} \\ &= \sum_{s_{j,t+1}} \frac{\pi(s_{j,t+1} \mid s_{jt}, a_{jt}(s^t))}{\sum\limits_{s_{k,t+1}} \frac{\phi_{k,t+1}(s^t)}{\phi_{j,t+1}(s^t)}\pi(s_{k,t+1} \mid s_{kt}, a_{kt}(s^t))} \\ &\leq \sum_{s_{j,t+1}} \sum_{s_{k,t+1}} \frac{\phi_{j,t+1}(s^t)}{\phi_{k,t+1}(s^t)}\pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)\end{aligned}$$

where the last line comes from Jensen's inequality. ■

Proposition 3 is a result that holds in all economies presented in this paper. In this particular economy, however, the implications are described in the following Proposition.

Proposition 4 $\frac{u_c(c_{jt}(s^t))}{u_c(c_{it}(s^t))}$ *is a submartingale.*

Proof. *Notice that*

$$\begin{aligned}\frac{u_c(c_{jt}(s^t))}{u_c(c_{it}(s^t))} &= \frac{1}{u_c(c_{it}(s^t))}\left[\frac{1}{u_c(c_{jt}(s^t))}\right]^{-1} = \\ &= \sum_{s_{i,t+1}} \frac{\pi(s_{i,t+1} \mid s_{it}, a_{it}(s^t))}{\sum\limits_{s_{j,t+1}} \frac{u_c(c_{i,t+1}(s^t,s_{t+1}))}{u_c(c_{j,t+1}(s^t,s_{t+1}))}\pi(s_{j,t+1} \mid s_{jt}, a_{jt}(s^t))}\end{aligned}$$

and therefore by Jensen's inequality

$$\begin{aligned}\frac{u_c(c_{jt}(s^t))}{u_c(c_{it}(s^t))} &\leq \sum_{s_{i,t+1}} \sum_{s_{j,t+1}} \frac{u_c(c_{j,t+1}(s^t, s_{t+1}))}{u_c(c_{i,t+1}(s^t, s_{t+1}))}\pi\left(s_{j,t+1} \mid s_{jt}, a_{jt}\left(s^t\right)\right)\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right) = \\ &= \sum_{s_{t+1}} \frac{u_c(c_{j,t+1}(s^t, s_{t+1}))}{u_c(c_{i,t+1}(s^t, s_{t+1}))}\pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)\end{aligned}$$

■

Proposition 4 determines the optimal consumption path. Notice that the submartingale result implies a persistent consumption inequality: once consumption of two agents start to diverge, this divergence will be long-lasting.

3 An endowment economy with unobservable bond markets

Take the same endowment economy but assume the planner cannot monitor the credit market. In this case, each agent can trade one-period bond, buying it or selling it at the observable price $p_t\left(s^t\right)$. The agent has the following budget constraint:

$$c_{it}\left(s^t\right) + p_t\left(s^t\right) b_{it}\left(s^t\right) \leq y_{it}\left(s^t\right) + \tau_{it}\left(s^t\right) + b_{i,t-1}\left(s^{t-1}\right)$$

The bond market must clear:

$$\sum_{i=1}^{N} b_{it}\left(s^t\right) = 0$$

Following Mele (2011), we now have to make sure that the effort and bond holding decisions of the agents are incentive compatible. We use the agent's first-order conditions with respect to effort and bond holding to characterize the optimal contract. The problem becomes:

$$P\left(s_0\right) = \max_{\left\{\left\{c_{it}(s^t), a_{it}(s^t)\right\}_{i=1}^{N}, p_t(s^t)\right\}_{t=0}^{\infty}} \left\{ \sum_{i=1}^{N} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \omega_i \left[u\left(c_{it}\left(s^t\right)\right) - \right.\right.$$

$$\left.\left. -\upsilon\left(a_{it}\left(s^t\right)\right)\right] \Pi\left(s^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right)\right\}$$

$$s.t. \quad \upsilon'\left(a_{it}\left(s^t\right)\right) = \sum_{j=1}^{\infty} \sum_{s^{t+j}|s^t} \beta^j \frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)} \left[u\left(c_{i,t+j}\left(s^{t+j}\right)\right) - \right.$$

$$\left. -\upsilon\left(a_{i,t+j}\left(s^{t+j}\right)\right)\right] \Pi\left(s^{t+j} \mid s^t, a^{t+j-1}\left(s^{t+j-1}\right)\right)$$

$$\forall i = 1, ..., N$$

$$p_t\left(s^t\right) u'\left(c_{it}\left(s^t\right)\right) = \beta \sum_{s^{t+j}|s^t} u'\left(c_{i,t+1}\left(s^{t+1}\right)\right) \pi\left(s_{t+1} \mid s_t, a_t\left(s^t\right)\right)$$

$$\forall i = 1, ..., N$$

$$\sum_{i=1}^{N} c_{it}\left(s^t\right) = \sum_{i=1}^{N} y_{it}\left(s^t\right)$$

We can now write down the Lagrangean of this problem. Substitute for the resource constraint and let $\phi \equiv \{\phi_i\}_{i=1}^N$,$\zeta \equiv \{\zeta_i\}_{i=1}^N$, $\varsigma \equiv \{c_i, a_i\}_{i=1}^N$, $\nu \equiv \{\lambda_i, \eta_i\}_{i=1}^N$:

$$
\begin{aligned}
& L\left(\varsigma^{\infty}, \nu^{\infty}, \phi^{\infty}, \zeta^{\infty}\right)= \\
& \quad=\sum_{i=1}^N \sum_{t=0}^{\infty} \sum_{s^t} \beta^t\left\{\phi_{it}\left(s^t\right)\left[u\left(c_{it}\left(s^t\right)\right)-\upsilon\left(a_{it}\left(s^t\right)\right)\right]-\lambda_{it}\left(s^t\right) \upsilon^{\prime}\left(a_{it}\left(s^t\right)\right)+\right. \\
& \quad\left.\quad+\left[p_t\left(s^t\right) \eta_{it}\left(s^t\right)-\zeta_{it}\left(s^t\right)\right] u^{\prime}\left(c_{it}\left(s^t\right)\right)\right\} \Pi\left(s^t \mid h_0, a^{t-1}\left(s^{t-1}\right)\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_{i,t+1}\left(s^t, s_{t+1}\right) & = \phi_{it}\left(s^t\right)+\lambda_{it}\left(s^t\right) \frac{\pi_{a_i}\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)}{\pi\left(s_{i,t+1} \mid s_{it}, a_{it}\left(s^t\right)\right)} \qquad \phi_{i0}\left(s_0\right)=\omega_i \\
\zeta_{i,t+1}\left(s^t, s_{t+1}\right) & = \eta_{it}\left(s^t\right) \qquad \zeta_{i0}\left(s_0\right)=0
\end{aligned}
$$

Notice that (5) implies that the marginal rate of substitution between today's consumption and tomorrow's consumption are equalized across agents. This is different from the previous simple endowment economy, where the marginal rate of substitution between today's consumption and tomorrow's consumption is governed by equation (4).

The following Proposition is the analog of Proposition 1. The proof is trivial and therefore not included.

Proposition 5 *The constrained efficient allocation solves the following functional equation*

$$
\begin{aligned}
W(s, \phi, \zeta) & = \min_{\nu} \max_{\varsigma, p}\left\{r(\varsigma, p, \nu, s, \phi, \zeta)+\beta \sum_{s^{\prime}} \pi\left(s^{\prime} \mid s, a\right) W\left(s^{\prime}, \phi^{\prime}\left(s^{\prime}\right), \zeta^{\prime}\right)\right\} \\
s.t. & \quad \phi_i^{\prime}\left(s^{\prime}\right)=\phi_i+\lambda_i \frac{\pi_{a_i}\left(s^{\prime} \mid s, a\right)}{\pi\left(s^{\prime} \mid s, a\right)} \qquad i=1, \ldots, N \\
& \quad \zeta_i^{\prime}=\eta_i \qquad i=1, \ldots, N
\end{aligned}
$$

where

$$
r(\varsigma, p, \nu, s, \phi, \zeta) \equiv \sum_{i=1}^N\left\{\phi_i\left[u\left(c_i\right)-\upsilon\left(a_i\right)\right]-\lambda_i \upsilon^{\prime}\left(a_i\right)+\left(p \eta_i-\zeta_i\right) u^{\prime}\left(c_i\right)\right\}
$$

Moreover, the operator

$$
\begin{aligned}
(TW)(s, \phi, \zeta) & = \min_{\nu} \max_{\varsigma, p}\left\{r(\varsigma, p, \nu, s, \phi, \zeta)+\beta \sum_{s^{\prime}} \pi\left(s^{\prime} \mid s, a\right) W\left(s^{\prime}, \phi^{\prime}\left(s^{\prime}\right), \zeta^{\prime}\right)\right\} \\
s.t. & \quad \phi_i^{\prime}\left(s^{\prime}\right)=\phi_i+\lambda_i \frac{\pi_{a_i}\left(s^{\prime} \mid s, a\right)}{\pi\left(s^{\prime} \mid s, a\right)} \\
& \quad \zeta_i^{\prime}=\eta_i \qquad i=1, \ldots, N
\end{aligned}
$$

is a contraction, and $W(\cdot,\phi,\zeta)$ is homogenous of degree 1, while the policy correspondences $\varsigma(\cdot,\phi,\zeta)$ and $\nu(\cdot,\phi,\zeta)$ are homogeneous of degree zero.

3.1 Characterization of the contract

Define $\Phi_t^{kj}(s^t) \equiv \frac{\phi_{kt}(s^t)}{\phi_{jt}(s^t)}$ and for any variable $x_{it}(s^t)$, let $\widetilde{x}_{it}(s^t) \equiv \frac{x_{it}(s^t)}{\phi_{it}(s^t)}$. Then the following Proposition defines the characteristics of the optimal contract.

Proposition 6 *A constrained-efficient allocation $\left\{\{c_{it}(s^t), a_{it}(s^t)\}_{i=1}^{N}\right\}_{t=0}^{\infty}$ satisfies the following*

$$\Phi_t^{jk}(s^t) = \left[\frac{u'(c_{jt}(s^t))}{u'(c_{kt}(s^t))}\right]^{-1} \cdot \frac{[p_t(s^t)\eta_{jt}(s^t) - \zeta_{jt}(s^t)]\,u''(c_{jt}(s^t))}{[p_t(s^t)\eta_{kt}(s^t) - \zeta_{kt}(s^t)]\,u''(c_{kt}(s^t))} \quad \forall k,j \tag{5}$$

$$\sum_{j=1}^{N} \eta_{jt}(s^t)\,u'(c_{jt}(s^t)) = 0 \tag{6}$$

Proof. *Take first order conditions with respect to consumption and bond price and rearrange.* ■

The differences with Section 2 are clear. First, in Section 2 we have found that $\Phi_t^{jk}(s^t) = \left[\frac{u'(c_{jt}(s^t))}{u'(c_{kt}(s^t))}\right]^{-1}$, while here we have a wedge between the ratio of Pareto weights and the inverse of the ratio of marginal utilities of consumption. This wedge depends on the price of bonds, the Lagrange multipliers associated to individual Euler equations, the new costates $\zeta_{jt}(s^t)$ with $j = 1, ..., N$ and the relative degree of concavity of the utility function (the ratio of second derivatives for consumption utility).

In the case of CRRA utility function for consumption, we can simplify the above result of equation (5):

Corollary 1 *Assume that $u(c_i) \equiv \frac{c_i^{1-\sigma_i}}{1-\sigma_i}$. Therefore*

$$\frac{c_{jt}}{c_{kt}} = \frac{\sigma_j}{\sigma_k} \cdot \frac{p_t(s^t)\widetilde{\eta}_{jt}(s^t) - \widetilde{\zeta}_{jt}(s^t)}{p_t(s^t)\widetilde{\eta}_{kt}(s^t) - \widetilde{\zeta}_{kt}(s^t)} \quad \forall k,j \tag{7}$$

Equation (7) shows that consumption inequality in this framework depends not only on the endogenous Pareto weights, but also on the shadow cost of the Euler equation. To fix ideas, imagine that $\sigma_j = \sigma_k$ and Pareto weights for agents j and k are identical at period t. Then,

$$\frac{c_{jt}}{c_{kt}} = \frac{p_t\left(s^t\right)\eta_{jt}\left(s^t\right) - \zeta_{jt}\left(s^t\right)}{p_t\left(s^t\right)\eta_{kt}\left(s^t\right) - \zeta_{kt}\left(s^t\right)} \tag{8}$$

which implies there can still be inequality due to different asset accumulation paths and different histories. Clearly, there are two reasons for inequality in this economy: one is induced by the incentive structure (governed by Pareto weights), where inequality is used as an incentive for increasing effort. The second source of inequality comes from wealth accumulation, since the contract wants to discourage it in order to avoid lower effort. The wedge in equation (5) reflects the second source of inequality.

Proposition 3 also holds in this economy: the ratio of Pareto weights is a submartingale. However, the implications are different from Section 2. Now it is the LHS of equation (5) that behaves as a submartingale: the ratio of marginal utilities of consumption multiplied by a wedge that depends on bond price, the Lagrange multiplier of the Euler equations and the new costate variables associated with them.

Moreover, equation (6) has the following consequence:

Corollary 2 *Assume $N = 2$. Define*

$$B_t\left(s^t\right) \equiv \frac{\sigma_1}{\sigma_2} \cdot \frac{p_t\left(s^t\right)\widetilde{\eta}_{1t}\left(s^t\right) - \widetilde{\zeta}_{1t}\left(s^t\right)}{p_t\left(s^t\right)\widetilde{\eta}_{2t}\left(s^t\right) - \widetilde{\zeta}_{2t}\left(s^t\right)}$$

Therefore

$$\frac{\eta_{1t}\left(s^t\right)}{\eta_{2t}\left(s^t\right)}\frac{u'\left(c_{1t}\left(s^t\right)\right)}{u'\left(c_{2t}\left(s^t\right)\right)} = -1 \tag{9}$$

$$c_{1t}\left(s^t\right) = \left[y_{1t}\left(s^t\right) + y_{2t}\left(s^t\right)\right]\frac{B_t\left(s^t\right)}{1 + B_t\left(s^t\right)} \tag{10}$$

The focus on a 2-agents economy helps understand the logic. Marginal utilities of consumption ratio is governed by the behavior of the Lagrange multipliers associated with individual Euler equations. The sign of $\eta_{it}\left(s^t\right)$ defines if an agent is the borrower or the lender in the economy. Remember that $\eta_{it}\left(s^t\right)$ is the Lagrange multiplier of Euler equation for agent i. Therefore, in the optimal contract, a positive sign of $\eta_{it}\left(s^t\right)$ implies that agent i will be willing to decrease her consumption today and increase it tomorrow, i.e. agent i wants to save. Viceversa, a negative sign of $\eta_{it}(s^t)$ indicates that agent i wants to borrow. The difference between $p_t(s^t)\eta_{it}(s^t)$ and $\zeta_{it}(s^t)$ drives the trades in the hidden assets market. What Corollary 2 says is that $\eta_{1t}(s^t)$ and $\eta_{2t}(s^t)$ have opposite sign. This implies that (except for period 0) also $\zeta_{1t}(s^t)$ and $\zeta_{2t}(s^t)$ have opposite signs. Therefore, in a 2-agents economy, as it is obvious, there will always be one agent indebted with the other. Which one will depend on the history of shocks that drive the value of $B_t(s^t)$.

3.2 The wedge

In this economy, the planner chooses the price of bonds, therefore

$$\frac{\beta\sum_{s^{t+1}|s^t} u'(c_{k,t+1}(s^{t+1}))\,\pi(s_{t+1}\mid s_t,a_t(s^t))}{u'(c_{kt}(s^t))}=\frac{\beta\sum_{s^{t+1}|s^t} u'(c_{i,t+1}(s^{t+1}))\,\pi(s_{t+1}\mid s_t,a_t(s^t))}{u'(c_{it}(s^t))}$$

The intertemporal wedge is given by

$$\beta\sum_{s^{t+1}|s^t} u'\left(c_{i,t+1}\left(s^{t+1}\right)\right)\pi\left(s_{t+1}\mid s_t,a_t\left(s^t\right)\right)-p_t\left(s^t\right)u'\left(c_{it}\left(s^t\right)\right)$$

Notice that in this economy the wedge can be positive or negative. In order to understand why, assume there are only two possible realization of the endowment, and imagine all agents but one have good shocks. Hence, the lucky agents can go in the market and buy bonds, then lending indirectly to the unlucky agent, who will be happy to sell and increase her consumption. The planner wants to avoid this behavior, since it makes more difficult to induce effort on the lucky agents. Therefore, the optimal contract would have a price that discourages the lucky agents to lend money and discourage the unlucky agents to borrow. Since many agents want to buy, the price of the bond must be so punitive that the lucky agents does not want to buy it, i.e. it must imply a loss and therefore must be higher than 1. It will therefore imply a negative intertemporal wedge.

TO BE COMPLETED

4 A production economy

To be completed

5 Numerical examples

5.1 Endowment economy

I solve the model with only two agents. Following the notation in Proposition 1, let $N = 2$, $\varsigma_i \equiv c_i$, $r^i(\varsigma_i, a_i, s) \equiv u(c_i) - v(a_i)$. Agents have the same utility function, the initial weights are the same and the parameter values are given in the following table:

α_i	ε_i	ν_i	σ_i	y_i^L	y_i^H	β	ω_i
0.5	2	0.5	2	.4	.6	0.95	0.5

Let $\theta \equiv \frac{\phi_2}{\phi_1}$. Figures 1 and 2 show that agent 1's consumption and continuation value are decreasing in θ for any possible state of the world while effort is increasing in θ. Obviously, the contrary is true for agent 2[3]. Therefore, the higher is θ, the larger the share of aggregate output that goes to agent 2.

Figure 3 and 4 show a sample path of 200 periods. Notice that θ, and therefore consumption inequality, are very persistent as Proposition 4 suggests. Finally, in Figure 5 I plot the Pareto frontier, which is decreasing and strictly concave.

5.2 Endowment economy with hidden wealth

The baseline parameterization is:

α_i	ε_i	ν_i	σ_i	β	ω_i
5	2	0.5	2	0.95	0.5

[3]Notice that, given the i.i.d. assumptions on shocks and the fact that shocks for the two agents have the same support, it turns out that $c_i^{LH} = c_i^{HL}$, $i = 1, 2$.

Figures 7 and 8 present optimal allocations for a sample path of realized shocks. In particular, in Figure 9 we can observe that asset positions become extreme quite soon[4].

Figures 10 and 11 present average allocations for 50000 simulations. One striking conclusion is that effort seems quite flat on average, also if it is decreasing for one agent and increasing for the other. From Figure 12 we can notice that it is not clear on average if there is one agent that tends to be always net debtor or net borrower.

I also show simulations for the case in which agents have different initial weights[5]. Figures 6 and 13 show sample path allocations for $\omega_1 = 0.45$ and $\omega_2 = 0.55$. Notice that the initial consumption inequality is very important in determining future inequality, due to strong persistence. Asset positions in Figure 14 become much more extreme than in previous case with same initial weight. Therefore, different initial weights imply big wealth inequality in the long run.

Finally averages over 50000 simulations in which agents have different initial weights are presented in Figures 15 and 16. Effort for high-weight agent tends to be lower, and this agent becomes permanently and hugely indebted with the other. Asset positions in Figure 17 become much more extreme than in previous case with same initial weight, confirming the behavior in the previous sample path.

6 Conclusions

I have provided a characterization of the optimal allocation for a simple endowment economy and for an endowment economy with hidden access to financial markets, when moral hazard is present. In all cases, the efficient contract can be characterized by a submartingale result related to the ratio of marginal utilities of consumption. In particular, in the simple endowment economy this quantity behaves as a submartingale. In an endowment economy with non-monitorable access to financial markets, the submartingale behavior is associated to the ratio of marginal utilities

[4]For a way to recover assets holdings from allocations, see Mele (2011), who builds on Abraham and Pavoni (2008)

[5]We can interpret different weights as asymmetric bargaining power in the contracting process, or different initial assets' positions.

of consumption multiplied by a wedge that depends on the bonds price and the shadow cost of the consumption-saving decision of each agent. The wedge can be positive or negative, according to the net position of the agents: if many of them want to save, then the wedge is positive. If instead many want to borrow, the wedge is negative.

Decentralization of the optimal allocation is left for future research.

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A Figures

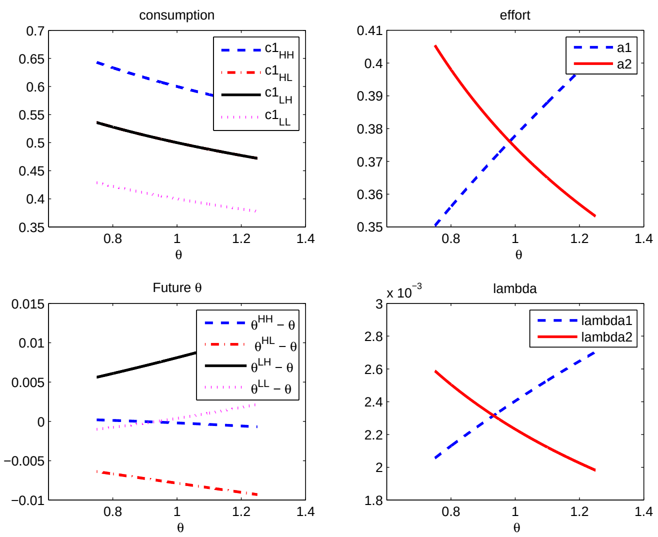

Figure 1: Risk sharing with moral hazard, policy functions (2 agents)

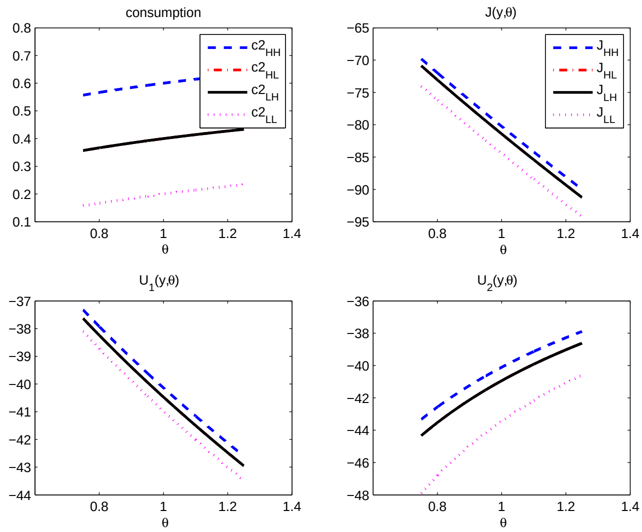

Figure 2: Risk sharing with moral hazard, policy functions (2 agents) (cont.)

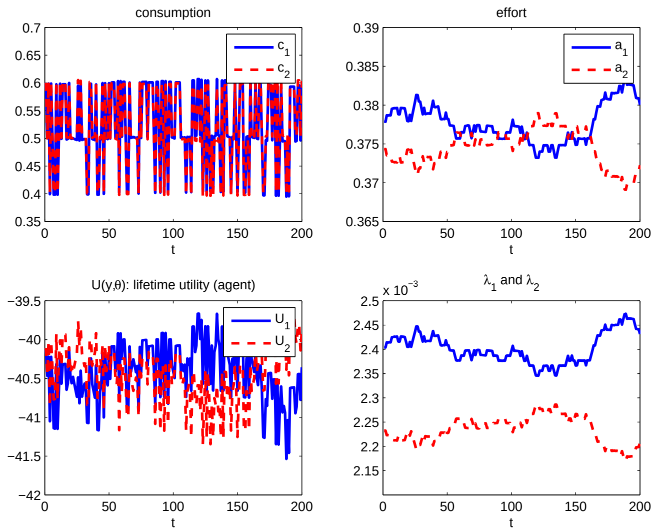

Figure 3: Risk sharing with moral hazard, sample path (2 agents)

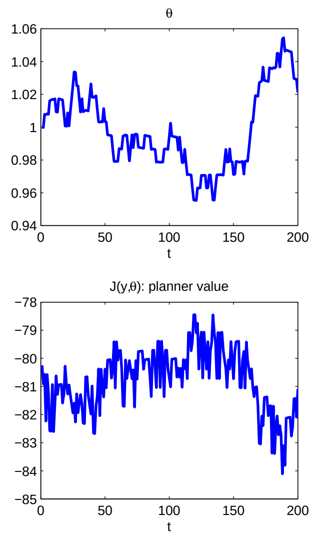

Figure 4: Risk sharing with moral hazard, sample path (2 agents) (cont.)

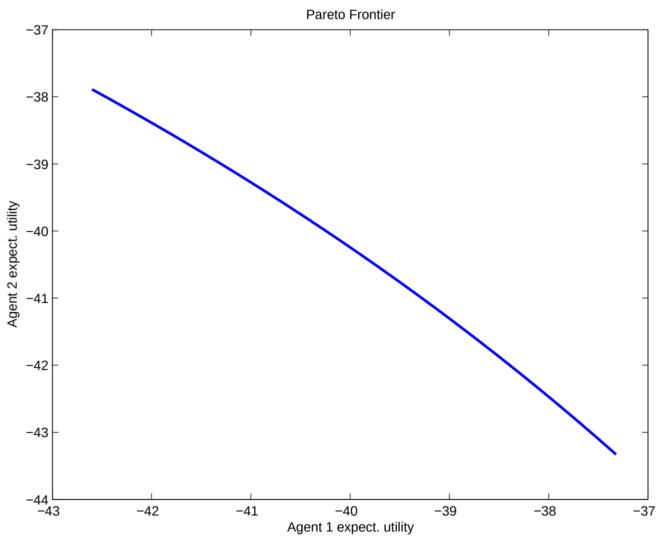

Figure 5: Risk sharing with moral hazard, Pareto frontier (2 agents)

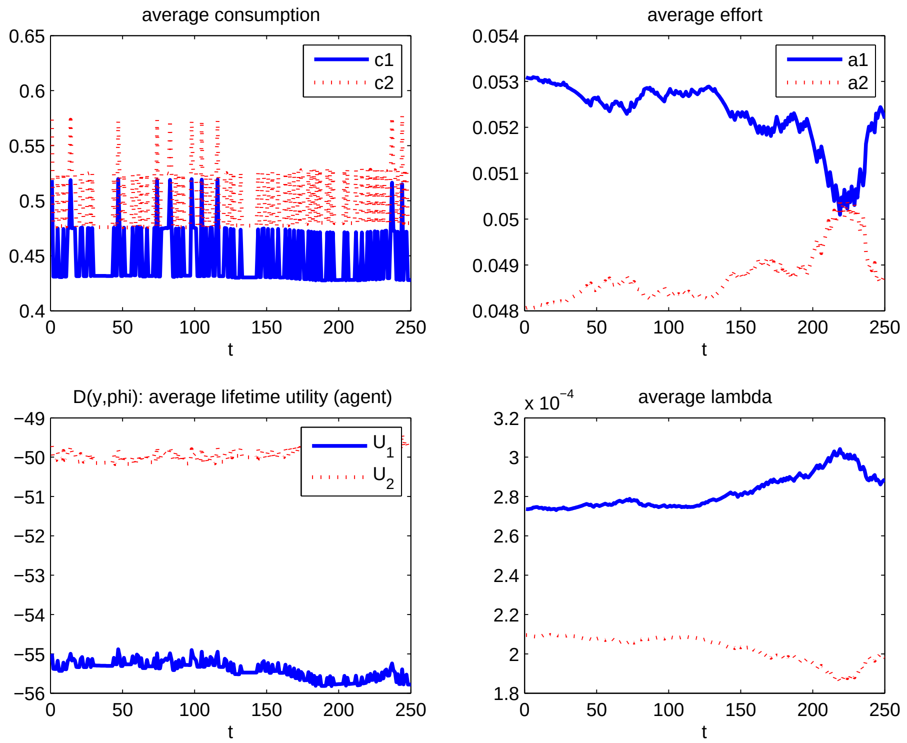

Figure 6: Endowment economy with hidden assets: sample path with different initial weights

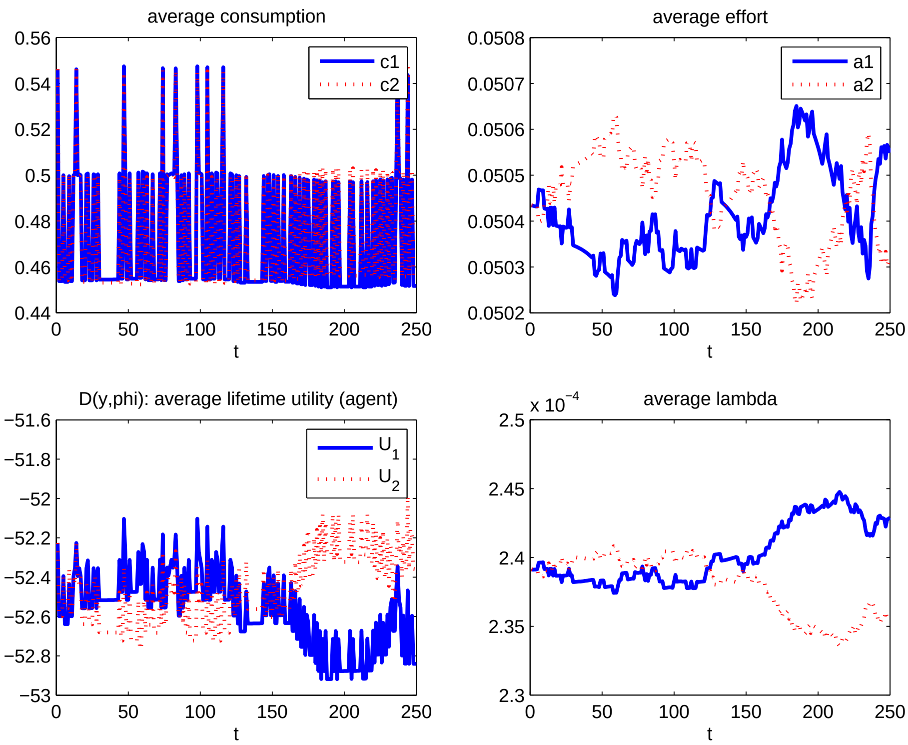

Figure 7: Endowment economy with hidden assets: sample path

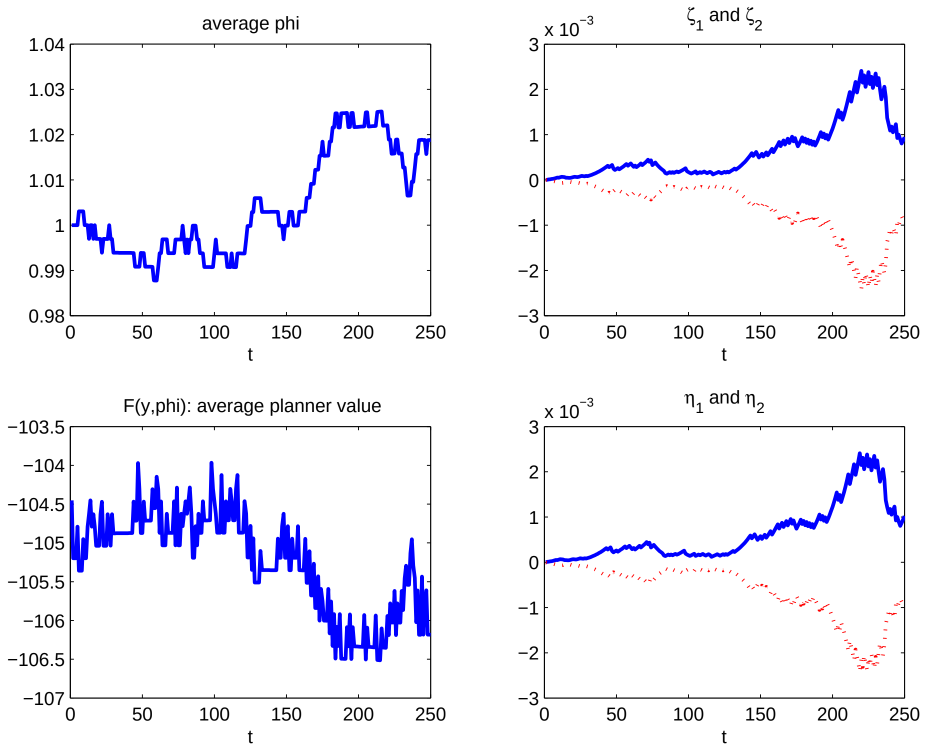

Figure 8: Endowment economy with hidden assets: sample path (cont.)

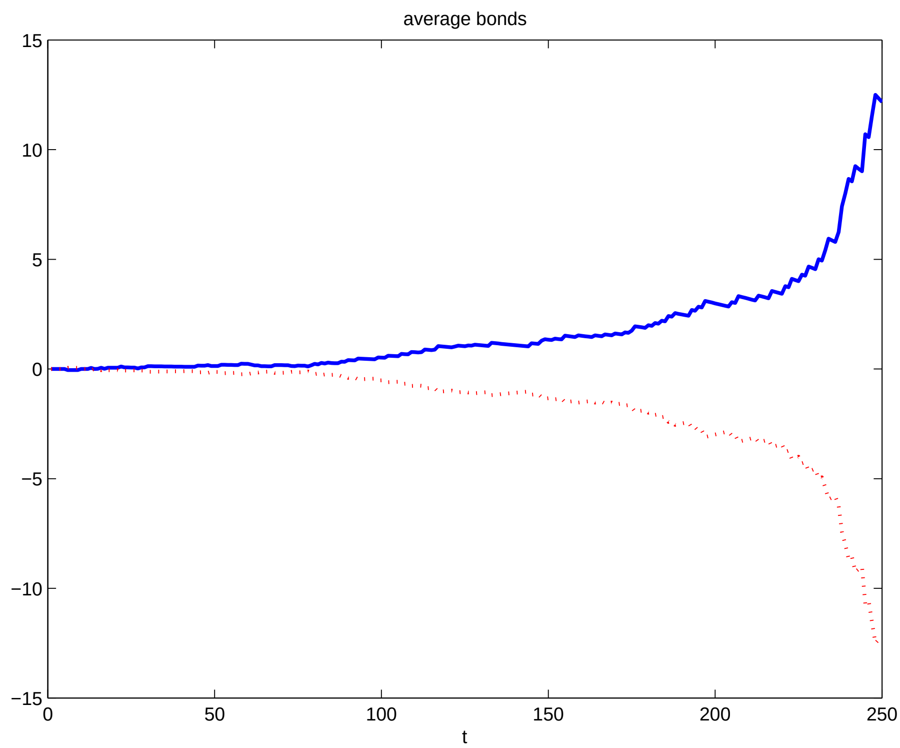

Figure 9: Endowment economy with hidden assets: sample path, bond positions

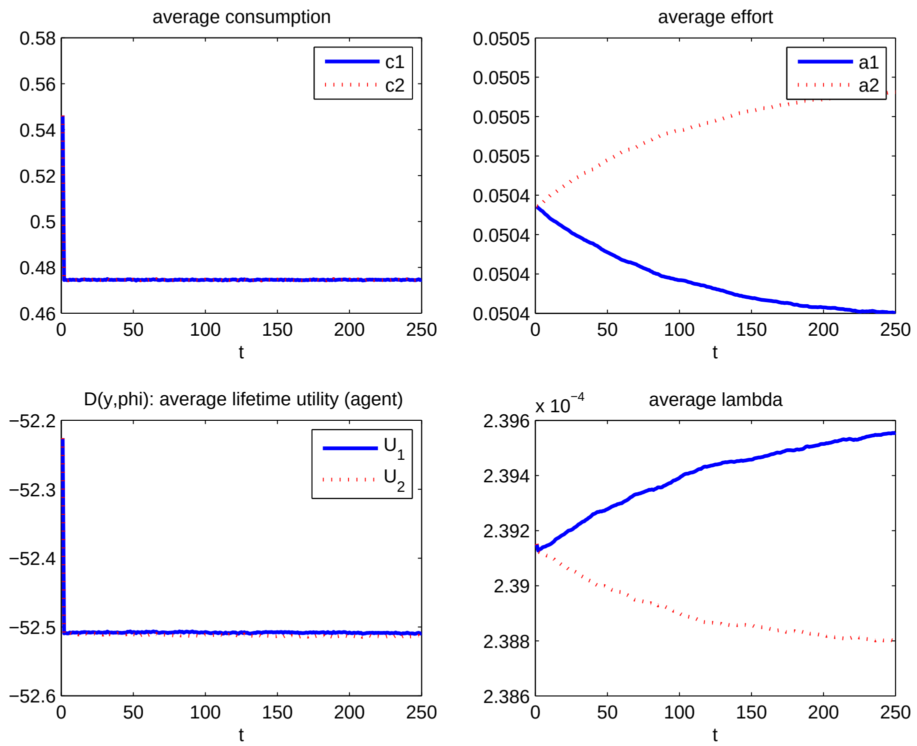

Figure 10: Endowment economy with hidden assets: average over 50000 simulations

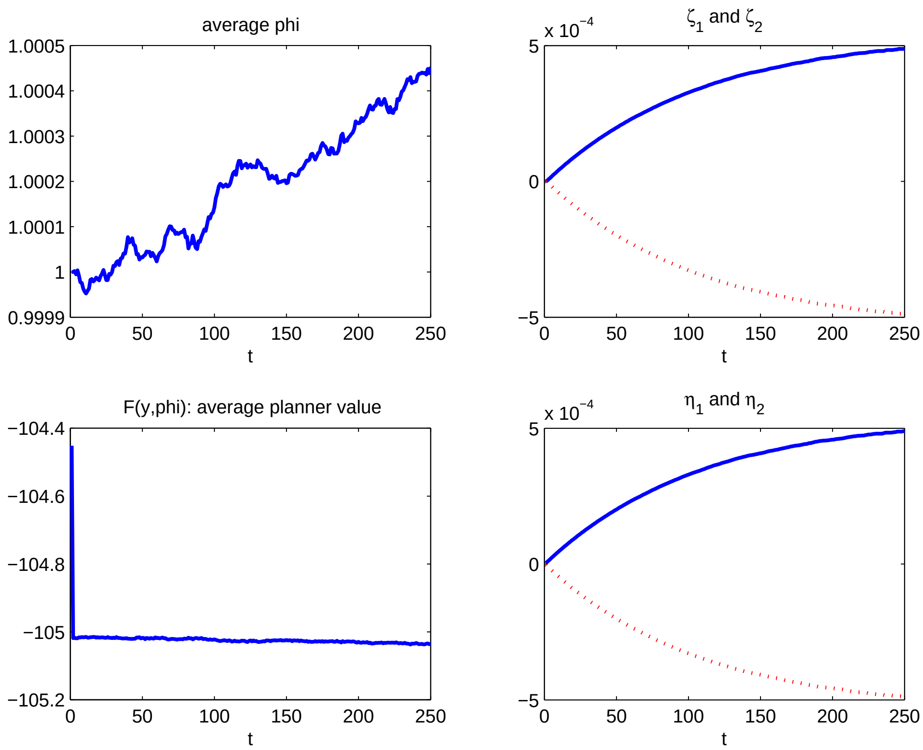

Figure 11: Endowment economy with hidden assets: average over 50000 simulations (cont.)

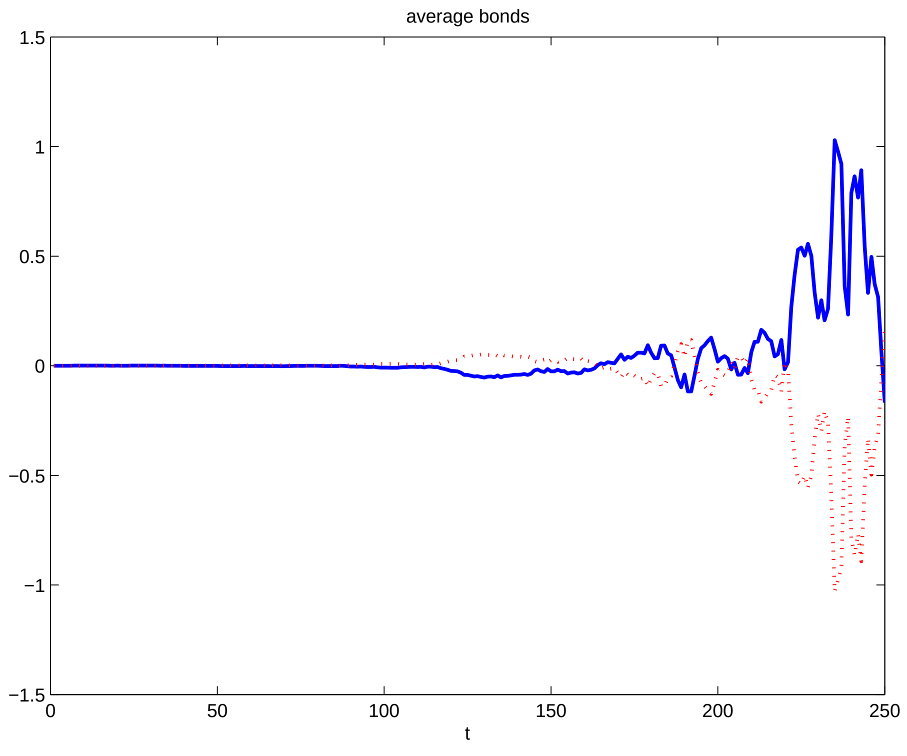

Figure 12: Endowment economy with hidden assets: average over 50000 simulations, bond positions

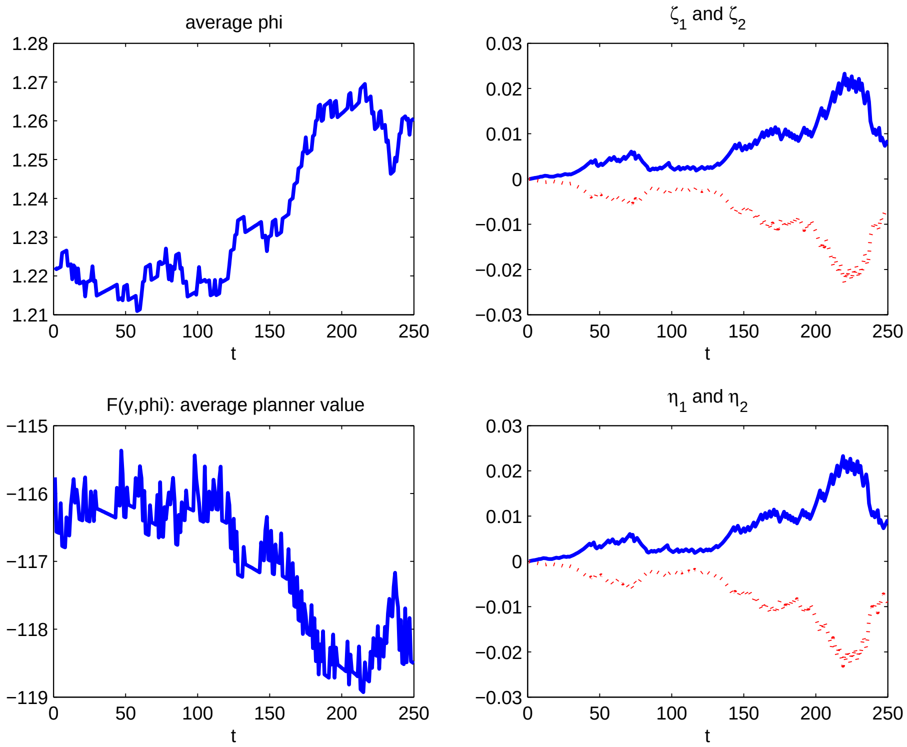

Figure 13: Endowment economy with hidden assets: sample path with different initial weights (cont.)

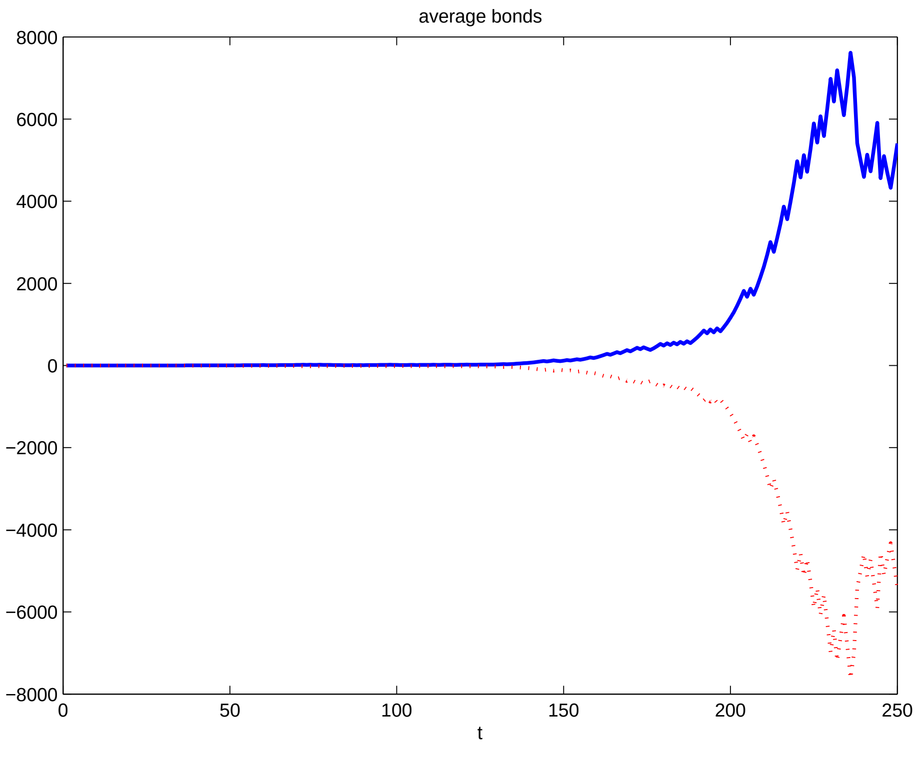

Figure 14: Endowment economy with hidden assets: sample path with different initial weights, bond positions

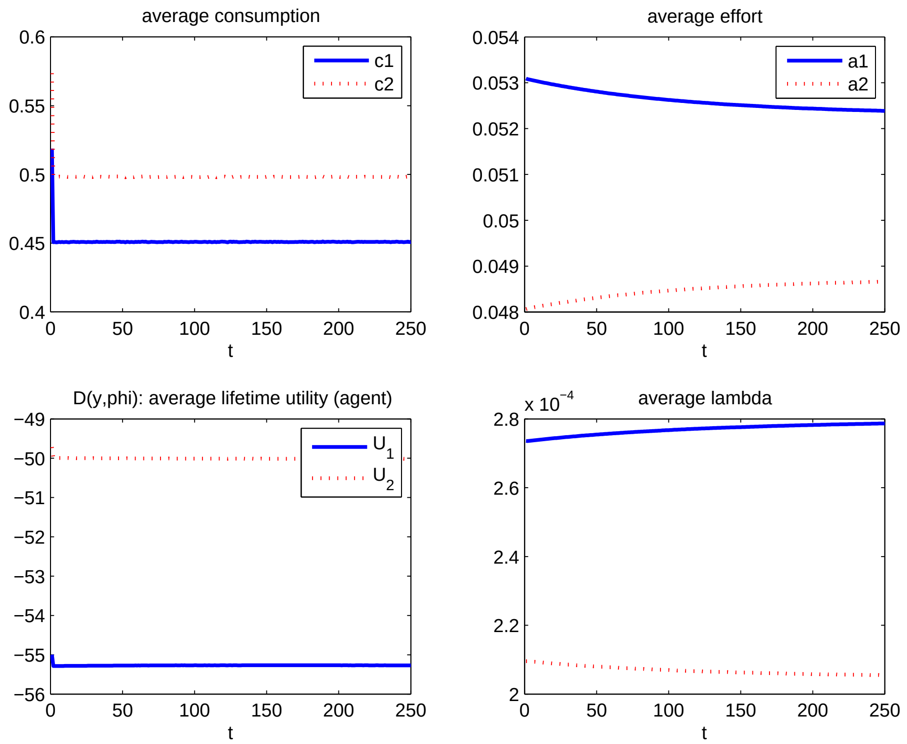

Figure 15: Endowment economy with hidden assets: average over 50000 simulations with different initial weights

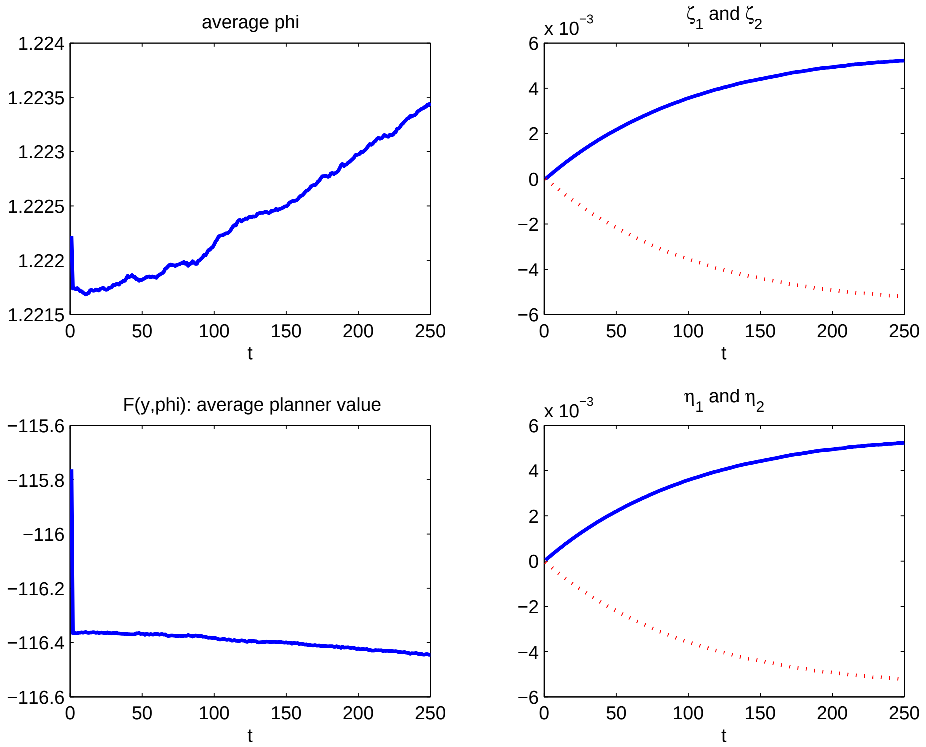

Figure 16: Endowment economy with hidden assets: average over 50000 simulations with different initial weights (cont.)

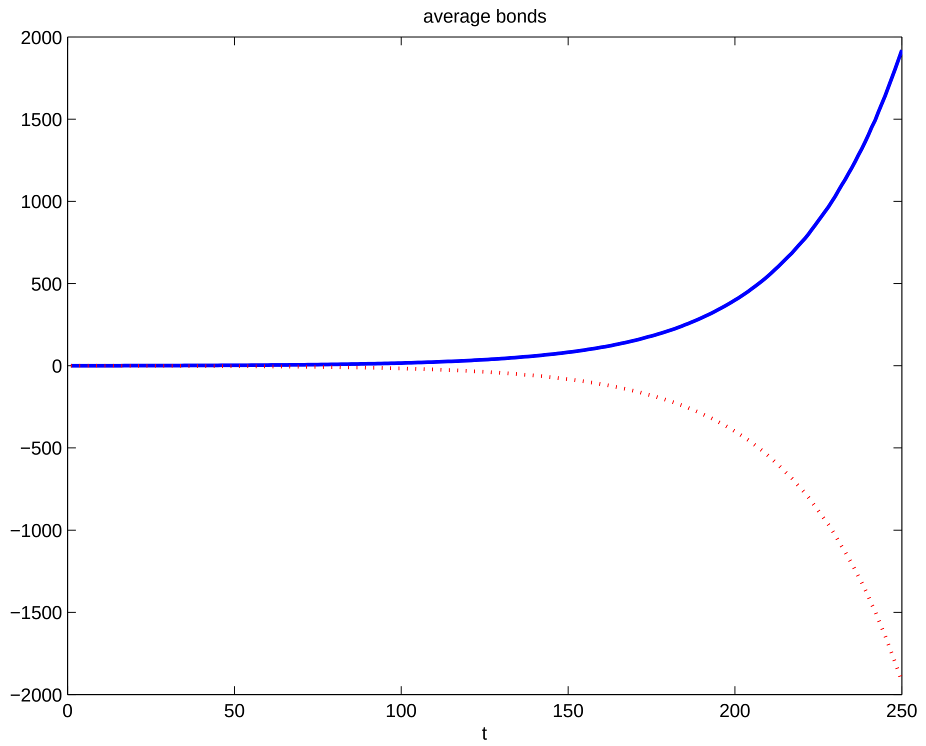

Figure 17: Endowment economy with hidden assets: average over 50000 simulations with different initial weights, bond positions