Microeconomic Theory I
Duality in Consumer Theory, Welfare Theory

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Indirect Utility Function

Indirect utility function

Definition

Indirect Utility Function:

$$V(p, w) = \max_{x} \{U(x) \mid px \leq w\} = U(x(p, w))$$

- It is the highest utility achievable given p and w.
- It is the utility at the demand for given p and w.
- It is the value of the utility maximization problem.

Properties of the Indirect Utility Function

Proposition

If U is continuous and LNS then $V(p, w)$ is

1. homogeneous of degree 0: $V(\alpha p, \alpha w) = V(p, w)$
   - why?
2. strictly increasing in w
   - why?
3. non-increasing in p
   - why?
4. continuous in p, w.

The Dual Problem: Expenditure Minimization

Expenditure Minimization Problem

Definitions

Given $U(x)$, the Expenditure Minimization Problem (EMP) is:

$$e(p, \bar{U}) \equiv \min_x px \quad \text{s.t. } U(x) \geq \bar{U}$$

is an **expenditure function**.

$$h(p, \bar{U}) \equiv \arg\min_x px \quad \text{s.t. } U(x) \geq \bar{U}$$

is a **Hicksian demand function**.

Expenditure Minimization Problem

- The *expenditure function* determines the *minimum wealth* required to reach a given level of utility, for given prices p.
- The *Hicksian demand function* determines the *bundle* that gives a required level of utility at the minimum cost, for given prices p.

Expenditure Minimization Problem

- In a utility maximization problem, we fix the budget constraint and we look for the highest indifference curve satisfying the constraint.
- In an Expenditure minimization problem, we fix the indifference curve and we look for the lowest budget line satisfying the constraint.
- The Expenditure Minimization problem is the the dual problem of the utility maximization problem.
- Note: $e(p, \bar{U}) = ph(p, \bar{U})$.

Duality

Proposition

Given $U(x)$ continuous, LNS, and $p \gg 0$,

1. if $x(p, w)$ solves UMP for $w > 0$, then
   $x(p, w)$ solves EMP when $\bar{U} = V(p, w) = U(x(p, w))$
   $e(p, V(p, w)) = w$
2. if $h(p, \bar{U})$ solves EMP for $\bar{U} > U(0)$, then
   $h(p, \bar{U})$ solves UMP when $w = ph(p, \bar{U}) = e(p, \bar{U})$
   $V(p, e(p, \bar{U})) = \bar{U}$.

$$x(p, w) = h(p, V(p, w))$$
$$h(p, \bar{U}) = x(p, e(p, \bar{U}))$$

Duality: Informal Proof

1) given a $x^\star$ solving UMP for some p, w, show that this $x^\star$ solves the *corresponding* EMP.

INFORMAL PROOF:

- Given a $x^\star \equiv x(\hat{p}, \hat{w})$ for some $\hat{p}$ and $\hat{w}$
- By contradiction, suppose $x^\star$ does not solve the EMP for prices $\hat{p}$ and $\bar{U} = U(x^\star)$.
- Then, there is a x' such that $U(x') \geq U(x^\star)$ and $\hat{p}x' < \hat{p}x^\star = \hat{w}$
- By LNS, there is a x'' with $\hat{p}x'' < w$ and $U(x'') > U(x') \geq U(x^\star)$. Contradiction!
- Therefore $e(\hat{p}, v(\hat{p}, \hat{w})) = \hat{p}x^\star$. By LNS, $\hat{p}x^\star = w$

Duality: Informal Proof

2) given a $y^\star$ solving EMP for some p, $\bar{U}$, show that this $y^\star$ solves the *corresponding* UMP.

Properties of the expenditure function

For $U(\cdot)$ continuous and LNS, $e(p, \bar{U})$ is

1. O^1 in p

Proof.

$$e(\alpha p, \bar{U}) = \min_x \left\{ \alpha p x \mid U(x) \geq \bar{U} \right\} = \\ = \alpha \min_x \{ px \mid \ldots \} = \alpha e(p, \bar{U})$$

□

Properties of the expenditure function

2. Increasing in p and strictly increasing in $\bar{U}$

Proof.

- Assume $p' > p$
  - $e(p', \bar{U}) = p' h(p', \bar{U}) \geq ph(p', \bar{U}) \geq ph(p, \bar{U}) = e(p, \bar{U})$
- Assume $\bar{U}' > \bar{U}$
  - By contradiction: $\bar{U}' > \bar{U}$ and $ph(p, \bar{U}') \leq ph(p, \bar{U})$
  - By continuity, $\exists x$ s.t. $\bar{U}' > U(x) > \bar{U}$ and $px < ph(p, \bar{U})$ (for example, take $x \equiv h(p, \bar{U}') - \varepsilon \begin{bmatrix} \vdots \end{bmatrix}$). Contradiction

□

Properties of the expenditure function

3. Concave in p

Proof.

$$\begin{aligned}
\alpha \in [0,1] \qquad p'' &= \alpha p + (1-\alpha)\, p' \\
ph(p'', \bar{U}) &\geq e(p, \bar{U}) = ph(p, \bar{U}) \\
p' h(p'', \bar{U}) &\geq e(p', \bar{U}) \\
e(p'', \bar{U}) &= p'' h(p'', \bar{U}) = \alpha ph(p'', \bar{U}) + (1-\alpha) p' h(p'', \bar{U}) \geq \\
&\geq \alpha e(p, \bar{U}) + (1-\alpha)\, e(p', \bar{U})
\end{aligned}$$

□

4. Continuous

Hicksian Compensation

- Duality: for given p, the same bundle x^* solves both problems
- After a price change, however, EMP gives a different solution than UMP

Definition

Hicksian Compensation: suppose that prices change from p to p'

- define $\bar{u} = V(p, w)$
- Hicksian Compensation: $e(p', \bar{u}) - w$

- wealth required so that, after the price change, the utility level reached by the consumer is unchanged.
- it is another way to compute substitution and income effect.

Hicksian Compensation

Proposition

$$(p' - p)(h(p', \bar{U}) - h(p, \bar{U})) \leq 0$$

i.e. the Hicksian substitution effect is negative.

Proof.

We know that

$$p' h(p', \bar{U}) \leq p' h(p, \bar{U})$$

$$p' \left(h(p', \bar{U}) - h(p, \bar{U})\right) \leq 0 \tag{1}$$

Similarly

$$ph(p, \bar{U}) \leq ph(p', \bar{U})$$

$$-p \left(h(p', \bar{U}) - h(p, \bar{U})\right) \leq 0 \tag{2}$$

the proposition follows from 1 and 2. □

*********** Math Aside **************

************ The Envelope Theorem *************

Theorem

If $f^*(r) = \max_x f(x, r)$ *then*

$$\frac{\partial f^*(r)}{\partial r} = \left.\frac{\partial f(x, r)}{\partial r}\right|_{x=x^*(r)}$$

where $x^*(r) = \arg\max_x f(x, r)$.

Proof.

$$f^*(r) = \max_x f(x, r) = f(x^*(r), r)$$

$$\frac{\partial f^*(r)}{\partial r} = \underbrace{\left.\frac{\partial f(x, r)}{\partial x}\right|_{x=x^*(r)}}_{0 \text{ (by FOC)}} \frac{\partial x^*(r)}{\partial r} + \left.\frac{\partial f(x, r)}{\partial r}\right|_{x=x^*(r)}$$

□

- We can use the envelope theorem for differentiable interior solutions
- It is often true for corner solutions, too

************* end of Math Aside **********

The Slutzky Equation: Income and Substitution Effect for Marginal Price Changes

Shephard's lemma

Proposition

If $U(x)$ is LNS, the solution to the expenditure minimization is interior, and $h(p,\bar{U})$ is a function then

$$h(p,\bar{U}) = \nabla_p e(p,\bar{U})$$
$$h_i(p,\bar{U}) = \frac{\partial e(p,\bar{U})}{\partial p_i} \quad \forall i \in \{1,\dots,L\}.$$

Shephard's lemma

Proof.

$$e(p,\bar{U}) = \min_{x,\lambda} \left\{ px - \lambda \left[\bar{U} - U(x) \right] \right\} \Longleftrightarrow \begin{cases} e(p,\bar{U}) = \min px \\ \text{s.t. } U(x) \geq \bar{U} \\ \quad (\text{LNS} \ \Rightarrow \text{"} = \text{"}) \end{cases}$$

$$\frac{\partial e(p,\bar{U})}{\partial p_i} = x_i^* = h_i(p,\bar{U}) \quad (\Leftarrow \text{envelope theorem})$$

□

Slutsky equation

Proposition

If $\succeq$ are continuous, strictly convex and "well-behaved", then

$$\frac{\partial x_i(p,w)}{\partial p_j} = \frac{\partial h_i(p,\bar{U})}{\partial p_j} - \frac{\partial x_i(p,w)}{\partial w} x_j(p,w)$$

where $\bar{U} = V(p,w)$.

- For an arbitrarily small Δp,

$$\text{total effect} = \text{substitution effect} + \text{income effect}$$

Slutsky equation

Proof.

We know that, for good i

$$h_i(p, \bar{U}) = x_i(p, e(p, \bar{U}))$$

differentiate with respect to p_j. We get

$$\frac{\partial h_i(p, \bar{U})}{\partial p_j} = \frac{\partial x_i(p, w)}{\partial p_j} + \frac{\partial x_i(p, w)}{\partial w} \underbrace{\frac{\partial e(p, \bar{U})}{\partial p_j}}_{h_j(p, \bar{U})}$$

By assumption $\bar{U} = V(p, w)$, so that

$$h_j(p, \bar{U}) = h_j(p, V(p, w)) = x_j(p, w)$$

Which concludes the proof.

Roy's identity

Proposition

If $\succeq$ are continuous, strictly convex and "well-behaved", then

$$x_j(p, w) = -\frac{\frac{\partial V(p,w)}{\partial p_j}}{\frac{\partial V(p,w)}{\partial w}}.$$

Roy's identity

Proof.

$$\text{define } \bar{U} = V(p, w) \text{ and } w = e(p, \bar{U})$$
$$\text{so that } V(p, e(p, \bar{U})) = \bar{U}$$

- Differentiate both sides with respect to p_j and rearrange:

$$\underbrace{\frac{\partial e(p, \bar{U})}{\partial p_j}}_{x_j(p,w)} = -\frac{\frac{\partial V(p,w)}{\partial p_j}}{\frac{\partial V(p,w)}{\partial w}}$$

□

Welfare Economics

Welfare Analysis

- So far, *positive analysis*: when prices change, describe how the consumer's behavior changes.
- In this section, *normative analysis*: normative analysis: after the price change, are consumers worse off or better off?
  - For example: evaluate a new policy, which is going to affect prices and income.
- To simplify the analysis, we assume that:
  - In period 1, budget set 1: p, w
  - After the policy change, budget set 2: p', w'

Problem

Find a sensible criteria to evaluate the effect of the change in budget set on the consumer's welfare.

How Can We Evaluate Changes in the Budget Set?

1. Comparing indirect utility
   - Example, if $V(p, w) - V(p', w') > 0$ the consumer is ...
   - Indirect utility not a good measure of welfare:
2. Using the expenditure function
   - **Equivalent variation**: $EV = e(p, V(p', w')) - w$
   - **Compensating variation**: $CV = w' - e(p', V(p, w))$

Equivalent Variation and Compensating Variation

- If p and p' only differ by one price (p_i) and $w = w'$, then

$$EV(p, p', w) = e(p, \bar{U}') - \underbrace{e(p', \bar{U}')}_{w'=w=e(p,\bar{U})} = \int_{p_i'}^{p_i} h_i(p, \bar{U}')dp_i$$

$$CV(p, p', w) = \underbrace{e(p, \bar{U})}_{w} - e(p', \bar{U}) = \int_{p_i'}^{p_i} h_i(p, \bar{U})dp_i$$

where $\bar{U}' \equiv V(p', w)$ and $\bar{U} \equiv V(p, w)$.

Note:

CV and EV have one drawback: they relies on knowing the indirect utility function.

Consumer Surplus

3. **Consumer surplus**: the area underneath the demand function.

$$\Delta CS = \int_{p_i'}^{p_i} x_i(p, w)dp$$

Remember

Consumer Surplus is only used to evaluate the welfare effect of a change in one price.

What is the Relation Between CS, EV, CV?

Remember:

- duality implies $h(p, V(p, w)) = x(p, w)$ and $h(p', V(p', w)) = x(p', w)$.
- Slutzky equation:

$$\frac{\partial x_i}{\partial p_i} = \frac{\partial h_i}{\partial p_i} - \frac{\partial x_i}{\partial w} x_i$$

- If we know the sign of $\frac{\partial x_i}{\partial w}$ we can determine the relative slope of x and h
  - if x_i is a normal good then $\frac{\partial x_i}{\partial w} > 0$, implying that $\frac{\partial x_i}{\partial p_i} < \frac{\partial h_i}{\partial p_i}$
  - if x_i is an inferior good then $\frac{\partial x_i}{\partial w} < 0$, implying that $\frac{\partial x_i}{\partial p_i} > \frac{\partial h_i}{\partial p_i}$

What is the Relation Between CS, EV, CV?

For a change in one price:

- For normal goods

$$CV \leq \Delta CS \leq EV$$

- For inferior goods

$$EV \leq \Delta CS \leq CV$$

- ΔCS is an approximation of CV and EV