Grounded by Gravity:
A Well-Behaved Trade Model with Industry-Level Economies of Scale

Online Appendix

Konstantin Kucheryavyy — U Tokyo
Gary Lyn — UMass Lowell
Andrés Rodríguez-Clare — UC Berkeley and NBER

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1. Algorithm for Finding All Equilibria Using Interval Analysis

The algorithm presented in this section is a simplified and adapted version of the algorithm described in Hansen and Walster (2003, Chapter 11). We have two major adaptations relative to Hansen and Walster (2003). First, we exploit the specific structure of our equilibrium system to remove parts of intervals for wages and labor allocations that cannot contain a solution (this is what Hansen and Walster (2003) call *hull consistency checks*; see Sections 10.3-10.5 in the book). Our second adaptation concerns the way we work with interval vectors that have zeros as (some of) their left endpoints. Evaluation of the Jacobian of our equilibrium system on such interval vectors yields intervals with infinite endpoints. In these cases an (interval) Newton method cannot be used. Consequently, we can only use bisection in these cases, and this is significantly more time consuming than a Newton method. To take advantage of a Newton method even in these cases, we first split interval vectors along all dimensions with zero endpoints in such a way that any component of any resulting interval vector is either very "narrow" and "close" to zero or is bounded away from zero. After that we never process components that are "narrow" and "close" to zero and apply a Newton method only to the components that are bounded away from zero. Details of both of the adaptations are described below.

In the algorithm description below we use math blackboard font to denote intervals,

interval vectors, and interval matrices: $\mathbb{X}$, $\mathbb{B}$, $\mathbb{J}$, etc. For brevity, we use the word "box" to mean "interval vector", and we use the word "interval" to mean a one-dimensional box. Components of boxes are denoted by subscripts: for example, the i-th component box $\mathbb{X}$ is denoted by $\mathbb{X}_i$ and is an interval. In this notation, $\mathbb{X} = (\mathbb{X}_1, \ldots, \mathbb{X}_N)^T$, where N is the dimension of $\mathbb{X}$. Lower and upper bounds of any interval $\mathbb{Y}$ are denoted by $\underline{Y}$ and $\overline{Y}$ correspondingly, so that $\mathbb{Y} = \left[\underline{Y}, \overline{Y}\right]$. In this notation an N-dimensional box $\mathbb{X}$ can be written as $\mathbb{X} = \left(\left[\underline{X}_1, \overline{X}_1\right], \ldots, \left[\underline{X}_N, \overline{X}_N\right]\right)^T$. For any box $\mathbb{X}$ its midpoint is given by

$$\operatorname{mid}(\mathbb{X}) \equiv \left(\frac{\underline{X}_1 + \overline{X}_1}{2}, \ldots, \frac{\underline{X}_N + \overline{X}_N}{2}\right)^T.$$

For any interval $\mathbb{X}$ its diameter is given by

$$\operatorname{diam}(\mathbb{X}) \equiv \overline{X}_1 - \underline{X}_1,$$

diameter of a box is the maximum over diameters of its components:

$$\operatorname{diam}(\mathbb{X}) \equiv \max_i \operatorname{diam}(\mathbb{X}_i).$$

Our system of equations and inequalities is given by expressions (3), (4) and (5) from the main text, plus an equation for normalization of wages: $\sum_i w_i = 1$. We transform this system in two ways. First, for all industries k with $\alpha_k < 0$ we leave only the equality part of the complementary slackness conditions and divide these equalities by $w_i^{-\varepsilon_k} L_{i,k}^{\alpha_k}$ for each country i.[1] Second, we divide all remaining complementary slackness conditions by $w_i^{-\varepsilon_k}$ for each country i, and use interval arithmetic to turn the inequality part of the complementary slackness conditions into equations. To see how the latter transformation can be done, consider an inequality $f(x) \geq 0$. We can write this inequality as $f(x) = [0, +\infty]$ or, equivalently, as $f(x) + [-\infty, 0] = 0$.[2] With these transformations, our

[1] Division by $w_i^{-\varepsilon_k} L_{i,k}^{\alpha_k}$ reduces the number of occurrences of w_i and $L_{i,s}$ in our equations. A general recommendation for calculation of interval function extensions is to minimize the number of occurrences of each variable. This typically helps to calculate sharper bounds on the range of values of a function over an interval.

[2] Here, the arithmetic is done over the set of extended real numbers $\mathbb{R} \cup \{-\infty, +\infty\}$. See Chapter 4 in Hansen and Walster (2003).

full equilibrium system is given by the following set of equations:

$$w_i^{1+\varepsilon_k} L_{i,k}^{1-\alpha_k} - \sum_n \frac{S_{i,k} \left[\tau_{ni,k}\right]^{-\varepsilon_k}}{\sum_l S_{l,k} L_{l,k}^{\alpha_k} \left[w_l \tau_{nl,k}\right]^{-\varepsilon_k}} \beta_{n,k} w_n \bar{L}_n = 0, \tag{1}$$

$$\text{for } i = 1, \ldots, N, \text{ and all } k \text{ such that } \alpha_k < 1;$$

$$\begin{cases} w_i^{1+\varepsilon_k} L_{i,k} - \sum_n \dfrac{S_{i,k} L_{i,k}^{\alpha_k} \left[\tau_{ni,k}\right]^{-\varepsilon_k}}{\sum_l S_{l,k} L_{l,k}^{\alpha_k} \left[w_l \tau_{nl,k}\right]^{-\varepsilon_k}} \beta_{n,k} w_n \bar{L}_n = 0, \\ w_i^{1+\varepsilon_k} - \sum_n \dfrac{S_{i,k} L_{i,k}^{\alpha_k - 1} \left[\tau_{ni,k}\right]^{-\varepsilon_k}}{\sum_l S_{l,k} L_{l,k}^{\alpha_k} \left[w_l \tau_{nl,k}\right]^{-\varepsilon_k}} \beta_{n,k} w_n \bar{L}_n + [-\infty, 0] = 0, \end{cases} \tag{2}$$

$$\text{for } i = 1, \ldots, N, \text{ and all } k \text{ such that } \alpha_k \geq 1;$$

$$\sum_k L_{i,k} - \bar{L}_i = 0, \quad i = 1, \ldots, N; \tag{3}$$

$$\sum_i w_i - 1 = 0, \quad i = 1, \ldots, N. \tag{4}$$

Inequalities for labor allocations, $L_{i,k} \geq 0$, are automatically satisfied in our algorithm, because we solve for equilibrium labor allocations using intervals with non-negative values only. Therefore, we do not include these inequalities into the system.

Let $\mathbb{IR}$ denote the set of intervals over the extended set of real number $\mathbb{R} \cup \{-\infty, +\infty\}$, $\mathbb{IR} = \{[\underline{x}, \overline{x}] \,|\, \underline{x}, \overline{x} \in \mathbb{R} \cup \{-\infty, +\infty\}, \underline{x} \leq \overline{x}\}$. Let $F : \mathbb{R}_+^{N+N*K} \to \mathbb{IR}^M$ be the vector-function corresponding to the left-hand sides of equations (1)-(4), where M is the number of equations. We need to formally define the image of F to be in $\mathbb{IR}^M$ because of inequalities (2) turned into equalities — evaluations of the LHS of the other equations produce degenerate intervals $[F_i, F_i]$. Interval arithmetic methods allow us to extend the domain of F to $\mathbb{IR}_+^{N+N*K}$ (here $\mathbb{IR}_+$ is the set of intervals over real numbers with nonnegative left endpoints). Such extension is called "natural extension" and is automatically handled by special software packages. Following the notation we use for intervals, we employ the same math blackboard font to denote the natural extension of F to intervals. That is, if $\mathbb{X}$ is a box corresponding to intervals for wages and labor allocations, then evaluation of the LHS of our equilibrium system (1)-(4) on box $\mathbb{X}$ produces box $\mathbb{F}(\mathbb{X})$. Similarly, evaluation of the Jacobian of our system on $\mathbb{X}$ produces interval matrix $\mathbb{J}(\mathbb{X})$. In the algorithm description below, we drop the argument $\mathbb{X}$ from the function and Jacobian notation, when it does not cause any confusion.

We say that the *equilibrium conditions are not violated* when evaluated on box $\mathbb{X}$, if $0 \in \mathbb{F}_i(\mathbb{X})$ for all i. As was mentioned in Section 3.3.2 of the main text, using interval arithmetic for evaluation of a function on a box generally results in overestimation of the true range of values of this function on the box. Overestimation occurs for at least two reasons: interval dependency and rounding errors due to finite precision arithmetic.[3] Overestimation is the reason why we use the term "equilibrium conditions are not violated" rather than "equilibrium conditions are satisfied". Because of overestimation, the fact that equilibrium conditions are not violated in a box does not imply that this box contains an equilibrium.

One of the most important parts of our algorithm is the set of *hull consistency checks*, HCC, that we perform in order to remove parts of intervals that cannot contain a solution. We use the following HCCs:

(HCC 1) For each industry k, given a set of intervals for labor allocations $\mathbb{L}_{1,k}, \ldots, \mathbb{L}_{N,k}$, find a country i with the widest interval $\mathbb{L}_{i,k}$. Calculate

$$\widetilde{\mathbb{L}}_{i,k} = \mathbb{L}_{i,k} \cap \left(\overline{L}_i - \sum_{j \neq i} \mathbb{L}_{j,k} \right).$$

If the resulting interval $\widetilde{\mathbb{L}}_{i,k}$ is empty, then the original set of intervals $\mathbb{L}_{1,k}, \ldots, \mathbb{L}_{N,k}$ does not contain a solution.

(HCC 2) Given a set of intervals for wages $\mathbb{W}_1, \ldots, \mathbb{W}_N$, find a country i with the widest interval $\mathbb{W}_i$. Calculate

$$\widetilde{\mathbb{W}}_i = \mathbb{W}_i \cap \left(1 - \sum_{j \neq i} \mathbb{W}_j \right).$$

If the resulting interval $\widetilde{\mathbb{W}}_i$ is empty, then the original set of intervals $\mathbb{W}_1, \ldots, \mathbb{W}_N$ does not contain a solution.

(HCC 3) Given a set of intervals for labor allocations, $\mathbb{L}_{1,k}, \ldots, \mathbb{L}_{N,k}$, and wages, $\mathbb{W}_1, \ldots, \mathbb{W}_N$, for each (i,k) use equations (1) for industries with $\alpha_k < 1$ to

[3] For example, in interval computations, multiplication $\mathbb{X} \cdot \mathbb{X}$ over interval $\mathbb{X} = [-1,1]$ is treated as if it were applied to two independent intervals. The result of this operation is interval $[-1,1]$, while the true range of values of function $f(x) = x \cdot x$ over interval $[-1,1]$ is $[0,1]$. This phenomenon is called *interval dependency*. For more details see, for example, Section 5.2 in Moore, Kearfott and Cloud (2009) or Section 2.4 in Hansen and Walster (2003).

calculate

$$\widetilde{\mathbb{L}}_{i,k} = \mathbb{L}_{i,k} \cap \left(\sum_n \frac{S_{i,k} \left[\tau_{ni,k}\right]^{-\varepsilon_k} \mathbb{W}_i^{-1-\varepsilon_k}}{\sum_l S_{l,k} \mathbb{L}_{l,k}^{\alpha_k} \left[\mathbb{W}_l \tau_{nl,k}\right]^{-\varepsilon_k}} \beta_{n,k} \mathbb{W}_n \bar{L}_n \right)^{\frac{1}{1-\alpha_k}},$$

and use equations (2) for industries with $\alpha_k \geq 1$ to calculate:

$$\widetilde{\mathbb{L}}_{i,k} = \mathbb{L}_{i,k} \cap \left(\sum_n \frac{S_{i,k} \mathbb{L}_{i,k}^{\alpha_k} \left[\tau_{ni,k}\right]^{-\varepsilon_k} \mathbb{W}_i^{-1-\varepsilon_k}}{\sum_l S_{l,k} \mathbb{L}_{l,k}^{\alpha_k} \left[\mathbb{W}_l \tau_{nl,k}\right]^{-\varepsilon_k}} \beta_{n,k} \mathbb{W}_n \bar{L}_n \right).$$

Next, for each i calculate

$$\widetilde{\mathbb{W}}_i = \mathbb{W}_i \cap \left(\frac{1}{\bar{L}_i} \sum_k \sum_n \frac{S_{i,k} \mathbb{L}_{i,k}^{\alpha_k} \left[\mathbb{W}_i \tau_{ni,k}\right]^{-\varepsilon_k}}{\sum_l S_{l,k} \mathbb{L}_{l,k}^{\alpha_k} \left[\mathbb{W}_l \tau_{nl,k}\right]^{-\varepsilon_k}} \beta_{n,k} \mathbb{W}_n \bar{L}_n \right).$$

If any of the resulting intervals $\widetilde{\mathbb{L}}_{i,k}$ or $\widetilde{\mathbb{W}}_i$ is empty, then the original set of intervals does not contain a solution.

(HCC 4) Given a set of intervals for labor allocations, $\mathbb{L}_{1,k}, \ldots, \mathbb{L}_{N,k}$, and wages, $\mathbb{W}_1, \ldots, \mathbb{W}_N$, for each (i,k) calculate

$$\mathbb{GMC}_{i,k} = \mathbb{W}_i \mathbb{L}_{i,k} - [0,1] \times \sum_n \beta_{n,k} \mathbb{W}_n \bar{L}_n.$$

If for any (i,k), $0 \notin \mathbb{GMC}_{i,k}$, then the given set of intervals for labor allocations and wages does not contain a solution. Here we exploit the fact that all trade shares are between 0 and 1.

In our simulations, introduction of the above set of consistency checks resulted in an almost 60-fold speedup of the algorithm relative to the algorithm with just the Newton and bisection steps.

We are now ready to describe the algorithm. When describing the algorithm, we aim at outlining the steps conceptually while skipping many of the details. The in-depth coverage of the details of the algorithm is contained in Hansen and Walster (2003), although in a more general context.

Algorithm Input:

(i) System of equations is given by (1)-(4).

(ii) Unknowns are intervals for labor allocations $L_{i,k}$ and wages w_i for $i = 1, \ldots, N$ and $k = 1, \ldots, K$. Initial intervals for labor allocations are $\left[0, \bar{L}_i\right]$ for each $L_{i,k}$; initial intervals for wages are $[0, 1]$ for each w_i. Thus, the initial (base) box is:

$$\mathbb{B} \equiv \Big(\underbrace{\left[0, \bar{L}_1\right], \ldots, \left[0, \bar{L}_1\right]}_{K \text{ times}}, \ldots, \underbrace{\left[0, \bar{L}_N\right], \ldots, \left[0, \bar{L}_N\right]}_{K \text{ times}}, \underbrace{[0,1], \ldots, [0,1]}_{N \text{ times}} \Big)^T.$$

Algorithm Output:

(i) List $\mathcal{R}$ of boxes consisting of intervals for labor allocations and wages such that for each box in $\mathcal{R}$: (a) diameters of its components are all smaller than a specified value $\varepsilon > 0$; (b) the equilibrium conditions evaluated on this box are not violated.

(ii) List $\mathcal{U}$ of boxes of the same size as $\mathcal{R}$: each box $\mathbb{X}$ in $\mathcal{R}$ has a corresponding box $\mathbb{Y}$ in $\mathcal{U}$. If $\mathbb{Y}$ is non-empty, then $\mathbb{X} \subseteq \mathbb{Y}$, and it was successfully verified that there exists a unique equilibrium within box $\mathbb{Y}$ and this equilibrium is in the box $\mathbb{X}$. In other words, box $\mathbb{Y}$ is a region of uniqueness.[4] If $\mathbb{Y}$ is empty, then the algorithm was not able to verify neither existence nor uniqueness of equilibrium within box $\mathbb{X}$.

(iii) List $\mathcal{L}$ of boxes that have not been processed. This array might be non-empty if the algorithm stops because the time or iteration limit is reached.

Algorithm Steps:

Step 0: Split the base box $\mathbb{B}$ into boxes $\mathbb{X}^{(1)}, \ldots, \mathbb{X}^{(M)}$ such that for any $\mathbb{X}^{(\ell)}$ we have that either $\mathbb{X}_i^{(\ell)} = [\underline{B}_i, \underline{B}_i + \delta]$ or $\mathbb{X}_i^{(\ell)} = [\underline{B}_i + \delta, \overline{B}_i]$ for $i = 1, \ldots, N*(K+1)$, and where $\delta > 0$ is small. Here $\underline{B}_j$ and $\overline{B}_j$ are endpoints of component j of the base box $\mathbb{B}$. As a result of this splitting, we deal with boxes such that their components are either bounded away from the left endpoints of $\mathbb{B}$ (that is, zeros), or, if they include the left endpoints of $\mathbb{B}$, then they are "narrow" and do not require any processing. Thus, after this splitting we neither do bisection nor try to apply the Newton step to the components $[\underline{B}_i, \underline{B}_i + \delta]$.

Create list $\mathcal{L} = \left\{\mathbb{X}^{(1)}, \ldots, \mathbb{X}^{(M)}\right\}$ and go to Step 1.

Step 1: If list $\mathcal{L}$ is empty, then finish. Otherwise, choose a box from $\mathcal{L}$ and call it $\mathbb{X}^{\mathrm{c}}$ ("c" for "current"). Remove $\mathbb{X}^{\mathrm{c}}$ from $\mathcal{L}$. Go to Step 2.

[4] Typically, this region of uniqueness is not much "larger" than box $\mathbb{X}$.

Step 2: Perform consistency checks HCC 1-4 on box $\mathbb{X}^{\mathrm{c}}$. If any of the consistency checks finds that there is no solution in $\mathbb{X}^{\mathrm{c}}$, stop any further processing of this box and go to Step 1. Otherwise, proceed further with current Step 2.

Consistency checks HCC 1-3 produce a new box $\widetilde{\mathbb{X}}^{\mathrm{c}}$. If $\widetilde{\mathbb{X}}^{\mathrm{c}}$ is a "significantly" reduced version of box $\mathbb{X}^{\mathrm{c}}$, then replace box $\mathbb{X}^{\mathrm{c}}$ with $\widetilde{\mathbb{X}}^{\mathrm{c}}$ and repeat current Step 2. Otherwise, replace box $\mathbb{X}^{\mathrm{c}}$ with $\widetilde{\mathbb{X}}^{\mathrm{c}}$ and go to Step 3.

Step 3: Evaluate the equilibrium system on the box $\mathbb{X}^{\mathrm{c}}$. If any of the equilibrium conditions is violated, stop any further processing of box $\mathbb{X}^{\mathrm{c}}$ and go to Step 1. Otherwise, go to Step 4.

Step 4: If $\mathrm{diam}\left(\mathbb{X}^{\mathrm{c}}\right)$ is "too large", go to Step 12, the bisection step. Otherwise, go to Step 5. We avoid applying the Newton step to "large" boxes, because this step is expensive and gives significant reductions of boxes only when they become relatively "small". The criterion for determining if the diameter of the current box is "too large" is described in Section 11.11 of Hansen and Walster (2003).

Step 5: Find all components of $\mathbb{X}^{\mathrm{c}}$ that are of type $[\underline{B}_i, \underline{B}_i + \delta]$. Let $\mathbb{X}^{\mathrm{cz}}$ ("z" for "zero") be the box with these components, and let $\mathbb{X}^{\mathrm{cl}}$ ("l" for "large") be the box with the rest of the components of $\mathbb{X}^{\mathrm{c}}$. For simplicity of exposition (and at the risk of abusing notation) let $\mathbb{X}^{\mathrm{c}}$ be represented as $\mathbb{X}^{\mathrm{c}} = \left(\mathbb{X}^{\mathrm{cl}}, \mathbb{X}^{\mathrm{cz}}\right)$.

Let $\mathbb{X}^{\mathrm{sl}} \equiv \left(\mathrm{mid}\left(\mathbb{X}^{\mathrm{cl}}\right), \mathrm{mid}\left(\mathbb{X}^{\mathrm{cl}}\right)\right)$, that is, $\mathbb{X}^{\mathrm{sl}}$ is a degenerate box that is a midpoint of box $\mathbb{X}^{\mathrm{cl}}$. Let $\mathbb{X}^{\mathrm{s}} \equiv \left(\mathbb{X}^{\mathrm{sl}}, \mathbb{X}^{\mathrm{cz}}\right)$.

Go to Step 6.

Step 6: Calculate the Jacobian matrix of the equilibrium system on box $\mathbb{X}^{\mathrm{c}}$. Call it $\mathbb{J}$.

Calculate the first-order Taylor expansion of the equilibrium system around $\mathbb{X}^{\mathrm{s}}$ using $\mathbb{J}$. Check the equilibrium conditions: if any of them is violated, stop processing the current box and go to Step 1. Otherwise, go to Step 7.

Step 7: In this step we are going to choose a part of the Jacobian matrix $\mathbb{J}$ to apply the Newton step to box $\mathbb{X}^{\mathrm{cl}}$. Recall that we have more equations than unknowns. So we are going to use only a subset of equations for the Newton step: one equation for one unknown.

For each component of $\mathbb{X}^{\mathrm{cl}}$ corresponding to country-i-industry-k labor allo-

cation choose the row of $\mathbb{J}$ corresponding to country-i-industry-k goods market clearing condition. For each component of $\mathbb{X}^{\mathrm{cl}}$ corresponding to country-i-wage for $i < N$ choose the component of $\mathbb{J}$ corresponding to country-i labor market clearing condition. If $\mathbb{X}^{\mathrm{cl}}$ contains a component for country-N wage, then choose the row of $\mathbb{J}$ corresponding to the wage normalization condition (4). Call the resulting Jacobian matrix $\mathbb{J}^{\mathrm{l}}$.

Go to Step 8.

STEP 8: Apply the Newton step to $\mathbb{X}^{\mathrm{cl}}$ using X^{sl} and $\mathbb{J}^{\mathrm{l}}$. This step produces a new box $\mathbb{X}^{\mathrm{nl}}$ such that either some components of $\mathbb{X}^{\mathrm{nl}}$ are empty or $\mathbb{X}^{\mathrm{nl}} \subseteq \mathbb{X}^{\mathrm{cl}}$. If some components of $\mathbb{X}^{\mathrm{nl}}$ are empty, then there is no solution in box $\mathbb{X}^{\mathrm{cl}}$. If this is the case then stop processing this box and go to Step 1. Otherwise, go to Step 9.

STEP 9: If box $\mathbb{X}^{\mathrm{nl}}$ is strictly inside of box $\mathbb{X}^{\mathrm{cl}}$, then for any point in box $\mathbb{X}^{\mathrm{cz}}$ there exists a unique solution within box $\mathbb{X}^{\mathrm{nl}}$. Record this fact: we will use box $\mathbb{X}^{\mathrm{c}}$ as a region of uniqueness when we find a solution inside of box $\mathbb{X}^{\mathrm{cl}}$ (that is, when we further reduce box $\mathbb{X}^{\mathrm{nl}}$ so that it is narrow enough). Go to Step 10.

STEP 10: If $\mathrm{diam}\left(\mathbb{X}^{\mathrm{nl}}\right) < \varepsilon$, the interval is narrow enough and we can record it as a solution. Replace box $\mathbb{X}^{\mathrm{c}}$ with $\left(\mathbb{X}^{\mathrm{nl}}, \mathbb{X}^{\mathrm{cz}}\right)$ and go to Step 14. If $\mathrm{diam}\left(\mathbb{X}^{\mathrm{nl}}\right) \geq \varepsilon$, go to Step 11.

STEP 11: If box $\mathbb{X}^{\mathrm{nl}}$ is a "significantly" reduced version of box $\mathbb{X}^{\mathrm{cl}}$, this means that the Newton step made good progress in reducing box $\mathbb{X}^{\mathrm{cl}}$. So, we should try to apply the Newton step again (and, actually, all the previous steps as well). Put $\left(\mathbb{X}^{\mathrm{nl}}, \mathbb{X}^{\mathrm{cz}}\right)$ in the list $\mathcal{L}$ and go to Step 1. If the progress was not "significant", replace box $\mathbb{X}^{\mathrm{c}}$ with $\left(\mathbb{X}^{\mathrm{nl}}, \mathbb{X}^{\mathrm{cz}}\right)$, and go to the bisection Step 12.

STEP 12: Let $i^* \equiv \mathrm{argmax}_i \,\mathrm{diam}\left(\mathbb{X}_i^{\mathrm{c}}\right)$. If $\mathrm{diam}\left(\mathbb{X}_{i^*}^{\mathrm{c}}\right) < \varepsilon$, the box is narrow enough. If that is the case, go to Step 14, otherwise go to Step 13.

STEP 13: Create two new boxes $\mathbb{X}^{'}$ and $\mathbb{X}^{''}$ by splitting box $\mathbb{X}^{\mathrm{c}}$ along dimension i^*. Boxes $\mathbb{X}^{'}$ and $\mathbb{X}^{''}$ are such that $\mathbb{X}_j^{'} = \mathbb{X}_j^{''} = \mathbb{X}_j^{\mathrm{c}}$ for all $j \neq i^*$, and $\mathbb{X}_{i^*}^{'} = [\underline{X}_{i^*}^{\mathrm{c}}, \mathrm{mid}\left(\mathbb{X}_{i^*}^{\mathrm{c}}\right)]$ and $\mathbb{X}_{i^*}^{''} = \left[\mathrm{mid}\left(\mathbb{X}_{i^*}^{\mathrm{c}}\right), \overline{X}_{i^*}^{\mathrm{c}}\right]$. Stop processing box $\mathbb{X}^{\mathrm{c}}$. Put boxes $\mathbb{X}^{'}$ and $\mathbb{X}^{''}$ into the list $\mathcal{L}$ and go to Step 1.

STEP 14: Put box $\mathbb{X}^{\mathrm{c}}$ into list $\mathcal{R}$. Get back to the representation of $\mathbb{X}^{\mathrm{c}}$ as $\left(\mathbb{X}^{\mathrm{cl}}, \mathbb{X}^{\mathrm{cz}}\right)$. If in Step 9 of some of the iterations we recorded the fact that there is a box $\widetilde{\mathbb{X}}^{\mathrm{cl}}$ such

that the Newton step resulted in a box $\widetilde{\mathbb{X}}^{\mathrm{nl}}$ that is strictly inside of $\widetilde{\mathbb{X}}^{\mathrm{cl}}$, and if $\mathbb{X}^{\mathrm{cl}} \subseteq \widetilde{\mathbb{X}}^{\mathrm{nl}}$, then $\widetilde{\mathbb{X}}^{\mathrm{cl}}$ is a region of uniqueness for $\mathbb{X}^{\mathrm{cl}}$ given any point from box $\mathbb{X}^{\mathrm{cz}}$. Put box $\left(\widetilde{\mathbb{X}}^{\mathrm{cl}}, \mathbb{X}^{\mathrm{cz}}\right)$ in the list $\mathcal{U}$. If there is no such box $\widetilde{\mathbb{X}}^{\mathrm{cl}}$, then put an empty box corresponding to $\mathbb{X}^{\mathrm{c}}$ in the list $\mathcal{U}$: uniqueness of solution inside of $\mathbb{X}^{\mathrm{c}}$ cannot be verified.

Go to Step 1.

We implemented the algorithm in Matlab. The source code is available online on our web sites. We use the INTLAB library for Matlab (Rump, 1999) for the implementation of interval arithmetic.

Several comments are in order. First, instead of normalizing one of the wages to 1, we use normalization (4). With this choice of normalization we can specify the base intervals for wages to be $[0, 1]$. If we were to normalize one of the wages to 1, then the base intervals for other wages would be $[0, \infty]$. While interval analysis methods can handle unbounded intervals, we find that in our case it is better to work with bounded intervals for unknowns to be able to apply the Newton method as often as possible.

Second, by Walras' Law we could drop the labor market clearing condition for one of the countries. However, we keep all conditions. It is typically better to keep as many conditions as possible when using interval analysis for solving a system of equations and/or inequalities. This allows discarding boxes that violate one of the conditions more often.

Third, the splitting of the base box $\mathbb{B}$ at Step 0 is a way to produce as many boxes as possible for which we can apply the Newton method. For small δ, we do not need to do any further bisection for components $[\underline{B}_i, \underline{B}_i + \delta]$, and we can apply the Newton method for components $[\underline{B}_i + \delta, \overline{B}_i]$ because they are bounded away from left endpoints of $\mathbb{B}$.

Uniqueness and Non-Uniqueness. Perhaps the most important detail of the algorithm that needs to be carefully explained is what we can conclude if for some box $\mathbb{X}$ from $\mathcal{R}$ the corresponding box $\mathbb{Y}$ from $\mathcal{U}$ is empty. There are several possibilities that are specific to our equilibrium system.

First, if $\mathbb{X}$ is the only element in $\mathcal{R}$, then we can conclude that $\mathbb{X}$ contains at least one equilibrium, because we know that at least one equilibrium exists. Unfortunately, we cannot formally conclude uniqueness. All we can say is that all equilibria are contained

within box $\mathbb{X}$. However, diameter of any box $\mathbb{X}$ from $\mathcal{R}$ is smaller than ε, which we set to $1\mathrm{e}{-10}$. In the context of our equilibrium system, the fact that all equilibria are within a box of diameter $1\mathrm{e}{-10}$ most likely means that the equilibrium is unique.

Second, it can happen that there are multiple boxes in $\mathcal{R}$, but all of them are contained within a box of a "small" diameter. This can happen if the algorithm cannot apply the Newton method around boxes in $\mathcal{R}$ and it needs to do bisection. Multiple boxes occur because of two factors. First, the solution region is approached by bisection from different sides. Second, when a box is near the solution region, but not necessarily includes the solution, this box can be quite narrow and the equilibrium conditions might still be non-violated due to overestimation of the range of values of functions that typically occurs when using interval arithmetic. If we continue bisecting such boxes with no solutions, we will eventually be able to discard them. However, this is very time consuming. In our simulations, we stop processing boxes when their diameter becomes smaller than $1\mathrm{e}{-10}$. Every time when our algorithm resulted in multiple boxes in $\mathcal{R}$, we found that the union of all of these boxes was contained in a box of diameter smaller than $1\mathrm{e}{-9}$. As in first case above, we conclude that this most likely means that the equilibrium is unique.

Third, it can happen that there are multiple boxes in $\mathcal{R}$, and some of them are at "large" distances from each other. In this case, we most likely have multiple solutions. However, we never encountered such situation in our simulations of economies with $\alpha_k \leq 1$ for all sectors.

Finally, it is worth mentioning that in the simulations our algorithm always produced a non-empty list $\mathcal{R}$.

References

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