

Topic Proposal:

Configuration spaces and algebraic functions

as discussed with Benson Farb

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1 Introduction

The prototypical example of a configuration space, the space of n points on a plane, is the classifying space of the braid group on n strands. Identifying the plane with complex numbers and points with roots of a polynomial, the same space also parametrizes square-free degree n complex polynomials. Operations on polynomials can be transformed into continuous maps of these spaces of roots and polynomials, and topological invariants of the spaces provide obstructions to solving polynomials. In this topic proposal we will discuss examples of how topology of configuration spaces and other spaces of polynomials interact.

Section 2 starts with basic definitions, and discusses cohomology of configuration space. Most of the results are on configuration spaces of the plane, but the last subsection involving computations is more general. In Section 3 we discuss how these invariants provide obstructions to solving polynomials. Section 4 defines algebraic functions, introduces Hilbert's 13th problem, and reuses the cohomology calculations to establish one result in that direction. Finally, Section 5 is on the space of non-singular polynomials on projective space, and uses the Vassiliev method of conical resolution to find the cohomology of these spaces.

2 Topology of configuration spaces

We will consistently identify $\mathbb{R}^2 = \mathbb{C}$ and all polynomials will be over \mathbb{C} . Define

$$\text{PConf}_n(\mathbb{C}) = \{(z_i) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}$$

and

$$\text{UConf}_n(\mathbb{C}) = \{\{z_1, \dots, z_n\} \subset \mathbb{C} \mid z_i \neq z_j \text{ for } i \neq j\} = \text{PConf}_n(\mathbb{C})/\mathfrak{S}_n,$$

where the symmetric group \mathfrak{S}_n acts on $\text{PConf}_n(\mathbb{C}) \subseteq \mathbb{C}^n$ by permuting coordinates. We can analogously define $\text{PConf}_n(X)$ and $\text{UConf}_n(X)$ for any space X , but for now it will suffice to consider $X = \mathbb{C}$. We will from now on omit X when $X = \mathbb{C}$.

We define the braid group $B_n = \pi_1(\text{UConf}_n)$ and the pure braid group $PB_n = \pi_1(\text{PConf}_n)$ on n strands. Since the action of \mathfrak{S}_n on PConf_n is free and proper discontinuous, the quotient map $\text{PConf}_n \rightarrow \text{UConf}_n$ is a covering map, so we get the associated exact sequence

$$1 \rightarrow PB_n \rightarrow B_n \rightarrow \mathfrak{S}_n \rightarrow 1.$$

The map from B_n to \mathfrak{S}_n takes a braid to the permutation induced on the ends, so this exact sequence conforms to the usual geometric description of braids and braid groups.

The obvious projection $\mathbb{C}^{n+1} \rightarrow \mathbb{C}^n$ restricts to a map $\text{PConf}_{n+1} \rightarrow \text{PConf}_n$, and this is a fiber bundle with fiber $\mathbb{C} \setminus \{z_1, \dots, z_n\}$ over (z_1, \dots, z_n) . Using the long exact sequence in homotopy, we get by induction that PConf_n and hence UConf_n are aspherical, or $K(\pi, 1)$, for every n . In particular, $H^*(\text{UConf}_n) = H^*(B_n)$, and $H^*(\text{PConf}_n) = H^*(PB_n)$. Since UConf_n is finite-dimensional, we get as another corollary that B_n is torsion-free.

2.1 Integral cohomology of PConf_n

Following [Arn69], we compute $H^*(\text{PConf}_n; \mathbb{Z})$ by induction on n .

Theorem 2.1 ([Arn69]). *$H^*(\text{PConf}_n; \mathbb{Z})$ is generated as a graded commutative algebra by $\omega_{ij} \in H^1$, for $1 \leq i, j \leq n$, $i \neq j$ with the relations*

$$\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0.$$

Sketch of proof. The fibration $\text{PConf}_{n+1} \rightarrow \text{PConf}_n$ has a global (continuous) section, for instance given by $z_{n+1} = 1 + \max_{i \leq n} |z_i|$, and the monodromy is trivial on fiber cohomology. This implies that the Leray–Serre spectral sequence has the E_2 page $E_2^{p,q} = H^p(\text{PConf}_n) \otimes H^q(\vee_n S^1)$. As for the differentials, $d_2 = 0$ due to the existence of the section and $d_r = 0$ for $r \geq 3$ by degree considerations. Therefore $E_\infty = E_2$, and by induction each term is free abelian, so there are no extension problems. See [Arn69] for more details. \square

In particular, each $H^i(\text{PConf}_n; \mathbb{Z})$ is free (see [Arn69] for explicit bases consisting of monomials in ω_{ij}). The generators ω_{ij} can be represented by the forms $\frac{dz_i - dz_j}{z_i - z_j}$ and evaluated on a braid counts the number of times z_i winds around z_j or vice versa.

The \mathfrak{S}_n action on cohomology can be described as $\sigma \cdot \omega_{ij} = \omega_{\sigma(i)\sigma(j)}$. Using transfer, we see that $H^1(\text{UConf}_n; \mathbb{Q}) = H^1(\text{PConf}_n; \mathbb{Q})^{\mathfrak{S}_n} = \mathbb{Q}$. For $i > 1$, it is known that $H^i(\text{UConf}_n; \mathbb{Z})$ is finite (see [Arn68]), so $H^i(\text{UConf}_n; \mathbb{Q}) = 0$.

2.2 Mod 2 cohomology

Now following [Fuk70] we describe briefly a cell-structure on UConf_n (or more precisely its one-point compactification), which lets us compute $H^*(\text{UConf}_n; \mathbb{F}_2)$, with somewhat explicit descriptions in terms of Stiefel–Whitney classes of a natural bundle on this configuration space.

The composition $B_n \rightarrow \mathfrak{S}_n \rightarrow O_n$ induces a map on classifying spaces $\text{UConf}_n \rightarrow \text{Gr}(n, \mathbb{R}^\infty)$ (the Grassmannian of n -planes in \mathbb{R}^∞). This pulls back the canonical bundle on $\text{Gr}(n, \mathbb{R}^\infty)$ to a

rank- n real vector bundle ξ_n on UConf_n . Since the composition $\text{PB}_n \rightarrow \text{B}_n \rightarrow \mathfrak{S}_n$ is trivial, this further pulls back to a trivial bundle on PConf_n and we have an alternate description of ξ_n as $(\text{PConf}_n \times \mathbb{R}^n)/\mathfrak{S}_n \rightarrow \text{PConf}_n/\mathfrak{S}_n$ (where \mathfrak{S}_n acts on the product diagonally). Associated to ξ_n we have the Stiefel–Whitney classes $w_i = w_i(\xi_n) \in H^i(\text{UConf}_n, \mathbb{F}_2)$.

Theorem 2.2 ([Fuk70]). $H^k(\text{UConf}_n; \mathbb{F}_2)$ has the basis $\eta(r_1, \dots, r_t)$ indexed by choices of non-negative integers r_1, r_2, \dots such that $\sum r_j 2^j \leq n$ and $k = \sum r_j (2^j - 1)$. The cup product is given by

$$\eta(r_1, \dots, r_t) \smile \eta(r'_1, \dots, r'_t) = \binom{r_1+r'_1}{r_1} \dots \binom{r_t+r'_t}{r_t} \eta(r_1+r'_1, \dots, r_t+r'_t).$$

$H^*(\text{UConf}_n; \mathbb{F}_2)$ is also generated as an algebra by w_1, \dots, w_n .

Sketch of proof. Stratify UConf_n by the multiplicities of the real parts of the n points. That is, given m_1, \dots, m_r summing to n , let $e(m_1, \dots, m_r)$ be the set of $\{z_1, \dots, z_n\}$ such that for some $x_1 < x_2 < \dots < x_r$, exactly m_j of the z_i 's have real part x_j . Then each $e(m_1, \dots, m_r)$ is homeomorphic to \mathbb{R}^{n+r} , and together define a cell-structure on $\text{UConf}_n \cup \{\infty\}$. The boundary maps are easy to describe combinatorially, with the unique 0-cell ∞ representing the degenerate cases of two points coinciding or one of the points going to $\infty \in \hat{\mathbb{C}}$. More explicitly, $e(m_1, \dots, m_s + m_{s+1}, \dots, m_r)$ is in the boundary of $e(m_1, \dots, m_s, m_{s+1}, \dots, m_r)$, with degree $\binom{m_s+m_{s+1}}{m_s}$ corresponding to the many ways in which the m_s points on the line $x = x_s$ and m_{s+1} points on $x = x_{s+1}$ can interleave when $x_s = x_{s+1}$.

From this, we can explicitly compute $H^i(\text{UConf}_n; \mathbb{F}_2)$. For the ring structure, we need to use that the sections $\text{PConf}_n \rightarrow \text{PConf}_{n+1}$ descend compatibly to homotopy classes of maps $\text{UConf}_m \times \text{UConf}_n \rightarrow \text{UConf}_{m+n}$ (embed $\mathbb{C} \sqcup \mathbb{C}$ inside \mathbb{C}). Defining $\text{B}_\infty = \varinjlim \text{B}_n$, we get that $H^*(\text{B}_\infty; \mathbb{F}_2) = \varprojlim (H^*(\text{B}_n; \mathbb{F}_2))$ is a Hopf algebra which surjects onto each $H^*(\text{B}_n; \mathbb{F}_2)$. Using general properties of Hopf algebras we get a description of the cup product on $H^*(\text{B}_\infty; \mathbb{F}_2)$ and hence each $H^*(\text{B}_n; \mathbb{F}_2)$.

The bundle ξ_n described above has the n generically transverse sections $s_k((x_1, \dots, x_n)) = \{(x_1, (\Re x_1)^k), \dots, (x_n, (\Re x_n)^k)\}$ (using the description of the total space as $(\text{PConf}_n \times \mathbb{R}^n)/\mathfrak{S}_n$), indexed by $k = 0, \dots, n-1$. These can be used to compute w_i , which generate the ring $H^*(\text{UConf}_n; \mathbb{F}_2)$ as well. \square

2.3 Other cohomology computations

For computing the cohomology of $\text{PConf}_n(X)$ or $\text{UConf}_n(X)$ for other spaces X , the main tool is the fiber bundle $\text{PConf}_{n+1}(X) \rightarrow \text{PConf}_n(X)$. When X is an open manifold, these fibrations also have sections like above. For a closed d -manifold X , Totaro provides in [Tot96] a spectral sequence converging to $H^*(\text{PConf}_n(X))$ with the E_2 page described in terms of $H^*(X)$ and $H^*(\text{PConf}_n(\mathbb{R}^d))$. Further, the differentials vanish when X is a projective variety, from Hodge-theoretic considerations. Some computations for configuration spaces on surfaces and spheres can be found in [Nap03].

Theorem 2.3 ([Tot96]). *If X is a smooth complex projective variety, then $H^*(\text{PConf}_n(X); \mathbb{Q})$ and $H^*(\text{UConf}_n(X); \mathbb{Q})$ can be computed from $H^*(X; \mathbb{Q})$.*

Contrastingly, the (rational) Betti numbers of $\text{PConf}_n(X)$ or $\text{UConf}_n(X)$ are in fact not determined by the rational Betti numbers of X . Fulton and MacPherson in [FM94] provide an explicit

counterexample: let $X = \mathbb{C}P^1 \times \mathbb{C}P^2$ and Y the total space of the $\mathbb{C}P^1$ bundle $\mathbb{P}(O(1) \oplus O(-1))$ over $\mathbb{C}P^2$. Then X and Y have the same Betti numbers but $\text{PConf}_3(X)$ and $\text{PConf}_3(Y)$ do not.

3 Relation with polynomials

Given a point $(z_1, \dots, z_n) \in \text{PConf}_n$, the polynomial $p(z) = (z - z_1) \dots (z - z_n)$ is square-free and hence the degree n discriminant Δ_n does not vanish on its coefficients. The polynomial p is monic and clearly depends only on the image $\{z_1, \dots, z_n\} \in \text{UConf}_n$. Conversely, given a monic polynomial with coefficients in $\text{Poly}_n := \mathbb{C}^n \setminus \{\Delta_n = 0\}$, it has n distinct roots, so we get a point in UConf_n .

More precisely, the *Viète map* $\Psi_n : \text{PConf}_n \rightarrow \text{Poly}_n$ given by

$$(z_1, \dots, z_n) \mapsto (\sigma_1(z_1, \dots, z_n), \sigma_2(z_1, \dots, z_n), \dots, \sigma_n(z_1, \dots, z_n)),$$

where σ_i are the elementary symmetric polynomials, descends to a continuous bijection and in fact a diffeomorphism $\text{UConf}_n \rightarrow \text{Poly}_n$. We will occasionally confuse these two spaces by this diffeomorphism.

In many senses, a method of finding roots of (square-free) polynomials is an inverse or partial inverse to either the diffeomorphism, or the Viète map. It then is unsurprising that obstructions to existence of solutions can be found or reformulated in terms of the topology of these spaces and maps.

As a first result, we establish the following well-celebrated theorem of Galois theory.

Theorem 3.1 (Abel–Ruffini). *The roots of the general polynomial of degree n cannot be written in terms of radicals and field operations.*

Instead of field extensions and Galois groups, we look at the topological analogues of covering spaces and monodromy groups. The splitting extension of the general quintic corresponds exactly to the cover $\text{PConf}_n \rightarrow \text{Poly}_n$ with monodromy group \mathfrak{S}_n , which is not solvable for $n \geq 5$. However, a solution in radicals would imply this group is a quotient of the monodromy group of a tower of cyclic covers, since attaching an n th root is the same as passing to a cover with monodromy $\mathbb{Z}/n\mathbb{Z}$. This proof is originally due to Arnol'd, for an existing and more detailed treatment, see [Žol00].

3.1 Finding roots approximately is hard too

Abel–Ruffini tells us that polynomials of high degree cannot be solved exactly using radicals. But in actual computations, we restrict ourselves to field operations, checking inequalities, and are only ever interested in computing the roots up to some error fixed beforehand. Following [Sma87], we show that an algorithm of this kind to find roots of a polynomial approximately would have to be sufficiently complicated, and the obstruction is directly implied by the result in Section 2.2.

Theorem 3.2 ([Sma87]). *For each d there is some $\epsilon > 0$ such that computing the roots of degree- d polynomials within ϵ has topological complexity at least $(\log_2 d)^{2/3}$.*

The precise definition of topological complexity can be found in [Sma87], but it is roughly the minimum number of if-then decisions needed to compute the roots using rational functions.

Sketch of proof. An approximate solution with complexity k is the same as a cover U_0, \dots, U_k of Poly_d and maps $f_i : U_i \rightarrow \text{PConf}_d$ such that for every $x \in U_i$, there is a point y within ϵ of $f_i(x)$ satisfying $x = \Psi_d(y)$. For a well-chosen ϵ , these can be improved to actual sections θ_i of Ψ_d on each (maybe shrunk) U_i . But then k is bounded below by the *cup length* of $\text{Ker } \Psi_d^*$, where Ψ_d^* is the map induced on cohomology (with arbitrary coefficients), and cup length is the maximal length of a non-zero cup product. By the results in Section 2.2, with \mathbb{F}_2 coefficients this cup length is greater than $(\log_2 d)^{2/3}$. \square

4 Algebraic functions

To expand on the themes in the discussion about Abel–Ruffini, we need to define the general notion of an algebraic function. We use the notation $\vec{x} = (x_1, \dots, x_m) \in \mathbb{C}^m$.

Definition 1. An n -valued *algebraic function* of m variables (over \mathbb{C}) is a polynomial $F(\vec{x}, y) = a_0(\vec{x})y^n + \dots + a_n(\vec{x})$, where $a_0, \dots, a_n \in \mathbb{C}[x_1, \dots, x_m]$ are polynomials, and $a_0 \neq 0$.

Such a polynomial is to be thought of as a function $f : \mathbb{C}^m \setminus \{a_0 = 0\} \rightarrow \text{Sym}^n(\mathbb{C})$ given by $\vec{x} \mapsto \{y \mid F(\vec{x}, y) = 0\}$. We also write $f : \mathbb{C}^m \xrightarrow{n} \mathbb{C}$. For example, n th root is an algebraic function of one variable, given by the polynomial $y^n - x$. Any rational function p/q is also an algebraic function, given by the polynomial $qy - p$.

Away from the vanishing set of a_0 and $\Delta_n(a_0, \dots, a_n)$, the polynomial $F(\vec{x}, y)$ has exactly n roots. Under the assumption that $\Delta_n(a_0, \dots, a_n)$ is not identically 0, we get an n -fold branched cover of \mathbb{C}^m given by the ‘graph’ $\Gamma(f) = \{(\vec{x}, y) \mid F(\vec{x}, y) = 0\}$.

By using resultants and Newton’s theorem on symmetric polynomials, it is not hard to see that sums, products, and more generally composition of algebraic functions is algebraic. We also have the *universal* (monic) n -valued algebraic function u_n , given by the polynomial $y^n + x_1 y^{n-1} + \dots + x_n$. By definition, any monic n -valued algebraic function is a composition of u_n with a polynomial map, and any n -valued algebraic function is a composition of u_n with a rational map. Note that for u_n , the branched cover $\Gamma(u_n)$ restricts to a cover on Poly_n , and the cover $\text{PConf}_n \rightarrow \text{UConf}_n = \text{Poly}_n$ factors through $\Gamma(u_n)$ — in fact PConf_n is the Galois closure of this smaller cover.

We can now restate the Abel–Ruffini theorem in terms of algebraic functions:

Theorem 4.1 (Abel–Ruffini). *The universal algebraic function u_n , for $n \geq 5$, cannot be written as a composition of rational functions and radicals.*

Even though u_5 is not representable in terms of radicals, it can be represented if we also allow the algebraic function of 1 variable given by the polynomial $y^5 + y + x$, also known as the Bring radical ([Jer58]). Using various substitutions under the name of *Tschirnhaus transformations*, the universal algebraic function u_n can be written in terms of algebraic functions of (at most) $n - r$ variables (along with field operations) as long as $n > (r - 1)!$ (see [Bra75]). Hilbert asked as the 13th problem in his list of 23 ([Hil02]) whether u_7 can be represented in terms of functions of at most 2 variables. Although the original formulation was in terms of *continuous* functions and answered in the positive by Kolmogorov and Arnol’d ([Kol56; Arn59]), it appears from later works that Hilbert intended an interpretation in algebraic functions ([Hil27; Abh97]), which is currently open.

More generally, we can consider the minimum number of variables needed for functions expressing u_n . The best general upper bound is the one found by Brauer ([Bra75]), which is known to not be tight, for example for $n = 9$. In this unrestricted version, the general problem remains widely open, with no non-trivial lower bounds. However, if we prohibit extraneous values, we have the following theorem of Arnol'd.

Theorem 4.2 ([Arn70, Theorem A]). *If n is a power of 2, then u_n cannot be represented in terms of algebraic functions of $< n - 1$ variables.*

Sketch of proof. To an n -valued algebraic function f , we associate Stiefel–Whitney classes as follows. On the complement $G(f)$ of the branch locus, we have an n -fold cover. This gives us the braid monodromy $\pi_1(G(f)) \rightarrow B_n$, and we pull-back $w_i = w_i(B_n)$ as in Section 2.2 to $w_i(f) \in H^i(G(f); \mathbb{F}_2)$.

Suppose the algebraic function $f : \mathbb{C}^k \xrightarrow{n} \mathbb{C}$ can be written (without extraneous values) in terms of algebraic functions of $\leq l$ variables. Then using general properties of covers, there is a single $\phi : \mathbb{C}^l \xrightarrow{n} \mathbb{C}$, and polynomial maps P, Q such that $f(x) = Q(\phi(P(x)), x)$. It follows that $w_i(f)$ is the pullback of $w_i(\phi)$. Now if $f = u_n$, then the braid monodromy is surjective, and when n is a power of 2, we get from Theorem 2.2 that $w_{n-1}(f) \neq 0$. However, $G(\phi)$ is an l -dimensional Stein manifold, so for $w_{n-1}(\phi)$ to be non-zero, we must have $l \geq n - 1$. \square

5 Non-singular polynomials and conical resolution

Above we described the topology of Poly_n and closely related spaces. We now describe the method of conical resolution, which can be used to compute the cohomology of spaces of polynomials subject to a broad class of non-degeneracy conditions (see [Gor14] for a much more general treatment), including Poly_n as a simple case. The method was introduced by Vassiliev in [Vas91], generalized in [Vas99] and again by Gorinov in [Gor05].

Following [Vas99], we demonstrate Vassiliev's method on the two examples of non-singular conics (in $\mathbb{C}P^2$) and non-singular cubic surfaces in $\mathbb{C}P^3$. The method consists of two main ideas. The first is to replace non-singular objects by singular objects via Alexander duality. The second is to exploit the combinatorics of the singular objects, using conical resolutions and topological order complexes, to obtain a space of the same homotopy type and a convenient filtration.

5.1 Quadrics in $\mathbb{C}P^2$

The space of homogeneous quadratic polynomials in three variables is isomorphic to \mathbb{C}^6 . Let Σ be the subset of singular polynomials. Let N be the space of non-singular conics in \mathbb{P}^2 . Then $\mathbb{C}^6 \setminus \Sigma \cong \mathbb{C}^\times \times N$, so the cohomology groups are related by the Künneth formula. By Alexander duality,

$$\tilde{H}^i(\mathbb{C}^6 \setminus \Sigma) \cong \tilde{H}_{11-i}(\Sigma),$$

where \tilde{H}_* denotes Borel–Moore homology, which in our case is the reduced homology of the one-point compactification.

The singular sets of the various elements in Σ are either a point (e.g. the polynomial XY), a line (e.g. X^2) or all of $\mathbb{C}P^2$ (e.g. 0). These sets are parametrized by $A_1 = \mathbb{C}P^2$, $A_2 = (\mathbb{C}P^2)^\vee \cong \mathbb{C}P^2$ and the one-point space $A_3 = \{\bullet\}$. Call a simplex in the join $A_1 * A_2 * A_3$ *coherent* if the vertices are incident on one another in $\mathbb{C}P^2$, and let Λ be the union of all the coherent simplices (it is obvious that a face of a coherent simplex is coherent). For $K \in \bigsqcup A_i$, define $\Lambda(K)$ to be the union of simplices all of whose vertices contain K , and let $L(K) \subset \Sigma$ be the subspace of \mathbb{C}^6 singular on K . Clearly $\Lambda(K)$ is a cone with vertex K and $L(K)$ has dimension 3, 1, or 0 respectively, for K being in A_1 , A_2 , or A_3 . Define $\sigma(K) = L(K) \times \Lambda(K)$ and $\sigma = \bigcup_K \sigma(K) \subset \Sigma \times \Lambda$ to be the conical resolution of Σ .

Proposition 5.1 ([Vas99, Proposition 2]). *The projection $\sigma \rightarrow \Sigma$ is a proper map and the induced map on one-point compactifications is a homotopy equivalence. In particular this induces isomorphisms on \tilde{H}_* .*

We have filtrations $F_1 \subset F_2 \subset F_3 = \sigma$ and $\Phi_1 \subset \Phi_2 \subset \Phi_3 = \Lambda$ by taking $F_i = \bigcup_{K \in A_j, j \leq i} \sigma(K)$ and $\Phi_i = \bigcup_{K \in A_j, j \leq i} \Lambda(K)$. It is clear from definition that $F_i \setminus F_{i-1}$ is a vector bundle over $\Phi_i \setminus \Phi_{i-1}$ with fiber $L(K)$ over $x \in \Lambda(K)$. Therefore the Borel–Moore homologies are related by Thom isomorphisms. Further, $\Phi_i \setminus \Phi_{i-1}$ fibers over A_i with fiber $\Lambda(K) \setminus \Phi_{i-1}$ over K .

This is enough information to compute the spectral sequences given by the filtration F_i , and it stabilizes on the E^2 page, with no extension problems. Tracing back the isomorphisms, we eventually get the following answer.

Theorem 5.2 ([Vas99, Proposition 1]). $H^i(N) = \{\mathbb{Z}, 0, 0, \mathbb{Z}/2, 0, \mathbb{Z}\}$.

5.2 Cubic surfaces in $\mathbb{C}P^3$

Theorem 5.3 ([Vas99, Theorem 4]). *The Poincaré polynomial (with \mathbb{Q} coefficients) of the space $N_{3,3}$ of non-singular cubic hypersurfaces in $\mathbb{C}P^3$ is equal to $(1 + t^3)(1 + t^5)(1 + t^7)$.*

Sketch of proof. The argument is broadly the same as above, with more involved calculations, and simpler arguments for the spectral sequence since we are using \mathbb{Q} coefficients. We again use that $\mathbb{C}^{20} \setminus \Sigma_{3,3} \cong \mathbb{C}^\times \times N_{3,3}$, where $\Sigma_{3,3}$ is the space of singular cubic polynomials in 4 variables.

The singularities of $f \in \Sigma_{3,3}$ can be of eleven types (a point, a pair of points, ..., a pair of intersecting lines, ..., a plane, ...), which we parametrize by the sets A_1, \dots, A_{11} , so that a singular set of type i can only be contained in a set of type j for $i \leq j$. Then the space $\sigma_{3,3}$ defined like above resolves $\Sigma_{3,3}$ and the map $\sigma_{3,3} \rightarrow \Sigma_{3,3}$ satisfies the conclusions of Proposition 5.1. We use the analogous filtration on $\sigma_{3,3}$. The spaces $F_i \setminus F_{i-1}$ are again vector bundles over $\Phi_i \setminus \Phi_{i-1}$, which in turn fiber over A_i with fiber $\Lambda(K) \setminus \Phi_{i-1}$.

These links (in the simplicial complex sense) are somewhat more complicated than before, and to determine their homology we need to use that $(S^2)^{*k}$ (the self join) is \mathbb{Q} -acyclic for $k \geq 2$ ([Vas99, Lemma 3]). For the complete E^1 page, see [Vas99, Figure 5]. All the terms are at most 1-dimensional, and since we have \mathbb{Q} -coefficients, the differential maps are either isomorphisms or 0. The E^∞ page can then be computed just from the facts that $\mathbb{C}^{20} \setminus \Sigma$ is a Stein manifold, and that its Poincaré polynomial must be divisible by $P(\mathbb{C}^\times, t) = 1 + t$. Tracing back the isomorphisms via Alexander duality and Künneth formula as before, we get the final result. \square

The Betti numbers from Theorem 5.3 coincide with those of $\mathrm{PGL}_4(\mathbb{C})$. Using a modification of the Vassiliev-Gorinov method to keep track of mixed Hodge structures on Borel–Moore homology, Tommasi proves in [Tom14] that this is true whenever the degree of the polynomials is high enough. More precisely, we have the following theorem.

Theorem 5.4 ([Tom14]). *Let X_d be the space of non-singular homogeneous polynomials of degree d in n variables. Then for $d > 2k - 1$, we have an isomorphism $H^k(X_d; \mathbb{Q}) \cong H^k(\mathrm{GL}_n(\mathbb{C}); \mathbb{Q})$.*

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