Microeconomic Theory I

5. Uncertainty

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Choice over Uncertain Outcomes

Basic concepts

Lotteries:

$$x = \langle q, \alpha \rangle$$

- q is a vector of outcomes
- α is a probability distribution ($\alpha_i \in [0,1]$, $\sum \alpha_i = 1$)

Example

$$q = \begin{bmatrix} \text{1 pig} \\ \text{3 bottles of wine} \\ \text{1 duck} \\ \text{1 cheese} \end{bmatrix} \quad \alpha = \begin{bmatrix} 1/2 \\ 1/4 \\ 1/8 \\ 1/8 \end{bmatrix}$$

- By defining X as a space of lottery, we can apply many of the concepts we already discussed to the case of choice over uncertain outcomes.

Basic concepts

- **Simple lotteries**: outcomes are deterministic
- **Compound lotteries**: some of the outcomes are lotteries

Example

$$m = \left\langle \begin{bmatrix} \text{1 pig} \\ \text{3 bottles of wine} \\ l \end{bmatrix}, \begin{bmatrix} \alpha_1 = 1/3 \\ \alpha_2 = 1/3 \\ \alpha_3 = 1/3 \end{bmatrix} \right\rangle$$

where l is a ticket to next year's raffle

$$l = \left\langle \begin{bmatrix} \text{1 duck} \\ \text{1 cheese} \end{bmatrix}, \begin{bmatrix} \alpha_1 = 1/2 \\ \alpha_2 = 1/2 \end{bmatrix} \right\rangle$$

Basic concepts

Example

We can define

$$\tilde{m} = \left\langle \begin{bmatrix} \text{1 pig} \\ \text{3 bottles of wine} \\ \text{1 duck} \\ \text{1 cheese} \end{bmatrix}, \begin{bmatrix} \alpha_1 = 1/3 \\ \alpha_2 = 1/3 \\ \alpha_3 = 1/6 \\ \alpha_4 = 1/6 \end{bmatrix} \right\rangle$$

- i.e. every compound lottery can be reduced to a simple lottery.
- We assume that every compound lottery is equivalent to its reduced simple lottery.

$$m \equiv \tilde{m}$$

i.e. all that matter are probabilities over final outcomes (we don't care how we get there).

Basic concepts

- **Space of Lotteries**:

$$L \equiv \left\{ \langle q, \alpha \rangle \mid q_i \in Q \;\; \forall i, \; \alpha_i \in [0,1] \, , \; \sum \alpha_i = 1 \right\}$$

  where Q is a set of deterministic outcomes.
- Same as our old X, we change notation to denote that we are working with lotteries.

Properties of preferences over lotteries

Definitions

- $\succeq$ over L are **rational** if they are complete and transitive.
- For x, $x' \in L$ let the **mixture of lotteries**

$$\alpha x \oplus (1-\alpha)x' \quad \alpha \in [0,1]$$

denote the compound lottery

$$\left\langle \begin{bmatrix} x \\ x' \end{bmatrix}, \begin{bmatrix} \alpha \\ 1-\alpha \end{bmatrix} \right\rangle$$

Properties of preferences over lotteries

Definition

$\succeq$ over L are **continuous** if $\forall x, x', x'' \in L$ the sets

$$\left\{\alpha \mid x'' \succeq \alpha x \oplus (1-\alpha)\, x'\right\}$$
$$\left\{\alpha \mid x'' \preceq \alpha x \oplus (1-\alpha)\, x'\right\}$$

are both closed in $[0, 1]$.

Properties of preferences over lotteries

Definition

$\succeq$ on L satisfy the **independence axiom** if $\forall x, x', x'' \in L$, with $x \succeq x'$, we have

$$\alpha x \oplus (1-\alpha) x'' \succeq \alpha x' \oplus (1-\alpha) x'' \quad \forall \alpha \in (0,1) \ .$$

Strong assumption

It implies linear indifference curves.

- Take $x \sim x'$
- Using the definition above for $x = x''$

$$\alpha x \oplus (1-\alpha) x'' = x \sim \alpha x' \oplus (1-\alpha) x$$

- Straight indifference curves

Expected utility

- Similarly to what we already did, we can define a utility function $U : L \to \mathbb{R}$ representing the preferences over lotteries.
- However, we can look for a utility function representation that exploits the specific structure of lotteries.

Definitions

Bernoulli utility $u : Q \to \mathbb{R}$ is utility defined over the deterministic outcomes.

The **expected utility** of a lottery $\langle q, \alpha \rangle$ is given by $EU(x) = \alpha u(q)$.

Expected utility

Theorem

If $\succeq$ are continuous, rational and satisfy the independence axiom over L then

$$\exists U : Q \to \mathbb{R} \;\; s.t. \;\; x \succeq x' \iff EU(x) \geq EU(x') \; .$$

- i.e. **there exist a utility function** representation having the **expected utility** form
- This is a very powerful theorem but we do not prove it now
  - independence axiom imply indifference curves that are linear and *parallel to each other*.

Expected utility

Exercise

$$x = \left\langle \begin{bmatrix} 1 \text{ cow} \\ 1 \text{ pig} \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \right\rangle \qquad x' = \left\langle \begin{bmatrix} 1 \text{ cow} \\ 1 \text{ pig} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$$

We know that $U(x) = 4$, $U(x') = 3$ and U is an Expected Utility.

$$x'' = \left\langle \begin{bmatrix} 1 \text{ cow} \\ 1 \text{ pig} \end{bmatrix}, \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} \right\rangle$$

Calculate $U(x'')$.

Expected utility

$\tilde{U}(x) = \ln(U(x)) \quad \forall x \in L$

- $\tilde{U}$ represents the same preferences as U

$$\forall x, x' \in L,\ \tilde{U}(x) \geq \tilde{U}(x') \Longleftrightarrow x \succeq x'$$

  i.e. it is **also a utility function** over lotteries
- but $\tilde{U}$ is **not an Expected Utility.**

$$ln(EU(x)) = \ln(\alpha U(q)) \neq \alpha ln U(q)$$

- Only linear affine transformations on an expected utility are also expected utilities

Allais paradox: Independence is a strong assumption

1. $q_1 = 2500000\$$
2. $q_2 = 500000\$$
3. $q_3 = 0$

Compare:

$$x_1 = \left\langle \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle \quad x_1' = \left\langle \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, \begin{bmatrix} .10 \\ .89 \\ .01 \end{bmatrix} \right\rangle$$

Compare:

$$x_2 = \left\langle \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, \begin{bmatrix} 0 \\ .11 \\ .89 \end{bmatrix} \right\rangle \quad x_2' = \left\langle \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, \begin{bmatrix} .10 \\ 0 \\ .90 \end{bmatrix} \right\rangle$$

Allais paradox

- In experimental settings, people usually:

Allais paradox

- According to EU theory, if $x_1 \succ x_1'$, then

$$u(q_2) > .11u(q_2) + 0.89u(q_3)$$

$$0.89u(q_2) > 0.89u(q_3)$$

- adding $0.1u(q_1) + 0.01u(q_3)$ to both sides, we get

$$0.1u(q_1) + 0.89u(q_2) + 0.01u(q_3) > 0.10u(q_1) + 0.90u(q_3)$$

  which implies $x_2 \succ x_2'$

- Most people violate Independence Axiom (and EU theory).

Machina's paradox

- q_1: a trip to Venice
- q_2: watching a movie about Venice
- q_3: staying at home and not watching a movie about Venice

- Generally

$$q_1 \succ q_2 \succ q_3$$

Machina's paradox

Compare:

$$x = \left\langle \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, \begin{bmatrix} .999 \\ .001 \\ 0 \end{bmatrix} \right\rangle \; x' = \left\langle \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, \begin{bmatrix} .999 \\ 0 \\ .001 \end{bmatrix} \right\rangle$$

Lotteries over monetary outcomes: risk

Expected utility with money

- $Q \subset \mathbb{R}_+$; all deterministic outcomes are money
- To simplify, we assume that the benoullli utility $u(x) : Q \to \mathbb{R}$ is strictly increasing.

Definition

$E(x) = \alpha q$ is the **expected value of a lottery**.

Expected utility with money

Definitions

A person with utility function U is

- **risk neutral** on L if $\forall x \in L \quad EU(x) = U(E(x))$
- **risk averse** on L if $\forall x \in L$, with x is a risky lottery (i.e. $\alpha \ll 1$) $EU(x) < U(E(x))$.

The **certainty equivalent** is a deterministic outcome that gives the same utility of a lottery.

$$CE(x) : \quad U(CE(x)) = EU(x)$$

Expected utility with money

Proposition

Assume a person with benoullli utility function U (strictly increasing) is an expected utility maximizer. Then the following statements are equivalent:

1. The person is risk averse
2. U is strictly concave
3. $CE(x) < E(x)$ for all risky lotteries.

Application: Insurance

- Initial wealth w
- With probability p, the consumer loses L
- (S)he can purchase insurance, which pays q in case there is a loss at a cost πq
- The agent's utility is:
- What is the optimal demand for insurance?
- Let us assume we have an interior solution...
  - but: what are the possible corner solutions?

Application: Insurance

- This implicitly defines the demand for insurance: for given price, how much insurance the consumer wants to purchase.

Application: Insurance

- Let's look at the firm side. Expected profits:
- **Assumption:** Competition $\Rightarrow$ new firms enter market until expected profits are 0
- Price of insurance in equilibrium:

Application: Insurance

- Back to the consumer's problem:

$$\frac{U'(w-L+(1-\pi)\,q)}{U'(w-\pi q)}=\frac{1-p}{p}\frac{\pi}{1-\pi}=1 \quad \text{if } \pi=p$$

$$U'(w-L+(1-\pi)\,q)=U'(w-\pi q)$$
$$\Updownarrow$$
$$w-L+(1-\pi)\,q=w-\pi q$$

$$L=q$$

- If the premium is actuarially fair, a risk averse consumers buys insurance to cover the potential loss entirely (**full insurance**: no variation in income across realization of the lottery)

Measuring the attitude toward risk

Risk aversion

- The curvature of $U(x)$ determines risk attitude: $U(x)$ concave $\iff$ risk aversion
- We will show that the strength of the curvature of $U(x)$ determines the strength of risk aversion.
- Main problem: how to measure the curvature of $U(x)$?
  - Can we use the value of $U''(x)$?

Absolute risk aversion

Definition

$$r(w) = -\frac{U''(w)}{U'(w)}$$

is **absolute risk aversion** (a local measure of risk aversion).

Interpretation: measure of risk aversion for very small additive bets

- Start with a given w, fix a probability p
- Construct a lottery in the following way: with probability p, give the consumer x_1, with probability $(1-p)$, give the consumer x_2 (x_1 and x_2 can be negative)
- Call *acceptance set* the set of x_1, x_2 such that the consumer accepts the bet.
- If the acceptance set of consumer A is contained in the acceptance set of consumer B, we say that A is more risk averse than B

Absolute risk aversion

More formally:

- Find $x_2(x_1)$ such that the consumer is indifferent between accepting and refusing the bet

$$pU(w + x_1) + (1 - p)\, U(w + x_2(x_1)) \equiv U(w)$$

- Let's caracterize the shape of the acceptance set near $(0, 0)$ (i.e. for small bets):

Absolute risk aversion

- $x_2(x_1)$ goes through $(0,0)$ and has a negative slope

$$\begin{aligned} x_2'(x_1): &\quad pU'(w+x_1)+(1-p)\,U'(w+x_2(x_1))x_2'(x_1)=0 \\ x_2'(0): &\quad pU'(w)+(1-p)\,U'(w)x_2'(0)=0 \\ &\quad x_2'(0)=-\tfrac{p}{1-p} \end{aligned}$$

i.e. the slope of $x_2(x_1)$ near $(0,0)$ does not depend on the utility function (i.e. it is the same for all consumers, and all preferences)

Absolute risk aversion

$$x_2''(x_1) : pU''(w + x_1) + (1-p)\,U''(w + x_2(x_1))\left(x_2'(x_1)\right)^2 + \\ + (1-p)\,U'(w + x_2(x_1))x_2''(x_1) = 0$$

$$x_2''(0) : pU''(w) + (1-p)\,U''(w)\left(\frac{p}{1-p}\right)^2 + (1-p)\,U'(w)x_2''(0) = 0$$

$$x_2''(0) = \frac{p}{(1-p)^2}\underbrace{\left[-\frac{U''(w)}{U'(w)}\right]}_{r(w)}$$

i.e. take two consumers A and B, if $-\frac{U_A''(w)}{U_A'(w)} > -\frac{U_B''(w)}{U_B'(w)}$, near $(0,0)$ the acceptance set of A is included in the acceptance set of B $\Rightarrow$ A is locally more risk averse.

Relative risk aversion

Definition

$$\rho(w) = -\frac{U''(w)}{U'(w)}w$$

is **relative risk aversion**.

- Start with a given w
- Multiplicative bets:
  - With probability p, multiply w by x_1
  - With probability $1-p$, multiply w by x_2
- Find $x_2(x_1)$ s.t.

$$pU(x_1 w) + (1-p)\,U(x_2(x_1)w) \equiv U(w)$$

Relative risk aversion

- Always goes through $(1,1)$
- Negative slope
- $x_1,\, x_2 > 0$

$$x_2'(1) = -\frac{p}{1-p}$$
$$x_2''(1) = \underbrace{-\frac{wU''(w)}{U'(w)}}_{\rho(w)} \frac{p}{(1-p)^2}$$

- The same logic behind absolute risk aversion applies here

Risk aversion

Proposition

Consider two utility functions $U_A(x)$ and $U_B(x)$. The following statements are equivalent:

1. $r_A(w) > r_B(w) \quad \forall w$
2. $\rho_A(w) > \rho_B(w) \quad \forall w$
3. There exists an increasing and concave function $\varphi(\cdot)$ such that $U_A(x) = \varphi(U_B(x)) \, \forall x \in \mathbb{R}$
4. $CE_A(L) < CE_B(L) \quad \forall L$ (L: lottery).