Salvetti Complex Construction for Manifold Reflection Arrangements

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Abstract

Artin groups are a natural generalization of braid groups and are closely related to Coxeter groups in the following sense. There is a faithful representation of a Coxeter group W as a linear reflection group on a real vector space. The group acts properly and fixes a union of hyperplanes. The W-action extends as a covering space action to the complexified complement of these hyperplanes. The fundamental groups of the complement and that of the orbit space are respectively the pure Artin group and the Artin group. For finite Coxeter groups Deligne proved that the associated complement is an aspherical space. Later, using the Coxeter group data Salvetti gave a construction of a cell complex which is a W-equivariant deformation retract of the complement. This construction was independently generalized by Charney and Davis to the Artin groups of infinite type. These cell complexes are important because a lot of algebraic properties of Artin groups were discovered using their combinatorial and topological aspects.

In this thesis we consider Coxeter groups that appear as groups of diffeomorphisms. To be precise, given a smooth manifold, a reflection is an order-2 auto-diffeomorphism that locally behaves like usual Euclidean reflections. Under suitable topological conditions the discrete subgroup generated by finitely many such 'manifold reflections' is a Coxeter group. We use this group data to introduce the notion of reflection arrangements on manifolds. These arrangements are a collection of finitely many codimension-1 submanifolds such that locally they resemble a hyperplane arrangement and provide a stratification of the manifold with combinatorial properties analogous to those in the classical case. Since the action of such a group is smooth it extends naturally to the tangent bundle; this observation leads to the definition of a tangent bundle complement as a generalization of the complexified complement.

The main aim of this thesis is to demonstrate how the combinatorial data of the reflection group can be used to build a regular cell complex with the same homotopy type as that of the tangent bundle complement. We show that this homotopy equivalence respects the reflection group action hence we also obtain a combinatorial model for the quotient space. These results lead to a manifold theoretic generalization of Artin groups. The proof of our main theorem uses an equivariant version of the nerve lemma. We use the incidence relations among the strata of the manifold to construct an open cover of the tangent bundle complement that is invariant under the group action. We further show that this open cover satisfies the hypotheses of the nerve lemma. Our work is a generalization of the seminal result of Salvetti and to best of our knowledge is new.

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Contents

Introduction

Coxeter groups are a class of abstract groups that admit a specific form of finite presentation. Generated by order-2 elements, they most commonly occur as reflection groups. In fact, the initial definition by Coxeter [7] was as a generalization of 2-dimensional reflection groups, for instance the dihedral groups. Closely related to and corresponding to each Coxeter group, we have another finitely presented abstract group, called an Artin group or a generalized braid group.

Every Coxeter group does occur as a group acting via generating reflections on Euclidean space (Theorem 1.2), and we can understand this action extended to the 'complexified complement'. More specifically, if W is a Coxeter group acting on $\mathbb{R}^n$ as a reflection group, then it also acts on $\mathbb{C}^n$ via the same linear maps, and on a certain maximal open subset M_W, the action is free. The space M_W is connected, and the W-action produces a covering space quotient $N_W = M_W/W$. The fundamental group of N_W is the Artin group corresponding to W. Understanding the topological details of these spaces is an important tool in the study of Artin groups.

Related to the reflection group actions on $\mathbb{R}^n$ is the notion of hyperplane arrangements. Each generating reflection fixes a hyperplane, and in fact the union of these contain all the fixed points. Hence, it is exactly the complexifications of these hyperplanes that need to be removed from $\mathbb{C}^n$ to obtain a free action. Further, the hyperplanes and their intersections provide a stratification of $\mathbb{R}^n$, and the combinatorial relations in this stratification also capture the group W. We mostly restrict ourselves to hyperplane arrangements that arise from Coxeter groups, though the theory of general hyperplane arrangements is interesting in its own right.

Salvetti [25, 26] showed that the homotopy type of M_W and N_W can be recovered from the incidence relations of the stratification and from the combinatorial data in the group W. Specifically, he provided constructions of a cell complex along with a W-action, from either of these two sets of data, that is equivariantly homotopy equivalent to M_W.

A generalization of hyperplane arrangements to smooth manifolds was introduced by Deshpande in [16]. In this context, a Coxeter group is realized as a discrete group of diffeomorphisms, generated by (suitably defined) reflections. The reflections fix codimension-1 submanifolds, and these again stratify the smooth manifold. The notion of complexified complement is replaced by the tangent bundle complement (Definition 4.4), which has analogous properties with regard to the action.

Under certain topological conditions on the stratification of the manifold, we construct in this context a simplicial complex, from either the stratification data, or the group theoretic data.

The constructions follow that of the Salvetti complex given in [24]. We show that this complex is equivariantly homotopy equivalent to the tangent bundle complement.

Finally, starting with a central reflection arrangement on $\mathbb{R}^n$, we consider cases where the intersection with S^{n-1} satisfy our definition of a manifold arrangement. In this case, we compute the cohomology of the tangent bundle complement, using results by Arnol'd and Brieskorn [2, 5]. For the essential arrangements (where the only point fixed by the whole group is the origin), we only obtain the cohomology groups, in all but the top degree.

The main original result of this thesis is the homotopy equivalence of the Salvetti complex and tangent bundle complement for manifold reflection arrangements arising from Coxeter group actions (Theorem 4.3). The key ingredient of the proof is an equivariant form of the nerve lemma, which is particularly suited to identifying the homotopy types of derived complexes, like the Salvetti complex. We find an appropriate open cover of the tangent bundle complement, using the assumption that the stratification results in a regular cell complex, with strata diffeomorphic to simple convex polytopes.

Chapter-wise organization

Chapter 1. We start by defining Coxeter groups and discussing their action on Euclidean space via reflection groups. Section 1.1 discusses the basics of Coxeter and Vinberg systems, and the existence of a reflection representation. Section 1.2 lists some of the combinatorial properties of the Coxeter presentations. The induced hyperplane arrangement and the stratification of the Tits cone is covered in Section 1.3. We introduce the complexified complement, and note its connection with the Artin group via the fundamental group in Section 1.4.

Chapter 2. This chapter explores in more details the braid and Artin groups and their connections to the various topological spaces previously introduced. The example of the symmetric group and classical braid groups is explained in Section 2.1. Section 2.2 introduces the constructions of the Salvetti complex in the Euclidean case, and discusses the homotopy equivalence with the complexified complement. Section 2.3 is a brief discussion of the related $K(\pi, 1)$ conjecture, which says that the complexified complement is an Eilenberg-MacLane space for the pure Artin group.

Chapter 3. This chapter introduces groups acting on manifolds via reflections. Section 3.1 provides the setup and basic assumptions for defining reflections on smooth manifolds. In Section 3.2, we explain that discrete groups of diffeomorphisms generated by reflections must have Coxeter presentation, and introduce the *universal construction* of Vinberg, which allows us to recover the manifold from the action on the fundamental chamber. We explain an important application of this setup in Section 3.3 by sketching the construction of exotic aspherical manifolds discovered by Davis.

Chapter 4. This chapter contains the crux of our work. We prove an analogue of Salvetti's theorem on the homotopy type of the complexified complement. Section 4.1 sets up most of the basics and discusses the stratification of the manifold, and Section 4.2 defines the tangent bundle complement. Section 4.3 constructs the Salvetti complex in the manifold context, and contains the proof of the main theorem. An analogue of the group theoretic construction of the Salvetti complex is developed in Section 4.4, and we also show here the equivalence with the previous construction. However this construction, unlike the Euclidean case, is not completely independent of the action data. Most of the results in this chapter are part of [8], which is available on arXiv.

Chapter 5. In this chapter, we discuss linear actions of Coxeter groups on Euclidean space and the induced reflection arrangement of the unit sphere. The arrangements that arise are shown to satisfy the assumptions in [17] and we find a formula for the homotopy type of the tangent bundle complement, and compute its integral cohomology. For the arguments in the rest of the chapter, we require a different construction of the Salvetti complex, which we describe in Section 5.1. Section 5.3 deals with the case of the symmetric group acting by permuting coordinates, and Section 5.4 deals with the general case of essential actions.

Chapter 6. Finally, in this chapter, we consider three possible topics to be potentially developed. We discuss some vanishing results for cohomology, which we hope to apply to sphere arrangements from the previous chapter; a different generalization of the complexified complement which can involve the theory of microbundles; and a possible extension of the $K(\pi, 1)$ conjecture to the manifold context.

Chapter 1

Coxeter groups and the reflection representation

Coxeter groups, named after H.S.M. Coxeter, are a class of abstract groups possessing a particular type of finite presentation. The generators are of order 2, and typically arise as reflections. Identifying reflections which generate a certain Coxeter group provides the major tools to study these groups. We begin with a brief introduction to Coxeter groups; we state only those properties that are relevant to the core of this thesis. Most of the definitions and results in this chapter are adapted from Paris' survey of the $K(\pi, 1)$ conjecture [24]. However, Humphreys' book [21] is a good reference for the interested reader.

1.1 Basic definitions

Let S be a finite set. A *Coxeter matrix* over S is a square matrix $(m_{st})_{s,t \in S}$ such that, $m_{ss} = 1$ for each $s \in S$, and for $s \neq t$, $m_{st} = m_{ts}$ is either an integer ≥ 2 or ∞. A Coxeter matrix is usually represented by its *Coxeter graph*, $\Gamma = \Gamma(M)$. This is a labeled, undirected graph on the set of vertices S — two vertices s, t are joined by an edge if $m_{st} \geq 3$, and this edge is labeled with m_{st} if $m_{st} \geq 4$.

Corresponding to a Coxeter graph Γ, we have the following finitely presented group:

$$W_\Gamma = \left\langle S \;\middle|\; (st)^{m_{st}} \text{ for } m_{st} \neq \infty \right\rangle .$$

A group with such a presentation is called a *Coxeter group*. The pair (W_Γ, S), along with the presentation, is called a *Coxeter system*. The cardinality of S is called the *rank* of the Coxeter system.

Remark 1.1. Since $m_{ss} = 1$, $s^2 = 1$ for each $s \in S$. In fact each $s \in S$ has order 2 in W_Γ. More generally, as shown in [3], if $m_{st} = \infty$, then the element st has infinite order, and otherwise st has order exactly m_{st}. So given a Coxeter system (W, S), the presentation (and Γ) is uniquely determined.

Let $\Pi(a, b; m)$ denote the length m product $abab \cdots$ (the final term is a if m is odd, and b if m is even).

(a) The symmetric group – $\mathfrak{S}_{n+1}$ (b) The dihedral group – D_{2n} (c) The infinite dihedral group – D_∞

Figure 1.1: Some examples of Coxeter graphs

Remark 1.2. Since each $s \in S$ has order 2, the relation $(st)^{m_{st}} = 1$ in the presentation of W_Γ above could be equivalently replaced by

$$\Pi(s, t; m_{st}) = \Pi(t, s; m_{st}) .$$

Relations of this form are called *braid relations*. A presentation of a group W as a Coxeter group with relations in either of these forms is called a *Coxeter presentation* of W.

Let Γ be a Coxeter graph. Let $\Sigma = \{\sigma_s \mid s \in S\}$ be a set in bijection with S. Then we have another finitely presented group A_Γ:

$$A_\Gamma = \left\langle \Sigma \;\middle|\; \Pi(\sigma_s, \sigma_t; m_{st}) = \Pi(\sigma_t, \sigma_s; m_{st}) \text{ for } m_{st} \neq \infty \right\rangle .$$

The group A_Γ is the *Artin group* corresponding to Γ, and (A_Γ, Σ) is called an *Artin system*.

The map $\Sigma \to S, \sigma_s \mapsto s$ induces a (surjective) group map $\theta : A_\Gamma \to W_\Gamma$. The kernel of θ is called the *pure (or colored) Artin group*, and is denoted by CA_Γ. Thus we have, associated to the Coxeter graph Γ, an exact sequence of groups:

$$1 \to CA_\Gamma \to A_\Gamma \xrightarrow{\theta} W_\Gamma \to 1 . \tag{1.1}$$

Example 1.1. The symmetric group on $n+1$ symbols, denoted $\mathfrak{S}_{n+1}$, has the Coxeter group presentation

$$\left\langle s_1, \ldots, s_n \;\middle|\; s_i^2, (s_i s_{i+1})^3, (s_i s_j)^2 \text{ for } |i-j| \geq 2 \right\rangle ,$$

where s_i stands for the transposition $(i, i+1)$. The corresponding Coxeter graph is shown in Fig. 1.1a.

Example 1.2. The dihedral group of order $2n$, denoted D_{2n}, has the Coxeter group presentation $\langle r, s | r^2, s^2, (rs)^n \rangle$. The corresponding Coxeter graph is shown in Fig. 1.1b.

Example 1.3. The Coxeter group with presentation $\langle r, s | r^2, s^2 \rangle$ is called the infinite dihedral group, and is isomorphic to $\mathbb{Z}/2 * \mathbb{Z}/2$. The corresponding Coxeter graph is shown in Fig. 1.1c.

Definition 1.1. A Coxeter system or the corresponding Coxeter graph Γ is said to be of *spherical type* if the group W_Γ is finite.

For instance, among the three examples above, the first two are of spherical type, while the third is not.

Remark 1.3. The spherical Coxeter groups can be classified into three infinite families — $\mathbf{A}_n, \mathbf{B}_n, \mathbf{D}_n$ for $n \geq 1$, and seven others — $\mathbf{E}_{6,7,8}$, $\mathbf{F}_4$, $\mathbf{H}_{3,4}$ and $\mathbf{I}_2$, where the subscript is the rank. In particular, the Coxeter group of type $\mathbf{A}_n$ is the symmetric group $\mathfrak{S}_{n+1}$, as described above. See [21, Chapter 4] for details.

Remark 1.4. Although Artin groups are always infinite, an Artin group A_Γ is said to be of *finite type* if Γ is of spherical type. Otherwise it is of *infinite type*.

Let I be a open convex cone in a finite dimensional real vector space V (possibly the whole of V). A *hyperplane arrangement* in I is a family $\mathcal{A}$ of hyperplanes of V that intersect I, and are locally finite at points of I. That is, each point $x \in I$ has a neighborhood that intersects only finitely many hyperplanes H from $\mathcal{A}$.

Let V be a finite dimensional real vector space. Then a *reflection* on V is defined to be an order-2 linear map that fixes a hyperplane H. The reflection need not be orthogonal to H, and so is not determined by H. Suppose $\overline{C}$ is a closed convex cone in V with non-empty interior C. A *wall* of $\overline{C}$ is a hyperplane generated by a facet of $\overline{C}$. Let $H_1, \ldots, H_n$ be the walls of $\overline{C}$ and let r_i be a reflection on V fixing H_i. Let W be the subgroup of $\mathrm{GL}(V)$ generated by $S = \{s_1, \ldots, s_n\}$. Then (W, S) is called a *Vinberg system* if $wC \cap C = \varnothing$ for each $w \in W$. In that case, S is called the *canonical generating system* of W and $\overline{C}$ is called the *fundamental chamber* of (W, S).

Theorem 1.1 (Vinberg, [31])**.** *Let (W, S) be a Vinberg system with fundamental chamber $\overline{C}$. Set*

$$\overline{I} = \bigcup_{w \in W} w\overline{C}.$$

Then the following statements hold:

i. *(W, S) is a Coxeter system.*
ii. *$\overline{I}$ is a convex cone with non-empty interior.*
iii. *The interior I of $\overline{I}$ is stable under the action of W and W acts properly discontinuously on I.*
iv. *Let $x \in I$ be such that $W_x = \{w \in W \mid wx = x\}$ contains an element other than 1. Then $rx = x$ for some reflection $r \in W$.*

The cone I is called the *Tits cone* of the Vinberg system (W, S). Denote by $\mathcal{R}$ the set of reflections in W. For $r \in \mathcal{R}$, let H_r be the hyperplane fixed by r. Set $\mathcal{A} = \{H_r \mid r \in \mathcal{R}\}$. Then $\mathcal{A}$ is a hyperplane arrangement in I by Theorem 1.1. This is called the *Coxeter arrangement* of (W, S).

Remark 1.5. The Vinberg system (W, S) comes with a specific representation of the Coxeter system (W, S). Note that in general, such a representation is not unique.

Remark 1.6. If Γ is of spherical type, the cone I is forced to be the whole of V.

Conversely, we describe how to start with a Coxeter system and obtain a representation as a Vinberg system. Let (W, S) be a Coxeter system. Let V be the real vector space with basis $\{e_s\}$ indexed by $s \in S$. Define the symmetric bilinear form $B : V \times V \to \mathbb{R}$ as follows:

$$B(e_s, e_t) = \begin{cases} -\cos\left(\dfrac{\pi}{m_{st}}\right), & \text{if } m_{st} \neq \infty, \\ -1, & \text{otherwise.} \end{cases}$$

For $s \in S$, define $\rho_s \in \mathrm{GL}(V)$ by $\rho_s(x) = x - 2B(x, e_s)e_s$. Then ρ_s is a reflection on V, and the mapping $s \mapsto \rho_s$ induces a representation $W \to \mathrm{GL}(V)$ (see [3]). This is called the *canonical representation* of (W, S).

We also have the *dual representation* $\rho^* : W \to \mathrm{GL}(V^*)$ given by $\rho^*(w) = \rho(w^{-1})^t$. For $s \in S$, define the hyperplane $H_s = \{\alpha \in V^* \mid \alpha(e_s) = 0\}$ in V^*. Let $\overline{C_0} = \{\alpha \in V^* \mid \alpha(e_s) \geq 0\}$.

Theorem 1.2 (Tits [30], Bourbaki [3]). *With notation as above:*

i. The canonical representation ρ and the dual representation ρ^* are faithful.

ii. The set $\overline{C_0}$ is a simplicial cone with walls H_s for $s \in S$. The transformation $\rho^*(s)$ is a reflection on V, which fixes H_s. For all $w \in W$, $\rho^*(w)C_0 \cap C_0 = \varnothing$, where C_0 is the non-empty interior of $\overline{C_0}$.

In particular, $(\rho^*(W), \rho^*(S))$ is a Vinberg system whose associated Coxeter system is (W, S).

1.2 Combinatorics of words and subgroups

Let Γ be a Coxeter graph and (W, S) its Coxeter system. Let S^* be the free monoid on S, and $q : S^* \to W$ the canonical homomorphism mapping S identically. The map q is surjective since each generator $s \in S$ has order 2 and hence is its own inverse. A word $\mu \in S^*$ is called an *expression* of $w \in W$ if $q(\mu) = w$. The length of w, denoted $\lg(w)$ is the minimum length over all expressions of w.

An expression $\mu = s_1 \cdots s_l$ of w is called *reduced* if $l = \lg(w)$. We say two words $\mu, \mu' \in S^*$ are related by an *elementary M-transformation* if for some $\nu_1, \nu_2 \in S^*$, $s, t \in S$ with $m = m_{st} \neq \infty$,

$$\mu = \nu_1 \Pi(s,t;m) \nu_2; \quad \mu' = \nu_1 \Pi(t,s;m) \nu_2 .$$

Theorem 1.3 (Tits [29]). *Let $w \in W$, and let μ, μ' be two reduced expressions of w. Then μ and μ' are related by a finite sequence of elementary M-transformations.*

Let (A, Σ) be the Artin system of Γ. By definition, the map $q : S^* \to W$ factors as the composition $S^* \xrightarrow{\tilde{q}} A \xrightarrow{\theta} W$ where $\tilde{q} : S^* \to A$ is induced by $s \mapsto \sigma_s$. By Theorem 1.3, two reduced expressions of w have the same image under $\tilde{q}$. Using this, we can define a set theoretic section $\tau : W \to A$ of θ by setting $\tau(w) = \tilde{q}(\mu)$, where μ is any reduced expression of w.

Theorem 1.4 (Bourbaki [3]). *Let (W, S) be a Coxeter system.*

i. Let $w \in W$, $s \in S$, and $\mu = s_1 \cdots s_l$ be a reduced expression of w, so that $\lg(w) = l$. Then either $\lg(ws) = \lg(w) + 1$ and hence μs is a reduced expression of ws, or for some $1 \leq i \leq l$, $w = s_1 \cdots \hat{s}_i \cdots s_l s$.

ii. Let $w \in W$ and $s, t \in S$. If $\lg(ws) = \lg(tw) = \lg(w) + 1$ and $\lg(tws) < \lg(ws)$, then $ws = tw$.

For $X \subset S$, set $M_X = (m_{st})_{s,t \in X}$, which is a Coxeter matrix, and $\Gamma_X = \Gamma(M_X)$ the associated Coxeter graph. If W_X is the subgroup of W generated by X, then (W_X, X) is the Coxeter system of Γ_X. W_X is called the *standard parabolic subgroup* of W generated by X. The cosets wW_X, for $w \in W$ are called *parabolic subgroups* of W. We say an element $w \in W$ is (X, Y)-*minimal* if it is of minimal length in the double coset $W_X w W_Y$.

Proposition 1.5. *Let (W, S) be a Coxeter system.*

i. *Let $X, Y \subset S$. Then each double coset $W_X w W_Y$ has a unique (X, Y)-minimal element.*

ii. *Let $X \subset S$ and $w \in W$. Then the following are equivalent:*

   a) *w is $(\varnothing, X)$-minimal;*
   b) *for each $s \in X$, $\lg(ws) > \lg(w)$;*
   c) *for each $u \in W_X$, $\lg(wu) = \lg(w) + \lg(u)$.*

iii. *Let $X \subset S$ and $w \in W$. Then the following are equivalent:*

   a) *w is $(X, \varnothing)$-minimal;*
   b) *for each $s \in X$, $\lg(sw) > \lg(w)$;*
   c) *for each $u \in W_X$, $\lg(uw) = \lg(u) + \lg(w)$.*

Let $X \subset S$ and $\mu = s_1 \cdots s_l$ be a reduced expression of $w \in W_X$. Then $s_1, \ldots, s_l \in X$.

1.3 Stratification induced by hyperplane arrangements

Consider a Vinberg system (W, S), with closed fundamental chamber C, and induced hyperplane arrangement $\mathcal{A}$. The intersections of hyperplanes in $\mathcal{A}$ produce closed subspaces of smaller dimension, which are (point-wise) fixed by more elements of W. For instance, for $s, t \in S$, $H_s \cap H_t$ is fixed by the standard parabolic subgroup generated by $\{s, t\}$. In fact, the elements of W that fix $H_s \cap H_t$ (point-wise *or* globally, see the theorem below) are exactly those belonging to this subgroup. Similar observations hold for more than two generators.

Restricting these subspaces to the cone C, we get faces of C of various dimensions (the meaning of a face of a polytopal cone is intuitively clear). So we expect a labeling of faces of C by standard parabolic subgroups of W. Further, this labeling should be extensible to the faces of wC by applying w to the labels, thereby obtaining the parabolic subgroups. This is exactly captured in the theorem below, due to Vinberg.

Let I be a non-empty open convex cone in a finite dimensional real vector space V. Let $\mathcal{A}$ be a hyperplane arrangement in I. A *chamber* of $\mathcal{A}$ is a connected component of $I \setminus (\bigcup \mathcal{A})$. Let $\mathcal{C}(\mathcal{A})$ denote the set of chambers of $\mathcal{A}$. For $H \in \mathcal{A}$, set $I^H = I \cap H$, and

$$\mathcal{A}^H = \left\{ H' \in \mathcal{A} \setminus \{H\} \;\middle|\; H' \cap I^H \neq \varnothing \right\}.$$

Then I^H is a non-empty open convex cone in H, and $\mathcal{A}^H$ is a hyperplane arrangement in I^H.

Definition 1.2. A chamber of $\mathcal{A}$ is a *codimension*-0 *face* of $\mathcal{A}$. Inductively for $d > 0$, a codimension-d face of $\mathcal{A}$ is a codimension-$(d-1)$ face of $\mathcal{A}^H$ for some $H \in \mathcal{A}$.

In particular, a *codimension*-1 *face* of $\mathcal{A}$ is a chamber of some $\mathcal{A}^H$. Denote by $\mathcal{F}(\mathcal{A})$ the collection of all faces of $\mathcal{A}$. Note that I is partitioned by $\mathcal{F}(\mathcal{A})$. For $F \in \mathcal{F}(\mathcal{A})$, denote by $\overline{F}$ the closure of F in I. Then we equip $\mathcal{F}(\mathcal{A})$ with the partial order $\leq$ defined by $F \leq G$ if $F \subseteq \overline{G}$.

For $F \in \mathcal{F}(\mathcal{A})$, the *support* of F, denoted by $|F|$, is the subspace of V generated by F. Set $\mathcal{A}_F = \{H \in \mathcal{A} \mid F \subset H\}$. Then for $H \in \mathcal{A}$, $H \cap |F|$ is a hyperplane in $|F|$ iff $H \notin \mathcal{A}_F$. Now, let $I^F = I \cap |F|$, and

$$\mathcal{A}^F = \left\{ H \cap |F| \;\middle|\; H \in \mathcal{A} \setminus \mathcal{A}_F \right\}.$$

Then I^F is an open convex cone in $|F|$, $\mathcal{A}^F$ is a hyperplane arrangement in I^F and F is a chamber of $\mathcal{A}^F$. On the other hand, $\mathcal{A}_F$ is a finite hyperplane arrangement in I and $\bigcap \mathcal{A}_F = |F|$, where the empty intersection (when F is a chamber) is understood to be V.

Given a Coxeter graph Γ and its Coxeter system (W,S), set $\mathcal{S}^f_\Gamma = \{X \subseteq S \mid W_X \text{ is finite}\}$, and $\mathcal{P}^f_\Gamma = \left\{ wW_X \;\middle|\; X \in \mathcal{S}^f, w \in W \right\}$. The collection $\mathcal{P}^f_\Gamma$ is ordered by inclusion.

Theorem 1.6 (Vinberg [31])**.** *Let (W,S) be a Vinberg system, let C_0 be its fundamental chamber, and $\mathcal{A}$ its Coxeter arrangement. Let Γ be the Coxeter graph of (W,S) viewed as a Coxeter system.*

i. Let $\mathcal{F}(C_0) = \left\{ F \in \mathcal{F}(\mathcal{A}) \;\middle|\; F \leq C_0 \right\}$. Then there is a bijection $\iota : \mathcal{S}^f_\Gamma \to \mathcal{F}(C_0)$ such that

$$\bigcap_{s \in X} H_s = \left|\iota(X)\right|$$

for all $X \in \mathcal{S}^f_\Gamma$. Moreover,

a) for $X, Y \in \mathcal{S}^f_\Gamma$, $X \subset Y$ iff $\iota(Y) \leq \iota(X)$;

b) for $X \in \mathcal{S}^f_\Gamma$, the stabilizer $\left\{ w \in W \;\middle|\; w\iota(X) = \iota(X) \right\}$ of $\iota(X)$ is equal to W_X, and elements of W_X fix $\iota(X)$ pointwise.

ii. There is a bijection $\tilde{\iota} : \mathcal{P}^f_\Gamma \to \mathcal{F}(\mathcal{A})$ defined by $\tilde{\iota}(wWx) = w(\iota(X))$. Moreover, for $X, Y \in \mathcal{S}^f_\Gamma$ and $u, v \in W$, $uW_X \subset vW_Y$ iff $\tilde{\iota}(vW_Y) \leq \tilde{\iota}(uW_X)$ iff $X \subset Y$ and $u \in vW_Y$.

iii. Let $X \subset S$ and $w \in W$. Then w is $(\varnothing, X)$-minimal iff for every reflection $r \in W_X$, H_r does not separate C_0 and $w^{-1}C_0$.

In particular, restricting $\tilde{\iota}$ to $W = \{wW_X \mid X = \varnothing\}$, we have a bijection $W \to \mathcal{C}(\mathcal{A})$.

1.4 The complexified complement

Let $\mathcal{A}$ be a hyperplane arrangement in an open convex cone I in an Euclidean space V. For a hyperplane $H \in \mathcal{A}$, $H \otimes \mathbb{C}$ is a (complex) hyperplane in $V \otimes \mathbb{C}$, and hence has real codimension 2. Identifying $V \otimes \mathbb{C}$ as $V \times V$, we get induced identifications $H \otimes \mathbb{C} = H \times H$. Define $M(\mathcal{A})$ as the complement of all the complexified hyperplanes:

$$M(\mathcal{A}) = (I \times I) \setminus \bigcup_{H \in \mathcal{A}} (H \times H)\,.$$

This is a connected manifold of dimension $2 \cdot \dim V$. Note that if $I = V$, then

$$M(\mathcal{A}) = (V \otimes \mathbb{C}) \setminus \bigcup_{H \in \mathcal{A}} (H \otimes \mathbb{C})\,.$$

If (W,S) is a Vinberg system, and $\mathcal{A}$ is its Coxeter arrangement, then set $M(W,S) = M(\mathcal{A})$. By Theorem 1.1, W acts freely and properly discontinuously on $M(W,S)$. Let

$$N(W,S) = M(W,S)/W\,.$$

The quotient map $M(W,S) \to N(W,S)$ is a regular cover.

The following theorem identifies the induced exact sequence of fundamental groups with the canonical exact sequence (1.1) of the Coxeter system (W,S). The special case of spherical type Coxeter systems was proved by Brieskorn [5], and the general case is due to Van der Lek:

Theorem 1.7 (Van der Lek, [23]). *Let (W, S) be a Vinberg system, and Γ the associated Coxeter graph of (W, S) viewed as a Coxeter system. Then $\pi_1(N(W, S))$ is isomorphic to A_Γ, $\pi_1(M(W, S))$ is isomorphic to CA_Γ, and the short exact sequence* (1.1) *is isomorphic to the one associated with the regular covering map $M(W, S) \to N(W, S)$:*

$$1 \to \pi_1(M(W, S)) \to \pi_1(N(W, S)) \to W \to 1 .$$

Chapter 2

Topological aspects of Artin groups

The classical braid group was introduced by Emil Artin as a tool to study knots and links. They abstractly capture the composition of physical braids on finitely many strings, and have applications in several mathematical subjects. Artin groups are a natural generalization of braid groups, just as Coxeter groups are a generalization of the symmetric groups. For different perspectives and applications of braids, see the book by Hansen [20].

2.1 Braid groups

Recall that associated to a Coxeter graph Γ, we have an exact sequence (1.1)

$$1 \to CA_\Gamma \hookrightarrow A_\Gamma \twoheadrightarrow W_\Gamma \to 1$$

of groups. An important and classical example of W is the symmetric group $\mathfrak{S}_n$ on n elements, which has the following presentation:

$$\mathfrak{S}_n = \left\langle s_1, \ldots, s_{n-1} \;\middle|\; s_i^2, s_i s_j = s_j s_i \text{ for } |i-j| \geq 2, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \right\rangle,$$

where s_i denotes the transposition $(i, i+1)$. Evidently, the Coxeter matrix is on the set $S = \{1, \ldots, n-1\}$, with

$$m_{ij} = \begin{cases} 1 & \text{if } i = j\,, \\ 3 & \text{if } |i-j| = 1\,, \\ 2 & \text{otherwise}\,. \end{cases}$$

Classically, the corresponding Artin group is known as the *braid group* on n strands, and is denoted B_n. The pure (or colored) Artin group is known as the *colored braid group* on n strands, and is denoted by PB_n. The canonical presentation of B_n is given by:

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| \geq 2, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \right\rangle.$$

Intuitively, the braid group can be thought of as all arrangements of n strings with fixed endpoints, that are always transverse to some fixed plane. The composition is given by attaching two such arrangements end to end. More formally, we can consider an embedding i of $\{1, \ldots, n\}$

into the open 2-disk D and take all isotopies between i and itself, modulo the equivalence given by isotopies of the pair $(I \times D, \{0,1\} \times D)$. Then the composition is given by the usual concatenation of homotopies.

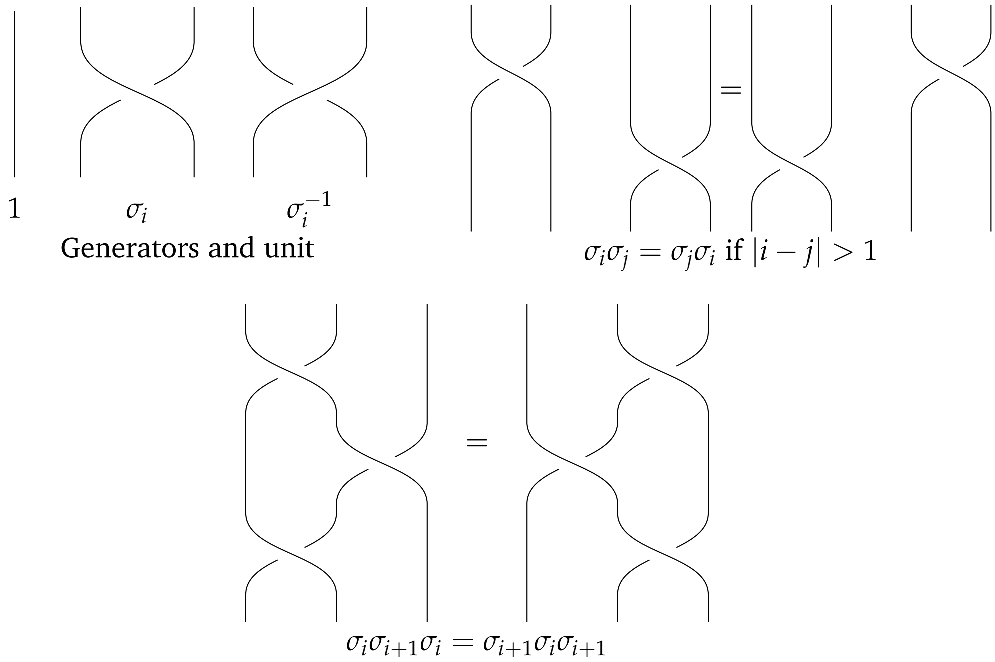

Figure 2.1: Braids and their relations

The standard representation of $\mathfrak{S}_n$ in $\mathrm{GL}_n(\mathbb{R})$ by permuting coordinates gives rise to a Vinberg system, with fundamental chamber

$$C_0 = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; x_1 \geq x_2 \geq \cdots \geq x_n \right\}.$$

The generating reflections are the transpositions $(i, i+1) \in \mathfrak{S}_n$, fixing the hyperplane given by $x_i = x_{i+1}$, which is a wall of C_0.

For this Vinberg system, all the reflections are given by the transpositions $(i, j) \in \mathfrak{S}_n$, which fix the hyperplane given by $x_i = x_j$. So we have by definition,

$$M(\mathfrak{S}_n) = M(\mathfrak{S}_n, \{s_1, \ldots, s_{n-1}\}) = \mathbb{C}^n \setminus \Delta,$$

where $\Delta = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; z_i = z_j \text{ for some } i \neq j \right\}$ is the super-diagonal. The quotient space $N = M(\mathfrak{S}_n)/\mathfrak{S}_n$ is the *braid space* on n strings, and has $\pi_1(N) = B_n$. In fact this is one of the common alternate definitions of B_n.

2.2 Salvetti complex

Let $\mathcal{A}$ be a hyperplane arrangement in a non-empty convex cone I in V. Recall that for $F \in \mathcal{F}(\mathcal{A})$, we define $\mathcal{A}_F = \{H \in \mathcal{A} \mid F \subset H\}$, which is another arrangement in I. Now for $F \in \mathcal{F}(\mathcal{A})$ and $C \in \mathcal{C}(\mathcal{A})$, let C_F be the chamber of $\mathcal{A}_F$ containing C.

Let

$$\mathrm{Sal}_0(\mathcal{A}) = \{(F, C) \in \mathcal{F}(\mathcal{A}) \times \mathcal{C}(\mathcal{A}) \mid F \leq C\}.$$

Define the relation $\preceq$ on $\mathrm{Sal}_0(\mathcal{A})$, easily checked to be a partial order, by

$$(F, C) \preceq (G, D) \text{ if } F \leq G \text{ and } C_F \subseteq D_G.$$

Definition 2.1. The *Salvetti complex* of $\mathcal{A}$, denoted $\mathrm{Sal}(\mathcal{A})$, is defined to be the geometric realization of the derived complex of $\mathrm{Sal}_0(\mathcal{A})$.

Remark 2.1. Recall that the derived complex (or the order complex) of a poset is an abstract simplicial complex whose n-simplices correspond to chains of length n in the poset.

If $\mathcal{A}$ is the Coxeter arrangement of a Vinberg system (W, S), then there is an obvious action of W on $\mathrm{Sal}_0(\mathcal{A})$, and this extends to an action on $\mathrm{Sal}(\mathcal{A})$. The following theorem was proved by Salvetti [26] for finite Coxeter groups and independently by Charney and Davis [6] for arbitrary Coxeter groups.

Theorem 2.1. *There is a homotopy equivalence $f : \mathrm{Sal}(\mathcal{A}) \to M(\mathcal{A})$. If $\mathcal{A}$ is the Coxeter arrangement of a Vinberg system (W, S), then f can be chosen to be W-equivariant, and induces a homotopy equivalence $\bar{f} : \mathrm{Sal}(\mathcal{A})/W \to M(\mathcal{A})/W = N(W, S)$.*

Starting with a Coxeter system (W, S) and the associated Coxeter graph Γ, we have an alternate definition of the Salvetti complex of Γ. Recall that $\mathcal{S}^f$ is the collection of subsets of S that generate finite subgroups of W. On $W \times \mathcal{S}^f$, consider the relation $\preceq$ as follows:

$$(u, X) \preceq (v, Y) \text{ if } X \subset Y, v^{-1}u \in W_Y, \text{ and } v^{-1}u \text{ is } (\varnothing, X)\text{-minimal}.$$

Lemma 2.2. *The relation $\preceq$ is a partial order on $W \times \mathcal{S}^f$.*

Proof. [24, Lemma 3.2]. □

Now again, define the *Salvetti complex* of Γ, denoted $\mathrm{Sal}(\Gamma)$, to be the geometric realization of the derived complex of $(W \times \mathcal{S}^f, \preceq)$. The group W acts on $W \times \mathcal{S}^f$ on the first coordinate, and this action respects the order $\preceq$. Hence again, we get an action of W on $\mathrm{Sal}(\Gamma)$. Identifying a Vinberg system as a Coxeter system, we have that the two constructions are equivalent:

Theorem 2.3. *Suppose $\mathcal{A}$ is the Coxeter arrangement of a Vinberg system (W, S), and let Γ be the associated Coxeter graph. Then there is an order preserving bijection between $\mathrm{Sal}_0(\mathcal{A})$ and $W \times \mathcal{S}^f$. This induces a W-equivariant homeomorphism $\mathrm{Sal}(\mathcal{A}) \to \mathrm{Sal}(\Gamma)$, which descends to a homeomorphism $\mathrm{Sal}(\mathcal{A})/W \to \mathrm{Sal}(\Gamma)/W$ on the quotients.*

This in particular shows that $\mathrm{Sal}(\mathcal{A})$, and hence the homotopy type of $M(W, S)$ and $N(W, S)$ does not depend on the representation of the Coxeter system (W, S) as a Vinberg system.

Example 2.1. Let $W = \mathbb{Z}/2 = \langle s | s^2 \rangle$ act on $\mathbb{R}$ by $x \mapsto -x$. Then $\mathcal{F}(\mathcal{A}) = \{\mathbb{R}_+, \mathbb{R}_-, \{0\}\}$, with obvious inclusion relations. The Salvetti complex and its quotient are shown in Fig. 2.2.

Example 2.2. Let $W = D_\infty = \langle s, t | s^2, t^2 \rangle$ act on $\mathbb{R}$ via $s \cdot x = -x$, $t \cdot x = 2 - x$. Then $\mathcal{C}(\mathcal{A}) = \{(n, n+1) \mid n \in \mathbb{Z}\}$ and $\mathcal{F}(\mathcal{A}) = \mathbb{Z} \cup \mathcal{C}(\mathcal{A})$, with obvious inclusions. We can also identify $\mathrm{Sal}(\mathcal{A}) \simeq \bigvee_{\mathbb{Z}} S^1$ and $\mathrm{Sal}(\mathcal{A})/W = S^1 \vee S^1$.

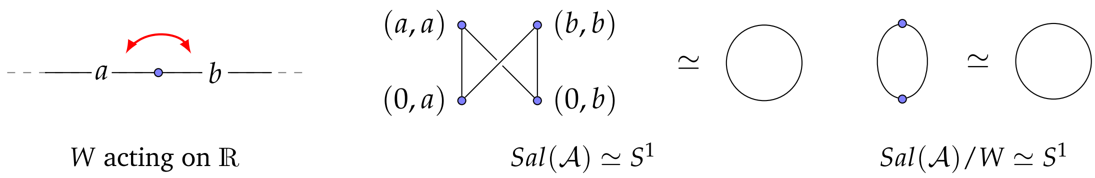

Figure 2.2: $W = \mathbb{Z}/2\mathbb{Z}$ acting on $\mathbb{R}$, and the corresponding Salvetti complex

2.3 The $K(\pi, 1)$ conjecture

Recall that a connected topological space X is the *Eilenberg-MacLane space* for a discrete group G if $\pi_1(X) = G$, and $\pi_n(X) = 0$ for $n \geq 2$. In this case we also say X is a $K(\pi, 1)$ space, or that X is aspherical. Eilenberg-MacLane spaces are important in understanding the group G. By Theorem 1.7, $\pi_1(N(W, S)) = A_\Gamma$, for any Vinberg system (W, S) with Coxeter graph Γ.

Conjecture 2.4 ($K(\pi, 1)$ conjecture). *Let (W, S) be a Vinberg system, and let Γ be the associated Coxeter graph. Then $N(W, S)$ is an Eilenberg-MacLane space for A_Γ.*

Since a cover of a space X is aspherical iff X is aspherical, this is equivalent to saying that $M(W, S)$ is an Eilenberg-MacLane space for CA_Γ. The most important result towards this conjecture is Deligne's 1972 proof [14] for all Artin groups of spherical type (Definition 1.1), which was conjectured and proven in some special cases by Brieskorn [5].

Theorem 2.5 (Deligne [14]). *If Γ is a Coxeter graph of spherical type, then* $\mathrm{Sal}(\Gamma)$ *is an Eilenberg-MacLane space.*

We will prove a further special case, when $W = \mathfrak{S}_n$ and $A = B_n$. The proof is due to Fadell and Neuwirth [18].

Theorem 2.6. *The space $M(\mathfrak{S}_n)$ is an Eilenberg-MacLane space for B_n.*

Proof. The proof is by induction on n. First, we identify $M(\mathfrak{S}_n)$ with $\mathbb{C}^n \setminus \Delta_n$ as before, where Δ_n is the super-diagonal of $\mathbb{C}^n$.

For $n = 2$, $M(\mathfrak{S}_2) = \left\{(z, w) \in \mathbb{C}^2 \;\middle|\; z \neq w\right\} \subset \mathbb{C}^2$. Note that the map $\mathbb{C} \times \mathbb{C}^* \to M(\mathfrak{S}_2)$ given by $(z, w) \mapsto (z, z + w)$ is a homeomorphism. Hence $M(\mathfrak{S}_2) \simeq S^1$, which is aspherical since its universal cover is $\mathbb{R}$, which is contractible.

For $n > 2$, consider the projection $p : \mathbb{C}^n \to \mathbb{C}^{n-1}$ to the first $n - 1$ coordinates. Restricted to $M(\mathfrak{S}_n)$, p maps onto $M(\mathfrak{S}_{n-1})$ and is a fiber bundle, and the fiber over $(\xi_1, \dots, \xi_{n-1})$ can be identified with $\mathbb{C} \setminus \{\xi_1, \dots, \xi_{n-1}\}$. Since for a point $(\xi_1, \dots, \xi_{n-1}) \in M(\mathfrak{S}_{n-1}) = \mathbb{C}^{n-1} \setminus \Delta_{n-1}$, the coordinates are all distinct (by definition), the fiber is $F = \mathbb{C} - \{(n-1) \text{ points}\}$, which has the same homotopy type as $\bigvee_{n-1} S^1$. This again has a contractible universal cover and hence is aspherical.

Now, from the fiber bundle $F \hookrightarrow M(\mathfrak{S}_{n-1}) \twoheadrightarrow M(\mathfrak{S}_n)$, we have the long exact sequence in homotopy groups:

$$\cdots \to \pi_k(F) \to \pi_k(M(\mathfrak{S}_n)) \to \pi_k(M(\mathfrak{S}_{n-1})) \to \pi_{k-1}(F) \to \dots$$

Since both F and $M(\mathfrak{S}_{n-1})$ (by induction) are aspherical, $\pi_k(M(\mathfrak{S}_n)) = 0$ for $k \geq 2$. □

Chapter 3

Coxeter groups acting on manifolds

3.1 Basic definitions

In this section we present a brief review of manifold reflection groups. These are subgroups of the diffeomorphisms group and are generated by finitely many reflections. It is interesting to note that the combinatorial and geometric properties of reflection groups are independent of the particular nature of the group action. For example, under reasonable (topological) assumptions, these manifold reflection groups have Coxeter presentation (see Theorem 3.2). Also the fixed point set behaves very much like Euclidean reflection arrangements. Moreover the combinatorics of the induced stratification is independent of the topology of the manifold. The main references for this section are Gutkin [19], Davis [9, Chapter 10] and Alekseevsky et al [1] (they consider subgroups of isometries of a complete Riemannian manifold). See also Straume [27] for similar results for groups generated by continuous reflections.

We begin by recalling some basics of group actions on manifolds, the main reference being Bredon's book [4]. Let X be a smooth, connected manifold without boundary of dimension l and let G be a discrete group of diffeomorphisms. We assume that the G-action on X is smooth and proper (i.e., properly discontinuous). In case the manifold X is open and G is infinite then we prefer that G acts cocompactly.

As the group action is proper, for every $x \in X$ the associated isotropy subgroup G_x is finite. There exists an open neighborhood U_x of x such that $gU_x \cap U_x = \varnothing$ for $g \in G \setminus G_x$. The orbit space X/G is Hausdorff. Each orbit $G(x)$ is closed and discrete and by the orbit-stabilizer theorem is in bijection with G/G_x.

Let H be a subgroup of G and let A be a left H-space. Then H acts on $G \times A$ via $h \cdot (g, a) = (gh^{-1}, ha)$. The *twisted product* of G and A, denoted by $G \times_H A$, is the orbit space of this H action on $G \times A$. We write $[g, x]$ for the equivalence class of (g, x). A G-*tube* about an orbit P is a G-equivariant embedding $\phi\colon G \times_H A \to X$ onto an open neighborhood of P in X. If A is homeomorphic to a disc, then the tube is an *equivariant tubular neighborhood* of P.

Definition 3.1. Let $x \in S \subset X$ be such that $G_x(S) = S$. Then S is called a *slice* at x, if the map

$$G \times_{G_x} S \to X$$

taking $[g, s]$ to gs is a G-tube about $G(x)$. A slice S is *linear* if it is G_x-equivariantly homeomorphic to a neighborhood of the origin in some linear representation of G_x on $\mathbb{R}^l$.

A group action on a smooth manifold is *locally smooth* (or locally linear) if every point has a linear slice. If a discrete group acts properly as well as smoothly on a manifold X then the action is locally smooth. Since each isotropy subgroup G_x is finite there is a G_x-invariant Riemannian metric on X (see [4, Chapter VI]). As the action is smooth the exponential map exp, defined (in terms of the invariant metric) on a neighborhood of the origin in $T_x(X)$, is G_x-equivariant. The exp map takes a small open disc about the origin homeomorphically onto a neighborhood of x. If the disc is small enough, then the neighborhood is a linear slice.

Another consequence of the locally smooth action is that the fixed point set of a finite subgroup of G is a *locally flat* submanifold of X embedded as a closed subset. Recall that Y is a locally flat k-submanifold of X if the pair (X, Y) is locally homeomorphic to $(\mathbb{R}^l, \mathbb{R}^k)$ around points of Y.

Definition 3.2. An involutive diffeomorphism r on X is a *reflection* if the r-action is locally smooth.

In case of $X = \mathbb{R}^l$ one can take a linear reflection to be as a locally smooth involution with the fixed subset a hyperplane that separates the ambient space. However, in general the fixed subset X_r could have components of different dimensions and need not separate the manifold. We say that a reflection r is *dissecting* if the fixed set X_r separates X. If the reflection is dissecting then X_r is a codimension-1 submanifold (not necessarily connected) and its complement has 2 connected components, which are interchanged by r [9, Lemma 10.1.3]. From now on we only consider dissecting reflections, unless otherwise stated.

We note that this assumption covers a fairly large class of examples. For instance, let Y be a codimension-1 submanifold of X embedded as a closed subset. Then the mod 2 homology exact sequence of the pair $(X, X \setminus Y)$ tells us that the complement $X \setminus Y$ has either 1 or 2 connected components. Moreover if $H_1(X, \mathbb{Z}_2) = 0$ then the complement has exactly two connected components. Hence, in case of simply connected manifolds or homology spheres (of dimension at least 2) all the reflections with connected codimension-1 fixed sets are dissecting. A general criterion for dissections is stated in the following lemma.

Lemma 3.1. *Let X be a connected l-manifold and let Y be a connected $(l-1)$-submanifold embedded in X as a closed subset. Then $X \setminus Y$ has exactly 2 connected components if and only if the inclusion induced homomorphism $H_c^{n-1}(X, \mathbb{Z}_2) \to H_c^{n-1}(Y, \mathbb{Z}_2)$ (on cohomology with compact support) is trivial.*

Proof. The proof is a simple diagram chase:

$$\begin{array}{ccccccc}
H_c^{n-1}(X, \mathbb{Z}_2) & \longrightarrow & H_c^{n-1}(Y, \mathbb{Z}_2) \cong \mathbb{Z}_2 & & & & \\
\cong \downarrow & & \downarrow \cong & & & & \\
H_1(X, \mathbb{Z}_2) & \longrightarrow & H_1(X, X \setminus Y, \mathbb{Z}_2) & \longrightarrow & \tilde{H}_0(X \setminus Y, \mathbb{Z}_2) & \longrightarrow & 0
\end{array}$$

The vertical isomorphisms are the duality isomorphisms. □

3.2 Structure of manifold reflection groups

We now explore the structure of a group of diffeomorphisms generated by finitely many reflections. Let W denote such a discrete group of diffeomorphisms whose action on X is proper and smooth. Let $\mathcal{R}$ denote the set of all reflections in W (i.e., conjugates of the generating reflections). For every $r \in \mathcal{R}$ the fixed set X_r is the *wall* associated to r (note that a reflection is uniquely determined by its wall). Denote the collection of walls in X by $\mathcal{A}_W = \{X_r \mid r \in \mathcal{R}\}$. Let $\mathcal{C}(\mathcal{A}_W)$ be the set of connected components of the complement $X \setminus \bigcup_{r \in R} X_r$. Elements of $\mathcal{C}(\mathcal{A}_W)$ are called *chambers*. A *wall of a closed chamber* C is a wall X_r such that $C \cap X_r$ is non-empty. In this case the set $C_r := C \cap X_r$ is a *mirror* of C. Two chambers are *adjacent* if they intersect in a common mirror.

Fix a closed chamber C and call it the *fundamental chamber* (note that this is a fundamental domain for the W action). Let S be the set of reflections s such that X_s is a wall of C; these are the *simple reflections*. Recall that a *mirror structure* on a space is a collection of closed subspaces indexed by a set [9, Chapter 5]. The family $(C_s)_{s \in S}$ is the *tautological mirror structure* of C and the chamber is called the *mirrored space* over S. For $x \in C$ we denote by $S(x)$ the set of walls that contain x; note that $S(x) = \varnothing$ if $x \in C^\circ$. The following important theorem justifies the term *Coxeter transformation groups* for such groups; see [9, Theorem 10.1.5], [1, Theorem 3.10] and [19, Theorem 1]. This theorem is indeed true for reflection groups acting on topological and generalized manifolds, see [27, Corollary 1.1].

Theorem 3.2. *Let W be a discrete group of diffeomorphisms acting properly and smoothly on a connected manifold X, let C be a fundamental chamber, let S denote the set of all simple reflections and for $s, r \in S$ let m_{sr} denote the order of sr. Then W is a Coxeter group with the presentation*

$$W = \langle S \mid s^2 = 1,\ (sr)^{m_{sr}} = 1 \forall s, r \in S, r \neq s, 2 \leq m_{sr} < \infty \rangle .$$

Next we describe the relationship between the fundamental chamber and the ambient manifold. The manifold X with W-action can be reconstructed from the fundamental chamber C using the *universal construction* of Vinberg [31] (called the basic construction in [9, Chapter 5]). Define an equivalence relation on $W \times C$ by

$$(g, x) \sim (h, y) \iff x = y \text{ and } g^{-1}x = h^{-1}x .$$

Denote the quotient space by $\mathcal{U}(W, C)$ and the elements by $[g, x]$. The natural W-action on the product descends to an action on $\mathcal{U}(W, C)$. The isotropy subgroup at $[g, x]$ is $gW_{S(x)}g^{-1}$. The map defined by sending $x \mapsto [1, x]$ is an embedding. The projection to the second factor descends to a retraction $p \colon \mathcal{U}(W, C) \to C$. The construction is universal in the sense that if D is another space with W-action and $f \colon C \to D$ is a map such that for all $s \in S, f(C_s) \subset D^s$ (here D^s the fixed set of s on D). Then there is a unique extension to a W-equivariant map $\tilde{f} \colon \mathcal{U}(W, C) \to D$ defined by sending $[g, x] \mapsto g \cdot f(x)$.

Theorem 3.3. *With the above notation the W-equivariant map $\mathcal{U}(W, C) \to X$ induced by the inclusion $C \hookrightarrow X$, is a homeomorphism.*

The collection $\mathcal{A}_W$ defines a stratification of the manifold; we now focus on exploring various properties of these strata. Recall that a second countable, Hausdorff space C is a *smooth l-manifold with corners* if it is equipped with an atlas of local charts from open subsets of C to open subsets of the standard l-simplicial cone $\mathbb{R}^l_+ := [0,\infty)^l$ so that the transition functions are diffeomorphisms. Given a point $x \in C$ and its local coordinates $(x_1, \dots, x_l) \in \mathbb{R}^l_+$ the number $c(x) = \big|\{i \mid x_i = 0\}\big|$is independent of the choice of coordinates and is known as the *depth of* x. For $0 \leq k \leq l$, a connected component of $c^{-1}(k)$ is a *pure stratum* of codimension k and a *stratum* is the closure of a pure stratum. A smooth manifold with corners is *nice* if each stratum of codimension 2 is contained in exactly two strata of codimension 1. As a consequence of niceness we have that a codimension-k stratum is also a smooth manifold with corners. Simplicial cones and simple convex polytopes are examples of smooth, nice manifolds with corners.

Definition 3.3. A *mirrored manifold with corners* is a smooth, nice manifold with corners C together with a mirror structure $(C_s)_{s\in S}$ on C indexed by some set S such that:

- each mirror is a disjoint union of closed codimension-1 strata of C;
- each closed codimension-1 stratum is contained in exactly one mirror.

The following theorem describes the structure a closed chamber [9, Proposition 10.1.9].

Theorem 3.4. *Let W be a Coxeter transformation group acting properly and smoothly on a smooth l-manifold X. Let C be a fundamental chamber endowed with its tautological mirror structure $(C_s)_{s\in S}$. Then C is a mirrored manifold with corners.*

There is a converse to the above theorem for which we need one more piece of terminology. The tautological mirror structure of C is said to be *W-finite* if for any subset $T \subseteq S$ for which the subgroup W_T is infinite, the corresponding intersection of mirrors $\bigcap_{s\in T} C_s$ is empty.

Theorem 3.5. *Let (W,S) be a Coxeter group and C be a mirrored manifold with corners with W-finite mirror structure $(C_s)_{s\in S}$. Then $\mathcal{U}(W,C)$ is a manifold and W acts properly and smoothly on it as a group generated by reflections.*

In [1], Alekseevsky et al consider Riemannian manifolds with the action of a discrete group of *isometries* generated by dissecting reflections. In this case the fundamental chamber is isometric to the quotient X/G, which is a Riemannian manifold with corners, with certain conditions involving angles along intersection of faces. They again show how to recover the manifold and the group action from the isotropy subgroup data along the faces.

3.3 Exotic aspherical manifolds

One of the important applications of this set-up is the seminal work of Davis on aspherical (that is, $K(\pi,1)$) manifolds. In particular he constructed closed aspherical manifolds whose universal cover is not homeomorphic to Euclidean space. We end this chapter with a brief sketch of this construction and refer the interested reader to [9, Section 10.5] for a detailed account.

Theorem 3.6 ([9, Theorem 10.5.1])**.** *In each dimension $n \geq 4$, there are closed aspherical manifolds M^n with universal cover $\widetilde{M}^n$ not homeomorphic to $\mathbb{R}^n$.*

We start with a finite simplicial complex L, which is a PL $(n-1)$-manifold, a homology sphere, and not simply connected (such exist for $n \geq 4$). Then L is the boundary of a compact contractible n-manifold C. Moreover, there is a Coxeter system (W, S) such that L is the abstract simplicial complex given by the poset $\mathcal{S}_{>\varnothing}$ of non-empty spherical subsets. We can construct a mirror structure on C (using that $\partial C = L$), such that $\mathcal{U}(W, C)$ is a contractible n-manifold with a W-action.

Recall that a space X is simply connected at infinity if for any compact subset K, there is a compact subset $K' \supseteq K$ such that any loop in $X - K'$ is nullhomotopic in $X - K$. Since L is chosen to not be simply connected, it turns out that $\mathcal{U}(W, C)$ is not simply connected at infinity. In particular, $\mathcal{U}(W, C)$ is not homeomorphic to $\mathbb{R}^n$ (it is easy to see that $\mathbb{R}^n$ is simply connected at infinity for $n \geq 3$).

Since W is finitely generated and embeds in a general linear group, it has a torsion-free subgroup, say Γ, of finite index. Now, all isotropy subgroups of W are finite, and hence Γ acts freely on $\mathcal{U}(W, C)$. So $M = \mathcal{U}(W, C)/\Gamma$ is a manifold, with universal cover $\mathcal{U}(W, C)$, in particular aspherical. Since W acts cocompactly on $\mathcal{U}(W, C)$, and Γ is of finite index, M is closed.

Chapter 4

Manifold reflection arrangements

In this chapter we focus on a particularly nice class of Coxeter group actions on smooth manifolds. Our aim is to introduce an analogue of the space $M(W,S)$ and study its homotopy type and that of its quotient by the Coxeter group action. We achieve our aim by extending the Salvetti complex construction. We begin by defining reflection arrangements for manifolds.

4.1 Definition and examples

First we isolate important characteristics of the finite reflection arrangements in $\mathbb{R}^l$ that help determine the topology of M_W:

1. the fixed point sets of reflections are codimension-1 subspaces that separate $\mathbb{R}^l$;
2. the intersections of these hyperplanes define a stratification of $\mathbb{R}^l$ into open polyhedral cones;
3. the geometric realization of the face poset of this stratification has the homotopy type of $\mathbb{R}^l$.

It is clear that in order to generalize the above properties to the setting of smooth manifolds we need to consider Coxeter transformation groups acting smoothly and properly. The walls (i.e., codimension-1 submanifolds) fixed by the reflections in W (since they are dissecting) will serve the purpose of reflecting hyperplanes.

Let W be such a Coxeter transformation group acting on X and let $\{H_1, H_2, \dots\}$ be the set of all connected components of walls. We denote by $\mathcal{L}$ the set of all non-empty intersections of H_i's and by $\mathcal{L}^d$ the subset of codimension-d intersections. For example, we have $\mathcal{L}^0 = \{X\}$ and $\bigcup \mathcal{L}^1 = \bigcup_i H_i$. For each $d \geq 0$ define the *set of codimension-d strata*

$$\mathrm{St}^d(X) = \text{connected components of } \bigcup \mathcal{L}^d \setminus \bigcup \mathcal{L}^{d+1}.$$

The set $\mathrm{St}(X) = \bigcup_{d\geq 0} \mathrm{St}^d(X)$ is equipped with the partial order given by the topological inclusion. Note that X is a disjoint union of the subsets in $\mathrm{St}(X)$. The reader can check that this defines a *stratification* in the sense of [28, Definition 2.1].

We would like the stratification of X induced by the reflecting submanifolds to satisfy condition (2) and the corresponding face poset to satisfy condition (3) above. From the previous section we know that each closed stratum is a nice manifold with corners, however, the resulting face poset need not realize the homotopy type of X. We propose the following definition.

Definition 4.1. Let X be a smooth l-manifold, let W be a Coxeter transformation group acting properly and smoothly and let R be the set of reflections in W. The *(manifold) reflection arrangement* corresponding to W is the finite collection

$$\mathcal{A}_W = \{X_r \mid r \in R\}$$

of walls fixed by W given that the following conditions are satisfied:

1. The stratification of X induced by the intersections of these submanifolds define the structure of a regular cell complex.
2. Each closed chamber, as a nice manifold with corners, is combinatorially equivalent to a simple, convex polytope.

The face poset of a nice manifold with corners is the set of all connected components of $c^{-1}(k)$ for all k and ordered by topological inclusion. Two nice manifolds with corners are said to be *combinatorially equivalent* if their face posets are isomorphic. In view of [32, Corollary 5.2], if P is the simple convex polytope combinatorially equivalent to the (closed) fundamental chamber $\overline{C}$ then there is a diffeomorphism of nice manifolds with corners $f : \overline{C} \to P$ that induces the given isomorphism on the face posets. In the language of [11], we assume that the closed chambers are Coxeter orbifolds of type (III).

We denote the face poset of a manifold reflection arrangement by $\mathcal{F}(\mathcal{A}_W)$ (we will drop the subscript W if the context is clear). According to Theorem 3.4, this cellular decomposition is simple, that is, a codimension-k cell is in the closure of k codimension-1 cells. As before, the codimension-0 faces are called *chambers*. The set of chambers of the arrangement $\mathcal{A}_W$ will be denoted by $\mathcal{C}(\mathcal{A}_W)$. From now on C will always denote an open chamber.

We say that a wall X_r of $\mathcal{A}$ *separates* two chambers C and D if they are contained in distinct connected components of $X \setminus X_r$. For two chambers C and D, the set of all walls that separate these two chambers is denoted by $\mathcal{R}(C, D)$. The *distance between two chambers* is the cardinality of the set $\mathcal{R}(C, D)$ and is denoted by $d(C, D)$.

Proposition 4.1. *Let X be an l-manifold and $\mathcal{A}$ be an arrangement of submanifolds, let C_1, C_2, C_3 be three chambers of this arrangement. Then,*

$$\mathcal{R}(C_1, C_3) = [\mathcal{R}(C_1, C_2) \setminus \mathcal{R}(C_2, C_3)] \cup [\mathcal{R}(C_2, C_3) \setminus \mathcal{R}(C_2, C_1)].$$

Definition 4.2. Let $\mathcal{A}_W$ be a manifold reflection arrangement in X, corresponding to W. For a point $x \in X$ the *local arrangement* at x is

$$\mathcal{A}_x := \{X_r \in \mathcal{A}_W \mid x \in X_r\}.$$

For a face $F \in \mathcal{F}(\mathcal{A}_W)$ the local arrangement at F is

$$\mathcal{A}_F := \{X_r \in \mathcal{A}_W \mid F \subseteq X_r\}.$$

A local arrangement $\mathcal{A}_F$ need not be a reflection arrangement in the sense of Definition 4.1. However it does define a stratification of X, which we denote by the pair $(X, \mathcal{A}_F)$. A *map of stratified spaces* is a continuous map that induces an order preserving map on the corresponding face posets (such maps are called strict morphisms in [28, Definition 2.15]). Define a map from $(X, \mathcal{A})$ to $(X, \mathcal{A}_F)$ by sending a face G to the face of $\mathcal{A}_F$ of least dimension that contains G. The reader can verify that this is a map of stratified spaces. We denote by π_F the induced map on the face posets. We also use the notation C_F for the chamber $\pi_F(C)$ of $\mathcal{A}_F$.

Definition 4.3. Given a face F and a chamber C denote by $F \circ C$ a chamber that satisfies:

1. $F \subseteq \overline{F \circ C}$,
2. $\pi_F(C) = \pi_F(F \circ C)$,
3. $d(C, F \circ C) = \min \{d(C, C') \mid C' \in \mathcal{C}(\mathcal{A}), F \subseteq \overline{C'}\}$.

The following result is evident.

Proposition 4.2. *The chamber $F \circ C$ always exists and is unique.*

The reader can verify that if $F \leq C$ then $F \circ C = C$ and for $F \leq F'$ we have $F' \circ (F \circ C) = F' \circ C$.

Before moving on we should convince the reader that there are examples other than the classical ones. Since each chamber C is simply connected and for every $s \in S$ the closed codimension-1 face C_s is non-empty the manifold $X \cong \mathcal{U}(W, C)$ is also simply connected if and only if for each spherical subset $\{s, t\}$ we have $C_s \cap C_t \neq \varnothing$ ([9, Theorem 9.1.3]). Moreover the manifold X is contractible if and only if for every non-empty spherical subset T the intersection $\cap_{s \in T} C_s$ is acyclic ([9, Theorem 9.1.4]).

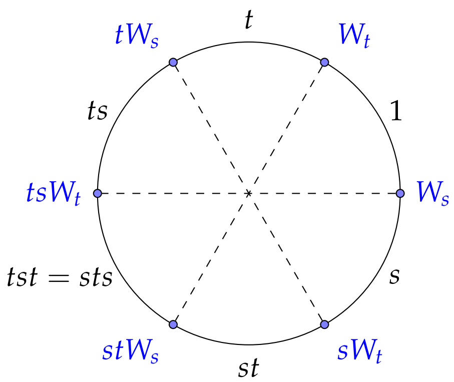

Figure 4.1: The S_3 action on S^1.

Example 4.1. Consider the dihedral group of order $2m$ generated by two generators, say $\{s, t\}$. Its natural reflection action on $\mathbb{R}^2$ fixes a union of $2m$ lines passing through the origin. The intersection of this line arrangement with the unit circle S^1 gives us a reflection arrangement on S^1. The fundamental chamber is a 1-simplex and its mirrors are two points labeled C_s and C_t. Note that the two element set $\{s, t\}$ is spherical but $C_s \cap C_t = \varnothing$. In particular, consider the action of S_3, the symmetric group on 3 letter on S^1 (Fig. 4.1). The chambers (the 6 arcs) are labeled by the group elements whereas the fixed submanifolds (the 6 vertices) are labeled by the one-generator parabolic subgroups.

Example 4.2. Let C be the 1-simplex and let s and t be the reflections across its endpoints and set $m_{st} = \infty$. The reader can verify that the group generated by these two elements is the infinite dihedral group $\mathbb{Z}/2 * \mathbb{Z}/2$ and the manifold $\mathcal{U}(W, C)$ is $\mathbb{R}^1$.

Example 4.3. If the fundamental chamber C is combinatorially equivalent to the unit simplex of dimension at least 2 and W is a finite Coxeter group of rank at least 3 then the manifold $\mathcal{U}$ is homeomorphic to the unit sphere (in fact, W can be made to act via isometries [10, Lecture 3]). In this case the manifold reflection arrangement is a collection of codimension-1 sub-spheres. All the positive-dimensional intersections are connected spheres.

Remark 4.1. Let $\mathbb{X}^l$ denote either $\mathbb{R}^l$ or S^l or the hyperbolic space $\mathbb{H}^l$. Let K be a (simple) convex polytope in $\mathbb{X}^l$ with nonobtuse dihedral angles (see [9, Section 6.3] for precise definition). Denote by S the set of reflections across the codimension-1 faces of K. The collection $\{K_s \mid s \in S\}$ of codimension-1 faces defines a mirror structure on K. If $K_s \cap K_t \neq \varnothing$ then it is a codimension-2 face with the dihedral angle $\frac{\pi}{m_{st}}$, for some integer $m_{st} \geq 2$. If such an intersection is empty then put $m_{st} = \infty$. It follows from [9, Theorem 6.4.3] that the group $W \subsetneq \mathrm{Isom}(\mathbb{X}^l)$ generated by S is a Coxeter group. The universal space $\mathcal{U}(W, K)$ is W-equivariantly homeomorphic to $\mathbb{X}^l$. The W-action is proper and K is a strict fundamental domain.

4.2 The tangent bundle complement

In the classical case the action of the reflection group on $\mathbb{R}^l$ extends to $\mathbb{C}^l$ by extension of scalars. Note that topologically $\mathbb{C}^l$ is the tangent bundle of $\mathbb{R}^l$. The Coxeter transformation group acts locally smoothly on the manifold and hence its action naturally extends to the tangent bundle. The fixed points of this action are the tangent bundles of the reflecting submanifolds. Hence the complement of the union of these tangent bundles over reflecting submanifolds is topologically a generalization of the complexified complement.

Definition 4.4. The *tangent bundle complement* associated with the manifold reflection arrangement $\mathcal{A}_W$ in X is denoted $M(\mathcal{A}_W)$ or M_W and defined as

$$M_W := TX \setminus \bigcup_{r \in R} TX_r.$$

Note that the Coxeter group action on M_W is a covering space action. Denote the orbit space of this action by N_W. It is clear that the group $\pi_1(M_W)$ is an analogue of the pure Artin group whereas $\pi_1(N_W)$ is an analogue of the Artin group in the sense that they fit in an exact sequence of the form (1.1).

4.3 The Salvetti complex construction

The aim is to prove that there is exists a regular cell complex that is W-equivariantly homotopy equivalent to the tangent bundle complement M_W.

Definition 4.5. The *Salvetti poset* of a manifold reflection arrangement $\mathcal{A}_W$ in an l-manifold X is the set

$$\mathrm{Sal}_0(\mathcal{A}_W) = \{(F,C) \in \mathcal{F}(\mathcal{A}_W) \times \mathcal{C}(\mathcal{A}_W) \mid F \le C\}$$

with the following relation $\preceq$ on $\mathrm{Sal}_0(\mathcal{A}_W)$ as a partial order

$$(F,C) \preceq (G,D) \text{ if } F \le G \text{ and } C_F \subseteq D_G\,.$$

The *Salvetti complex* of $\mathcal{A}_W$, denoted by $\mathrm{Sal}(\mathcal{A}_W)$, is defined to be the geometric realization of the derived complex of $(\mathrm{Sal}_0(\mathcal{A}_W), \preceq)$.

An alternative way of defining the Salvetti complex, in terms of parabolic subgroups is discussed and shown to be equivalent to this construction in Section 4.4.

Theorem 4.3. *There exists a W-equivariant homotopy equivalence $f : \mathrm{Sal}(\mathcal{A}_W) \to M_W$. This induces a homotopy equivalence $\bar{f} : \mathrm{Sal}(\mathcal{A}_W)/W \to M_W/W$.*

We use the same strategy as that of Salvetti [25] and also of Paris [24]; we use the *nerve lemma*. The proof closely imitates that of Theorem 3.1 in [24]. Since our arrangements are assumed to originate from a group action, we deal with only the equivariant case of the constructions. The principal step is to find a nerve of M_W that equivariantly corresponds to $\mathrm{Sal}_0(\mathcal{A})$, so that we can apply the equivariant nerve lemma.

Proposition 4.4 (nerve lemma, [24, Proposition 2.2])**.** *Let X be paracompact, and $\mathcal{U}$ an open cover of X such that every finite non-empty intersection of elements of $\mathcal{U}$ is contractible. Let G be a group acting freely and properly discontinuously on X, such that for $g \in G \setminus \{1\}$ and $U \in \mathcal{U}$, $g(U) \in \mathcal{U}$ and $U \cap g(U) = \varnothing$. Then G acts freely and properly discontinuously on $|N(\mathcal{U})|$, there exists a G-equivariant homotopy equivalence $f : |N(\mathcal{U})| \to X$, and this homotopy equivalence induces a homotopy equivalence $\bar{f} : |N(\mathcal{U})|/G \to X/G$.*

As the proof of the theorem is long and technical we begin by sketching the ideas and constructions used. Looking at the definition of the Salvetti complex, we would like to find a cover $\mathcal{U}$ of $M(\mathcal{A})$, indexed by $\mathrm{Sal}_0(\mathcal{A})$, such that non-empty finite intersections correspond to chains in the Salvetti poset. Further, we want the indexing to be equivariant (in particular, $\mathcal{U}$ should be invariant under W), and these finite intersections to be contractible (that is, $\mathcal{U}$ should be a *good cover*). Since $M(\mathcal{A})$ is a subspace of TX, we would like the open set corresponding to (F,C) to be a bundle over a neighborhood of F.

In the proof in [24], the linear structure of $\mathbb{R}^k$ is used to construct certain open subsets, denoted $\Delta(\gamma)$, of each face $F \in \mathcal{F}$ for every chain of faces γ ending at F. Here, we use the equivariant *convex structures* defined below to identify the faces with convex polytopes (Lemma 4.5), and pull back corresponding neighborhoods using these diffeomorphisms, denoting them by $\Theta(\gamma)$. These neighborhoods $\Theta(\gamma)$ are equivariant, and satisfy topological and combinatorial properties corresponding to those among γ (Lemmas 4.6 to 4.9).

Using these, we construct (open regular) neighborhoods $\omega(F)$ of each face F, as mentioned above. We again establish a number of elementary properties of these neighborhoods (Lemmas 4.10 to 4.13 and 4.20). In particular, these neighborhoods intersect if and only if the corresponding faces form a chain, and in that case, the intersection is contractible (Lemmas 4.14 to 4.19).

Next, we want to construct bundles $U(F, C)$ over $\omega(F)$ that somehow take into account the chamber C. In the Euclidean case, the open sets $U(F, C)$ can be defined as products with one factor $\omega(F)$. This uses that the tangent space at each point is isomorphic to the entire arrangement, and hence has identical stratifications.

Since we are working on a manifold, we use that at a point $x \in F$, $T_x X$ has an arrangement combinatorially isomorphic to $\mathcal{A}$ restricted to a neighborhood of x. Then, since $\omega(F)$ is contractible, with a nice retract to F, we use this to pull back the chamber in $T_x X$ corresponding to C to a bundle over $\omega(F)$. It remains to be checked that this makes sense as x varies and that $U(F, C)$, constructed thus, satisfies the required properties (Lemmas 4.21 to 4.28).

Definition 4.6. If $F \in \mathcal{F}(\mathcal{A})$ is a face of dimension k, define a *convex structure* on F to be a diffeomorphism $\varphi : P \to \overline{F}$ for some simple convex polytope $P \subset \mathbb{R}^k$.

Since both the spaces are manifolds with corners, any such diffeomorphism must also be a combinatorial equivalence. That is to say, faces of P are mapped to faces $F' \leq F$, and vice-versa, and $F'' \leq F'$ iff $\varphi^{-1}(F'') \leq \varphi^{-1}(F')$ for faces $F', F'' < F$.

Lemma 4.5. *There is a collection of convex structures φ_F for each $F \in \mathcal{F}(\mathcal{A})$ such that for any $F < F'$, if $P = \varphi_{F'}^{-1}(\overline{F})$, then $\varphi_{F'}|_P = \varphi_F$. Further, these can be chosen in an equivariant way, that is, such that $\varphi_{wF} = w \circ \varphi_F$ for $w \in W$.*

Proof. Fix a fundamental chamber $C_0 \in \mathcal{C}(\mathcal{A})$. By the remark after Definition 4.1, we can give C_0 a convex structure $\varphi = \varphi_{C_0}$. Now, for the chamber wC_0 for some $w \in W$, define the convex structure by $\varphi_{wC_0} = w \circ \varphi$.

For a face F of C_0, restrict φ to $\varphi^{-1}(\overline{F})$, that is, set $\varphi_F = \varphi|_{\varphi^{-1}(\overline{F})}$, and similarly for a face F of wC_0. It is enough to check well-definedness on faces F of C_0. But if F is a face of both C_0 and wC_0, then w must fix $\overline{F}$ point-wise (since F is fixed by a unique standard parabolic subgroup, see Lemma 4.30). Hence, on $\varphi^{-1}(\overline{F})$, $w \circ \varphi = \varphi$. $\square$

So fix a collection of convex structures φ_F as above. Let $\mathring{\Delta}^n$ denote the *open* unit simplex in $\mathbb{R}^{n+1}$ and Δ^n the closed unit simplex. If $x_0, x_1, \ldots, x_n$ are points contained in $\overline{F}$ for some face F, and $(t_0, t_1, \ldots t_n) \in \Delta^n$, then the point $\varphi_F\left(\sum t_i \varphi_F^{-1}(x_i)\right)$ does not depend on the choice of F. So we shall abuse notation and denote this by $\sum t_i x_i$, as long as x_i are contained in the closure of some face F. We then also have, for any $w \in W$, $w(\sum t_i x_i) = \sum t_i w(x_i)$.

Throughout the rest of this section we use the following definitions and notations. A *chain of length* $p+1$ in $\mathcal{F}(\mathcal{A})$ is a sequence $(F_0, \ldots, F_p)$ in $\mathcal{F}(\mathcal{A})$ such that $F_0 < F_1 < \cdots < F_p$. We define two partial orders on chains. Let $\gamma = (F_0, \ldots, F_p)$ and $\gamma' = (F'_0, \ldots, F'_q)$. First, $\gamma \leq \gamma'$ if $F_0 = F'_0$ and $\{F_0, \ldots, F_p\} \subseteq \{F'_0, \ldots, F'_q\}$. Second, $\gamma \preceq \gamma'$ if $p \leq q$, and for $0 \leq i \leq p$, $F_{p-i} = F'_{q-i}$. Note that if $F_0 = F'_0$ (in particular if $\gamma \leq \gamma'$) and $\gamma \preceq \gamma'$ then $\gamma = \gamma'$.

For $F \in \mathcal{F}(\mathcal{A})$, denote by $\mathrm{Chain}(F)$ the set of chains $(F_0, \ldots, F_p)$ such that $F = F_0$. More generally, let $\mathrm{Chain}(\gamma)$ denote the set of chains γ' such that $\gamma \leq \gamma'$. For a chain $\gamma = (F_0, F_1, \ldots, F_p)$, and $w \in W$, let $w\gamma$ be the chain $(wF_0, wF_1, \ldots, wF_p)$.

For each $F \in \mathcal{F}(\mathcal{A})$, fix a point $x(F) \in F$ so that $wx(F) = x(wF)$ for all $w \in W$. For a given chain $\gamma = (F_0, F_1, \dots, F_p)$, let $x_i = x(F_i)$ and define the following subset of F_p:

$$\Theta(\gamma) = \left\{ t_0 z_0 + t_1 x_1 + \cdots + t_p x_p \;\middle|\; z_0 \in F_0, (t_0, t_1, \dots, t_p) \in \mathring{\Delta}^p \right\}.$$

Lemma 4.6. *Let $\gamma = (F_0, F_1, \dots, F_p)$ be a chain for some $p > 0$ and let $\gamma_1 = (F_0, F_1, \dots, F_{p-1})$. Then*

$$\Theta(\gamma) = \left\{ tz + (1-t)x(F_p) \;\middle|\; z \in \Theta(\gamma_1), 0 < t < 1 \right\}.$$

Proof. Let $x_i = x(F_i)$. Using that $\mathring{\Delta}^p = \left\{ (tx, 1-t) \;\middle|\; x \in \mathring{\Delta}^{p-1}, 0 < t < 1 \right\}$, it is enough to note that by definition,

$$\Theta(\gamma_1) = \left\{ \left(t_0 z_0 + t_1 x_1 + \cdots + t_{p-1} x_{p-1} \right) \;\middle|\; z_0 \in F_0, (t_0, t_1, \dots, t_{p-1}) \in \mathring{\Delta}^{p-1} \right\}. \qquad \square$$

Lemma 4.7. *For any $w \in W$, and any chain γ,*

$$w\Theta(\gamma) = \Theta(w\gamma).$$

Proof. Let $x_i = x(F_i)$ and $u \in \Theta(\gamma)$. By definition, $u = t_0 z + \sum_{i=1}^{p} t_i x_i$ for some $z \in F_0$, and some $(t_0, t_1, \dots, t_p) \in \mathring{\Delta}^p$. But then if $x_i' = x(wF_i) = w(x_i)$, we must have that $wu = t_0 z + \sum_{i=1}^{p} t_i x_i'$, and so $wu \in \Theta(w\gamma)$.

Thus, $w\Theta(\gamma) \subseteq \Theta(w\gamma)$. Similarly, $w^{-1}\Theta(w\gamma) \subseteq \Theta(\gamma)$, and so we are done. $\square$

Lemma 4.8. *Let $\gamma = (F_0, F_1, \dots, F_p)$ and $\gamma' = (F_0', F_1', \dots, F_q')$ be two chains. If $p \leq q$ and $\Theta(\gamma) \cap \Theta(\gamma') \neq \varnothing$, then $\gamma \preceq \gamma'$.*

Proof. Let $u \in \Theta(\gamma) \cap \Theta(\gamma')$. Then $u \in F_p$ and $u \in F_q'$, and hence $F_p = F_q'$. Now we proceed by induction on p. If $p = 0$, we are already done, so assume $p \geq 1$ and the induction hypothesis. Let $\gamma_1 = (F_0, F_1, \dots, F_{p-1})$ and $\gamma_1' = (F_0', F_1', \dots, F_{q-1}')$.

Let $x = x(F_p)$. Then $u = tz + (1-t)x = t'z' + (1-t')x$ for some $0 < t, t' \leq 1$, $z \in \Theta(\gamma_1)$, and $z' \in \Theta(\gamma_1')$. Thus, z, z' are points in ∂F_p and are on the ray xu. Since x is an interior point of $\overline{F_p}$, this implies that $z = z'$, and hence $\Theta(\gamma_1) \cap \Theta(\gamma_1') \neq \varnothing$. By the induction hypothesis, $\gamma_1 \preceq \gamma_1'$ and hence $\gamma \preceq \gamma'$. $\square$

Lemma 4.9. *Let $\gamma = (F_0, F_1, \dots, F_p)$ and $\gamma' = (F_0', F_1', \dots, F_q')$ be two chains. If $\gamma \preceq \gamma'$, then $\Theta(\gamma') \subseteq \Theta(\gamma)$.*

Proof. If $p = q$, there is nothing to prove, so assume $p < q$. For $i = 1, 2, \dots, q$, let $x_i = x(F_i')$.

Now let $u \in \Theta(\gamma')$. Then for some $(t_0, t_1, \dots, t_q) \in \mathring{\Delta}^q$, and $z \in F_0'$,

$$u = t_0 z + \sum_{i=1}^{q} t_i x_i.$$

Then let $t_0' = \sum_{i=0}^{q-p} t_i > 0$ and

$$u' = \frac{1}{t_0'} \left[t_0 z + \sum_{i=1}^{q-p} t_i x_i \right],$$

which is a point in $\overline{F_{q-p}'} = \overline{F_0}$. Let $t_i' = t_{q-p+i}$ and note that $x(F_i) = x_{q-p+i}$. But now $(t_0', t_{q-p+1}, \dots, t_q)$ is a point in $\mathring{\Delta}^p$ and so $u = t_0' u' + \sum_{i=1}^{p} t_i' x(F_i) \in \Theta(\gamma)$. $\square$

For a given $F \in \mathcal{F}(\mathcal{A})$, set

$$\omega(F) = \bigcup_{\gamma \in \mathrm{Chain}(F)} \Theta(\gamma) .$$

More generally, for a chain γ,

$$\omega(\gamma) = \bigcup_{\gamma' \in \mathrm{Chain}(\gamma)} \Theta(\gamma') .$$

Lemma 4.10. *Let $\gamma = (F_0, F_1, \ldots, F_p)$ be a chain and $G \in \mathcal{F}(\mathcal{A})$ be any face with $\dim G > \dim F_p$. Then*

$$\overline{G} \cap \omega(\gamma) = \{tz + (1-t)x(G) \mid z \in \partial G \cap \omega(\gamma), 0 < t \leq 1\} .$$

Proof. Suppose $u \in \overline{G} \cap \omega(\gamma)$. If $u \in \partial G$, take $z = u$ and $t = 1$. Otherwise $u \in \Theta(\delta)$ for some $\delta = (G_0, G_1, \ldots, G_q) \in \mathrm{Chain}(\gamma)$ such that $G = G_q$. Since $F_0 = G_0$, and G is of higher dimension than F_0, $q > 0$. Define $\delta_1 = (G_0, G_1, \ldots, G_{q-1})$. Then by Lemma 4.6, for some $z \in \Theta(\delta_1)$ and $0 < t < 1$, $u = tz + (1-t)y$. But since $G \notin \gamma$, $\delta_1 \in \mathrm{Chain}(\gamma)$. Thus $z \in \Theta(\delta_1) \subseteq \omega(\gamma)$, and so the forward inclusion is proved.

For the reverse inclusion, let $z \in \partial G \cap \omega(\gamma)$. Clearly $z \in \omega(\gamma)$, so let $0 < t < 1$. Now let $\delta_1 = (G_0, G_1, \ldots, G_q) \in \mathrm{Chain}(\gamma)$ be such that $z \in \Theta(\delta_1)$. Then $z \in G_q$, and hence $G_q < G$. So $\delta = (G_0, G_1, \ldots, G_q, G)$ is a chain and since $\gamma \leq \delta_1 \leq \delta$, $\delta \in \mathrm{Chain}(\gamma)$. Now again by Lemma 4.6, $tz + (1-t)y \in \Theta(\delta) \subseteq \overline{G} \cap \omega(\gamma)$. □

Lemma 4.11. *Let $F \in \mathcal{F}(\mathcal{A})$. Then $\omega(F)$ is open in X.*

Proof. Let $d = \dim F$ and for $k \geq 0$, let X^{d+k} be the $(d+k)$-skeleton. We prove by induction on k that $X^{d+k} \cap \omega(F)$ is open in X^{d+k}. The set $X^d \cap \omega(F) = F$ is of course open in X^d, so assume $k \geq 1$ and the induction hypothesis. Consider a face G of dimension $d+k$ and let $x = x(G)$.

If $F \not< G$, then $\overline{G} \cap \omega(F) = \varnothing$. Otherwise, by the induction hypothesis, $B = \omega(F) \cap \partial G$ is open in ∂G. Then $\overline{G} \cap \omega(F) = \{tz + (1-t)x \mid z \in B, 0 < t \leq 1\}$ is open in $\overline{G}$. So $\omega(F) \cap X^{d+k}$ is open in X^{d+k}. □

Lemma 4.12. *Let $\gamma = (F_0, F_1, \ldots, F_p)$ and $\gamma' = (F'_0, F'_1, \ldots, F'_q)$ be two chains. If the set $\{F_0, F_1, \ldots, F_p\} \subseteq \{F'_0, F'_1, \ldots, F'_q\}$, then $\omega(\gamma') \subseteq \omega(\gamma)$.*

Proof. By Lemma 4.9, it is enough to show that for any $\delta' \in \mathrm{Chain}(\gamma')$, there is some $\delta \in \mathrm{Chain}(\gamma)$ such that $\delta \preceq \delta'$. Let $\delta' = (G_0, G_1, \ldots, G_r)$. Since $F_0 \in \{F'_0, F'_1, \ldots, F'_q\} \subseteq \{G_0, G_1, \ldots, G_r\}$, let $F_0 = G_s$ for some $s \leq r$. Then $\delta = (G_s, G_{s+1}, \ldots, G_r)$ works. □

Lemma 4.13. *If $u \in \omega(F)$ then there is a unique chain $\gamma \in \mathrm{Chain}(F)$ such that $u \in \Theta(\gamma)$.*

Proof. Let $\gamma, \gamma' \in \mathrm{Chain}(F)$ such that $u \in \Theta(\gamma) \cap \Theta(\gamma')$. Then by Lemma 4.8, without loss of generality, $\gamma \preceq \gamma'$. But since both are in $\mathrm{Chain}(F)$, we must have $\gamma = \gamma'$. □

Lemma 4.14. *Let $F, G \in \mathcal{F}(\mathcal{A})$. If $\omega(F) \cap \omega(G) \neq \varnothing$, then $F \leq G$ or $G \leq F$.*

Proof. Let $u \in \omega(F) \cap \omega(G)$. Then for some $\gamma = (F_0, F_1, \ldots, F_p) \in \mathrm{Chain}(F)$ and $\gamma' = (F'_0, F'_1, \ldots, F'_q) \in \mathrm{Chain}(G)$, $u \in \Theta(\gamma) \cap \Theta(\gamma')$. In particular, $F_0 = F$ and $F'_0 = G$. But by Lemma 4.8, either $\gamma \preceq \gamma'$ or $\gamma' \preceq \gamma$. Thus, either $F \leq G$ or $G \leq F$. □

By induction, we get:

Lemma 4.15. *Let $F_0, F_1, \ldots, F_p \in \mathcal{F}(\mathcal{A})$ be such that $\omega(F_0) \cap \omega(F_1) \cap \cdots \cap \omega(F_p) \neq \varnothing$. Then up to a permutation of indices, $F_0 \leq F_1 \leq \cdots \leq F_p$.*

Lemma 4.16. *Let $\gamma = (F_0, F_1, \ldots, F_p)$ be a chain. Then*

$$\omega(F_0) \cap \omega(F_1) \cap \cdots \cap \omega(F_p) = \omega(\gamma).$$

Proof. By Lemma 4.12, $\omega(\gamma) \subseteq \omega(F_0) \cap \omega(F_1) \cap \cdots \cap \omega(F_p)$. Conversely, let $u \in \omega(F_0) \cap \omega(F_1) \cap \cdots \cap \omega(F_p)$. Let $\delta_i \in \mathrm{Chain}(F_i)$ be such that, $u \in \Theta(\delta_i)$. We want $\delta_0 \in \mathrm{Chain}(\gamma)$ and hence $u \in \omega(\gamma)$. It is enough to show that $F_i \in \delta_0$ for each $i = 1, \ldots, p$. Now by Lemma 4.8, either $\delta_0 \preceq \delta_i$ or $\delta_i \preceq \delta_0$. Since $F_0 < F_i$, the former cannot hold, and hence $F_i \in \delta_0$. □

Lemma 4.17. *Let $\gamma = (F_0, F_1, \ldots, F_p)$ be a chain and let F_p have dimension d. Then $\omega(\gamma) \cap X^d \subseteq F_p$, where X^d is the d-skeleton of X.*

Proof. Consider $G \in \mathcal{F}(\mathcal{A})$ of dimension at most d, and let $u \in G \cap \omega(\gamma)$. Then there is some $\delta = (G_0, G_1, \ldots, G_q) \in \mathrm{Chain}(\gamma)$ such that $u \in \Theta(\delta)$. But then $G_q = G$ and since $F_p \subseteq \{G_0, G_1, \ldots, G_q\}$, we must have $F_p = G_r$ for some $r \leq q$. In particular, $F_p \leq G$. But then the dimension assumption forces $F_p = G$, so $u \in F_p$. □

Lemma 4.18. *Let $\gamma = (F_0, F_1, \ldots, F_p)$ be a chain. Then $\overline{F_p} \cap \omega(\gamma)$ is contractible.*

Proof. By Lemma 4.17, $\overline{F_p} \cap \omega(\gamma) = F_p \cap \omega(\gamma)$. Let $x_i = x(F_i)$ and

$$x = \frac{1}{p+1}(x_0 + x_1 + \cdots + x_p) \in \Theta(\gamma).$$

Let $u \in \Theta(\delta) \cap F_p$ where $\delta = (G_0, G_1, \ldots, G_q) \in \mathrm{Chain}(\gamma)$. Then $F_0 = G_0$ and $F_p = G_q$. Let $u = \sum_{j=0}^{q} t_j z_j$ where $z_0 \in F_0 = G_0$, and $z_j = x(G_j)$ for $0 < j \leq q$. Since $\delta \in \mathrm{Chain}(\gamma)$, each y_i for $i > 0$ occurs as some z_j. Let $t \in (0,1)$. Then $z_0' = tz_0 + (1-t)x_0 \in G_0$ and so $tx + (1-t)u$ is also a positive convex combination of z_0' and z_j for $j > 0$, and hence $tx + (1-t)u \in \Theta(\delta) \subseteq F_p \cap \omega(\gamma)$. So the segment ux is contained in $\omega(\gamma)$. This corresponds to star convexity under the convex structure of F_p and hence the contractibility follows. □

Lemma 4.19. *Let $\gamma = (F_0, F_1, \ldots, F_p)$ be a chain and let F_p have dimension d. Then $\omega(\gamma)$ deformation retracts to $w(\gamma) \cap X^d$, and hence is contractible.*

Proof. By induction, it is enough to show that $\omega(\gamma) \cap X^{n-k+1}$ deformation retracts to $\omega(\gamma) \cap X^{n-k}$ for $1 \leq k \leq n-d$.

Let $G \in \mathcal{F}(\mathcal{A})$ be of codimension $k-1$. It is then enough to show that $\omega(\gamma) \cap G$ deformation retracts to $\omega(\gamma) \cap \partial G$. But this is straightforward from Lemma 4.10. □

In particular, $\omega(F)$ deformation retracts to F for any $F \in \mathcal{F}(\mathcal{A})$. Further, the obvious choice of these deformation retracts is W-equivariant. That is, we can pick the deformation retracts $r_F^s : \omega(F) \to \omega(F)$, $s \in [0,1]$ such that for any $w \in W$, $w \circ r_F^s = r_{wF}^s \circ w$. In particular, letting $r_F = r_F^1 : \omega(F) \to F$, $r_{wF} \circ w = w \circ r_F$.

Lemma 4.20. *Let $w \in W$ and $F \in \mathcal{F}(\mathcal{A})$ be such that $(w \cdot \omega(F)) \cap \omega(F) \neq \varnothing$. Then $wF = F$.*

Proof. First, $\gamma \in \mathrm{Chain}(F)$ iff $w\gamma \in \mathrm{Chain}(wF)$. So, by Lemma 4.7, we have $w\omega(\gamma) = \omega(w\gamma)$. Now by Lemma 4.14, either $F \leq wF$ or vice versa. But since F and wF are of the same dimension, $F = wF$. □

To construct $U(F, C)$, fix the face F. By the local smoothness assumption, since F is contractible, $TX|_{\overline{F}}$ is W_F-equivariantly isomorphic to $\overline{F} \times \mathbb{R}^n$ with W_F acting on $\mathbb{R}^n$ as a linear reflection group. We will identify the two bundles by this isomorphism.

For a point $x \in F$ the tangent space T_xF has an intrinsically defined convex cone comprising of the 'inner' tangent vectors, this is called the *inner sector* (see [22, Section 2] and [1, Section 4.1]). Note that there exists a coordinate neighborhood of x on which the restriction of $\mathcal{A}_x$ is a hyperplane arrangement. The local smoothness assumption tells us that the local arrangement around x, under taking inner sectors, is combinatorially isomorphic to the arrangement in T_xX. Let C_F^T be the chamber in $\mathbb{R}^n$ (under the above identification) that is fiberwise given by the inner sector of C_F.

Now by definition of r_F, $T\omega(F)$ is W-equivariantly isomorphic to $r_F^*(TX|_F)$. Let the corresponding bundle map be $\tilde{r}_F : T\omega(F) \to F \times \mathbb{R}^n$ and define $U(F, C)$ to be $\tilde{r}_F^{-1}(F \times C_F^T)$. By construction, $U(wF, wC) = wU(F, C)$ for any $w \in W$.

For $x \in F$, denote $\tilde{r}_F^{-1}(r_F(x) \times C_F^T) \cap T_xX = U(F, C) \cap T_xX$ by $T_x^+C_F$. This is the inner sector of C_F at x.

Lemma 4.21. *Let $(F, C), (G, D) \in \mathrm{Sal}_0(\mathcal{A})$ and $F \leq G$. Let $x \in \omega(F) \cap \omega(G)$. Then $T_x^+C_F \subseteq T_x^+D_G$ iff $C_F \subseteq D_G$. Further, $T_x^+D_G$ is covered by the closures of $T_x^+C_F$ as C varies over the chambers containing F such that $C_F \subseteq D_G$.*

Proof. By construction, $r_G(x) \in \omega(F)$, $r_F(x) = r_F(r_G(x))$, and the homotopy between the identity of $\omega(F) \cap \omega(G)$ and $r_F|_{\omega(F)\cap\omega(G)}$ goes through $r_G|_{\omega(F)\cap\omega(G)}$. So it is enough to show the statement for $r_G(x)$. That is, we can assume $x \in G \cap \omega(F)$. In this case, $T_x^+D_G = D_G^T$.

But then, since $W_G \subseteq W_F$, and r_F is equivariant under W_F and in particular W_G, we have that the fiber isomorphism $T_xX \to T_{r_F(x)}X$ preserves the chambering by W_G. In particular, for a chamber $C \geq F$, $T_x^+C_F$ intersects D_G^T iff $T_x^+C_F \subseteq D_G^T$.

Now by a connectedness argument on $\omega(F) \cap \overline{G}$, D_G^T contains $T_x^+C_F$ iff it contains C_F^T. Using the definition of C_F^T and D_G^T on the tangent space of a point on $F \subseteq \overline{G}$, we see that this happens iff $C_F \subseteq D_G$. □

Lemma 4.22. *Let $(F, C) \in \mathrm{Sal}_0(\mathcal{A})$. Then $U(F, C) \subseteq M(\mathcal{A})$.*

Proof. Let $(x, v) \in U(F, C)$. Let $G \in \mathcal{F}(\mathcal{A})$ be such that $x \in G$. Since $x \in \omega(F)$, $F \leq G$. Suppose for some $w \in W$ that is not identity, $w(x, v) = (x, v)$. Then $wx = x$ and hence, by Lemma 4.20, $w \in W_F$. But then $r_F w = w r_F$ and hence $\tilde{r}_F w = w \tilde{r}_F$. But this means $\tilde{r}_F(v)$ is fixed by $w \in W_F$, which contradicts $\tilde{r}_F(v) \in C_F^T$. □

Lemma 4.23. *Let $(F, C), (G, D) \in \mathrm{Sal}_0(\mathcal{A})$. If $U(F, C) = U(G, D)$, then $(F, C) = (G, D)$.*

Proof. Since $U(F,C)$ is a bundle over $\omega(F)$, it determines $\omega(F)$. By construction, F is the unique minimal face contained in $\omega(F)$. So $F = G$.

So now by Lemma 4.21 we have that $C_F = D_F$. Suppose for the sake of contradiction that $C \neq D$. But then in some neighborhood of a point $x \in F$, C and D must be separated by some wall corresponding to a reflection $r \in W_F$. So C and D are in different components of $X \setminus X_r$. Since X_r occurs as a wall in $\mathcal{A}_F$, C_F cannot then be equal to D_F. □

Lemma 4.24. *We have*

$$M(\mathcal{A}) = \bigcup_{(F,C)\in \mathrm{Sal}_0(\mathcal{A})} U(F,C).$$

Proof. Let $(x,v) \in M(\mathcal{A})$. Let $F \in \mathcal{F}(\mathcal{A})$ be such that $x \in F$. Since (x,v) is not fixed by any non-identity element of W, $\tilde{r}_F(v)$ is not fixed by any non-identity element of W_F. Hence, for some $C \in \mathcal{C}(\mathcal{A})$, $F \leq C$ and $\tilde{r}_F(v) \in C_F^T$. Then $(x,v) \in U(F,C)$. □

Lemma 4.25. *Let $(F,C),(G,D) \in \mathrm{Sal}_0(\mathcal{A})$. If $U(F,C) \cap U(G,D) \neq \varnothing$, then either $(F,C) \preceq (G,D)$ or $(G,D) \preceq (F,C)$.*

Proof. We have, since $U(C,F)$ and $U(G,D)$ are bundles over $\omega(F)$ and $\omega(G)$ respectively, that $\omega(F) \cap \omega(G) \neq \varnothing$. Then by Lemma 4.14, without loss of generality, let $F \leq G$. Now by Lemma 4.21, we get $C_F \subseteq D_G$. □

By induction, we get:

Lemma 4.26. *Let $(F_0,C_0),\ldots,(F_p,C_p) \in \mathrm{Sal}_0(\mathcal{A})$. If $U(F_0,C_0) \cap \cdots \cap U(F_p,C_p) \neq \varnothing$, then up to a permutation of the indices we have $(F_0,C_0) \preceq \cdots \preceq (F_p,C_p)$.*

Lemma 4.27. *Let $(F_0,C_0),\ldots,(F_p,C_p)$ be a chain in $\mathrm{Sal}_0(\mathcal{A})$. Then $U(F_0,C_0) \cap \cdots \cap U(F_p,C_p)$ is non-empty and contractible.*

Proof. By Lemma 4.21 the intersection is a bundle over $\omega(F_0,\ldots,F_p)$ with fiber $(C_0)_{F_0}$, which is contractible. By Lemma 4.19, the base space is contractible. □

Lemma 4.28. *Let $(F,C) \in \mathrm{Sal}_0(\mathcal{A})$ and $w \in W \setminus \{1\}$. Then $wU(F,C) \cap U(F,C) = \varnothing$.*

Proof. If $w \notin W_F$, we are done by Lemma 4.20. If $w \in W_F$, we are done by Lemma 4.21. □

Proof of Theorem 4.3. We have constructed a family $\{U(F,C) \mid (F,C) \in \mathrm{Sal}_0(\mathcal{A})\}$ of open subsets of $M(\mathcal{A})$, with the following properties:

1. The assignment of $U(F,C)$ to (F,C) is one-to-one. This is Lemma 4.23.
2. The sets $U(F,C)$ cover $M(\mathcal{A})$. This is Lemma 4.24.
3. For $w \in W$, $wU(F,C) = U(wF,wC)$. This is by construction.

4. For $(F_0, C_0), (F_1, C_1), \ldots, (F_p, C_p) \in \mathrm{Sal}_0(\mathcal{A})$, the intersection

$$U(F_0, C_0) \cap U(F_1, C_1) \cap \cdots \cap U(F_p, C_p)$$

is non-empty if and only if, up to permutation, we have a chain

$$(F_0, C_0) \preceq (F_1, C_1) \preceq \cdots \preceq (F_p, C_p)\,.$$

For such a chain, the intersection of $U(F_i, C_i)$ is contractible. This is by Lemmas 4.26 and 4.27.

5. If $w \in W \setminus \{1\}$, then $wU(F, C) \cap U(F, C) = \varnothing$. This is Lemma 4.28.

Thus, the nerve of the cover $\{U(F, C)\}$ of $M(\mathcal{A})$ corresponds equivariantly to the order complex of $(\mathrm{Sal}_0(\mathcal{A}), \preceq)$. By the equivariant nerve lemma, we are done. □

4.4 Coxeter equipment and the Salvetti complex

Let X be a smooth l-manifold and $\mathcal{A}$ be a reflection arrangement corresponding to a Coxeter transformation group (W, S). As before, denote the fundamental chamber by C, which we would like to be a simple convex polytope, and its face poset by $\mathcal{F}(C)$.

Consider the surjective map σ from the mirrors of $\overline{C}$ onto the set of generators S of the Coxeter system (W, S) given by sending $C_s \mapsto s$. Let F be a k-face of C, since it is the intersection of k codimension-1 faces, say $\{C_{s_1}, \ldots, C_{s_k}\}$, we can extend the map σ from $\mathcal{F}(C)$ to the power set of S as follows:

$$F = C_{s_1} \cap \cdots \cap C_{s_k} \mapsto \{s_1, \ldots, s_k\}.$$

If for each face F the subgroup generated by $\sigma(F)$ is finite then the we call σ a *Coxeter equipment* of C by (W, S) (see [1, Section 4.2]). The reader can verify that a Coxeter equipment is injective and an order reversing poset map.

Definition 4.7. A subset T of S is said to be an *acceptable (spherical) subset* if the subgroup W_T is finite and the intersection $\bigcap_{s \in T} C_s$ is a non-empty face of C.

We denote by $\mathcal{S}^f_X$ the set of all acceptable subsets. The following lemmas follow from the universal construction of Vinberg (see [31]).

Lemma 4.29. *With the notation as before, the given Coxeter equipment $\sigma\colon \mathcal{F}(C) \to \mathcal{S}^f_X$ is a bijection. Moreover the following properties hold.*

i. For $T, U \in \mathcal{S}^f_X$ we have $T \subseteq U \iff \sigma^{-1}(U) \leq \sigma^{-1}(T)$.
ii. For $T \in \mathcal{S}^f_X$ the stabilizer of the face $F := \sigma^{-1}(T)$ is equal to W_T, and every element of W_T pointwise fixes F.

Proof. [24, Theorem 2.10] □

We set $\mathcal{P}_X^f := \{wW_T \mid w \in W \text{ and } T \in \mathcal{S}_X^f\}$ and order it by inclusion. Since every face in $\mathcal{F}(\mathcal{A})$ is of the form wF for some $w \in W$ and $F \in \mathcal{F}(C)$ one can extend the Coxeter equipment to whole of $\mathcal{F}(\mathcal{A})$ by sending $wF \mapsto wW_F$.

Lemma 4.30. *The above defined extension of the Coxeter equipment gives a bijection between $\mathcal{P}_X^f$ and $\mathcal{F}(\mathcal{A})$. Moreover, the following properties hold.*

i. *Let $v, w \in W$ and $T, U \in \mathcal{S}_X^f$. We have $vW_T \subseteq wW_U$ if and only if $\sigma^{-1}(wW_U) \leq \sigma^{-1}(vW_T)$.*
ii. *Let $v, w \in W$ and $T, U \in \mathcal{S}_X^f$. We have $vW_T \subseteq wW_U$ if and only if $T \subseteq U$ and $w^{-1}v \in W_U$.*
iii. *The restriction of the extended σ gives the bijection between the chambers and the elements of W.*

Before stating the next lemma we recall the following definition from Section 1.2.

Definition 4.8. Let (W, S) be a Coxeter system and T a subset of S. An element $w \in W$ is said to be $(\varnothing, T)$-*minimal* if any one of the following equivalent condition is satisfied.

i. The element w is of minimal length in wW_T;
ii. $\lg(ws) > \lg(w)$ for all $s \in T$;
iii. $\lg(wu) = \lg(w) + \lg(u)$ for all $u \in W_T$.

The following lemma ties the geometry of the arrangement to the combinatorics of W. Recall that for chambers C and D, $\mathcal{R}(C, D)$ is the set of all walls that separate them.

Lemma 4.31. *An element $w \in W$ is $(\varnothing, T)$-minimal for some $T \subseteq S$ if and only if for every reflection $r \in W_T$ the corresponding wall $X_r \notin \mathcal{R}(C, w^{-1}C)$.*

Lemma 4.32. *Let $\mathcal{A}$ be a manifold reflection arrangement in X associated with a Coxeter system (W, S) and the Coxeter equipment σ. Then the relation $\preceq$ defined as:*

$$(v, T) \preceq (w, U) \iff T \subseteq U, w^{-1}v \in W_U, w^{-1}v \text{ is } (\varnothing, T)\text{-minimal}$$

is a partial order on $\mathcal{S}_X^f \times W$.

Proof. [24, Lemma 3.2]. □

Note that the action of W on $\mathcal{S}_X^f \times W$ defined by $w \cdot (T, v) = (T, wv)$ is order preserving. Let C' be a chamber of the form wC and F' be one of its face. Define the map $\Phi\colon \mathrm{Sal}_0(\mathcal{A}) \to \mathcal{S}_X^f \times W$ as follows

$$\Phi(F', C') = \Phi(wF, wC) = (\sigma(F), w)\,.$$

Theorem 4.33. *The map Φ is an order reversing poset isomorphism.*

Proof. [24, Theorem 3.3]. □

Hence the map Φ induces a homeomorphism on the geometric realizations of the corresponding posets. Moreover, the reader can verify that the map is also W-equivariant. Combining the above result with Theorem 4.3 we get the following.

Corollary 4.34. *The geometric realization of $\mathcal{S}_X^f \times W$ is W-equivariantly homotopy equivalent to the associated tangent bundle complement M_W.*

Corollary 4.35. *The homotopy type of N_W depends on the combinatorial type of the fundamental chamber and the chosen Coxeter equipment.*

Chapter 5

Reflection arrangements in spheres

It is well known, in case of spheres, that the finite subgroups of isometries that are generated by reflections are Coxeter groups [9, Chapter 6]. Each reflection in this Coxeter transformation group fixes a codimension-1 subsphere. The complement of this collection is a disjoint union of 'spherical simplices' which are freely permuted by the action. Hence this collection is a *reflection arrangement* on the sphere (in the sense of Definition 4.1). The aim of this chapter is to understand the topology of the tangent bundle complement associated to reflection arrangements in spheres.

The case of 1-dimensional sphere is particularly easy to understand as we explain below. However, the case of higher-dimensional spheres is different; since they are simply connected the intersection of mirrors corresponding to every 2-element spherical subset is non-empty. Consequently all the 2-element subsets are acceptable and we have that $\pi_1(N_W) = A_W$ (that is, the Artin group corresponding to W) and $\pi_1(M_W) = CA_W$ (see [8, Theorem 4.13]). However, these are not $K(\pi,1)$-spaces and it would be interesting to figure out their topological properties and relationship with the standard $K(\pi,1)$ model.

We use results from [17] to find a closed form formula for homotopy type of the tangent bundle complement and analyze its cohomology. In particular, it is shown that for a class of sphere arrangements that exhibit certain antipodal symmetry (*mirrored arrangements*) the tangent bundle complement has the homotopy type of a hyperplane complement wedged with a certain number of spheres.

Definition 5.1. A finite collection $\mathcal{A}$ of codimension-1 sub-spheres in S^l is said to be a *mirrored arrangement* if every sub-sphere is invariant under the antipodal map and there exists a sub-sphere $S_0 \notin \mathcal{A}$ such that restriction of $\mathcal{A}$ to the two connected components of $S^l \setminus S_0$ results in hyperplane arrangement which are combinatorially isomorphic.

For such a mirrored arrangement we denote by $\mathcal{A}^+$ and $\mathcal{A}^-$ the restriction of $\mathcal{A}$ to either hemisphere. We have the following result from [17, Theorem 4.6].

Theorem 5.1. *Let $\mathcal{A}$ be a mirrored sphere arrangement in S^l and let $\mathcal{C}(\mathcal{A}^+)$ be the set of chambers of $\mathcal{A}^+$. Then the tangent bundle complement*

$$M(\mathcal{A}) \simeq \mathrm{Sal}(\mathcal{A}^+) \vee \bigvee_{|\mathcal{C}(\mathcal{A}^+)|} S^l.$$

In order to use the above result we first verify that the reflection arrangements are mirrored.

Theorem 5.2. *Let $\mathcal{A}_W$ be a reflection arrangement in S^l corresponding to a finite Coxeter group W. Then $\mathcal{A}_W$ is a mirrored arrangement.*

Proof. Consider $\mathcal{A}_W$ as the intersection of the unit sphere with the corresponding Coxeter arrangement in $\mathbb{R}^{l+1}$. The hyperplanes in this arrangement are orthogonal to the (pairs of) roots in the root system associated to W (see [21, Section 1.2]). Recall that the roots of a given root system are partitioned into two subsets; one consisting of 'positive' roots and the other consisting of 'negative' roots. More explicitly, choose a linear functional not vanishing on the root system. Declare a root positive if it pairs positively with the functional, else call it negative. Note that the hyperplane corresponding to the functional is not in the arrangement; it intersects each hyperplane in the arrangement in general position at the origin. Choose the equator cut out by this hyperplane as a choice for S_0. Now it is straightforward to check that the sphere arrangement is mirrored. □

In case of a reflection arrangement $\mathcal{A}_W$ (in a sphere) we can achieve a little more; we can give a precise description of $\mathrm{Sal}(\mathcal{A}_W^+)$ and interpret the number $|\mathcal{C}(\mathcal{A}^+)|$ in terms of exponents of W. In order to do that it will be convenient to have an alternative cell structure for the Salvetti complex (which is not simplicial).

5.1 A cell structure for the Salvetti complex

For notational simplicity let the reflection arrangement on S^l be denoted by $\mathcal{A}$, and the induced regular cell structure on S^l by $\mathcal{F}(\mathcal{A})$, as usual. It is well-known that isometries of S^l are precisely restrictions of orthogonal linear maps in $\mathbb{R}^{l+1}$. Hence, let the corresponding hyperplane arrangement in $\mathbb{R}^{l+1}$ be denoted by $\widetilde{\mathcal{A}}$.

We first look at the dual cell structure. For every face F fix a point $x(F) \in F$, say the *barycenter* of F. Note that $\overline{F}$ is homeomorphic to an appropriate-dimensional disc B_F and carries a regular cell structure given by faces $F' \leq F$. For every $G < F$ the barycenter $x(G)$ corresponds to a point y_G of B_F. If $\gamma = (G_0, \ldots, G_k)$ is a chain of faces (with $G_0 < \cdots < G_k$) of F then denote by γ_B the simplex that is the convex hull of the vertices $y_{G_0}, \ldots, y_{G_k}$. Let $\Delta(\gamma)$ be the image of γ_B under the chosen homeomorphism (it need not be the 'spherical convex hull' of $x(G_0), \ldots x(G_k)$). Finally, denote by F^* the union of all those $\Delta(\gamma)$'s that arise from chains beginning in F and call it the *dual cell* of F. The collection of all the dual cells defines a regular cell structure since link of each vertex is a sphere. Denote this dual cell structure by $\mathcal{F}^*(\mathcal{A})$.

We use the same notation $\mathcal{F}^*(\mathcal{A})$ for the face poset of this cell structure with the partial order $\preceq$. In this perspective, $\mathcal{F}^*(\mathcal{A})$ is the dual poset of $\mathcal{F}(\mathcal{A})$, that is, $G^* \preceq F^*$ iff $F \leq G$. Every k-face in $(S^l, \mathcal{F}(\mathcal{A}))$ corresponds to an $(l-k)$-cell in $(S^l, \mathcal{F} * (\mathcal{A}))$ for $0 \leq k \leq l$.

Note that a 0-cell C^* is a vertex of a k-cell F^* in $\mathcal{F}^*$ if and only if the closure $\overline{C}$ of the corresponding chamber contains the $(l-k)$-face F. The symbol $F^* \circ C^*$ will denote the vertex of F^* that is dual to the unique chamber closest to C.

Now given a sphere arrangement $\mathcal{A}$ in S^l construct a regular l-complex $\mathrm{Sal}(\mathcal{A})$ as follows. The 0-cells of $\mathrm{Sal}(\mathcal{A})$ correspond to 0-cells of $\mathcal{F}^*$, which we denote by the pairs $\langle C^*; C^* \rangle$. For each 1-cell $F^* \in \mathcal{F}^*$ with vertices C_1^*, C_2^*, take two homeomorphic copies of F^* denoted by $\langle F^*; C_1^* \rangle$ and $\langle F^*; C_2^* \rangle$. Attach these two 1-cells in $\mathrm{Sal}(\mathcal{A})_0$ (the 0-skeleton) such that

$$\partial \langle F^*; C_i^* \rangle = \{\langle C_1^*; C_1^* \rangle, \langle C_2^*; C_2^* \rangle\}$$

for $i = 1, 2$. We put an orientation on the 1-skeleton $\mathrm{Sal}(\mathcal{A})_1$ by directing each 1-cell $\langle F^*; C^* \rangle$ such that the initial vertex is $\langle C^*; C^* \rangle$.

By induction assume that we have constructed the $(k-1)$-skeleton of $\mathrm{Sal}(\mathcal{A})$, $1 \le k-1 < l$. To each k-cell $G^* \in \mathcal{F}^*$ and to each of its vertex C^* assign a k-cell $\langle G^*; C^* \rangle$ whose face poset is isomorphic to that of G^*. Let $\phi(G^*, C^*) \colon \partial \langle G^*; C^* \rangle \to \mathrm{Sal}(\mathcal{A})_{k-1}$ be the same characteristic map that identifies a $(k-1)$-cell $H^* \subseteq \partial G^*$ with the $(k-1)$-cell $\langle H^*; H^* \circ C^* \rangle \subseteq \partial \langle G^*; C^* \rangle$. Extend the map $\phi(G^*, C^*)$ to the whole of $\langle G^*; C^* \rangle$ and use it as the attaching map, hence obtaining the k-skeleton. The boundary of every k-cell is given by

$$\partial \langle F^*; C^* \rangle = \bigcup_{G^* \prec F^*} \langle G^*; G^* \circ C^* \rangle \ .$$

It is not difficult to see the the simplicial Salvetti complex in Section 4.3 is the barycentric subdivision of the cell complex above, so we end up with the same space. Note that in the above cell structure, k-cells of $\mathrm{Sal}(\mathcal{A})$ correspond to codimension-k faces in $\mathcal{F}(\mathcal{A})$. A similar construction works for the hyperplane arrangements $\widetilde{\mathcal{A}}$, with the same correspondence between faces and cells.

For the sake of completeness let us also describe the above cell structure in terms of the group data. A k-cell of the Salvetti complex corresponds to the pair (T, w) where T is a cardinality k acceptable subset and $w \in W$. The boundary relations are the same order relations described in Lemma 4.32.

5.2 The case of the unit circle

As described in Example 4.1, a Coxeter transformation group acting on S^1 is precisely the finite dihedral group D_{2m}. An action of D_{2m} fixes $2m$ points and consequently the tangent bundle complement is an infinite cylinder with $2m$ punctures. The Salvetti complex has $2m$ 0-cells with labels $(\varnothing, w)$ for every $w \in D_{2m}$. There are $4m$ 1-cells with labels of the form either $(\{s\}, w)$ or $(\{t\}, w)$ for $w \in D_{2m}$. The reader can verify that the boundary of a 1-cell, say $(\{s\}, w)$ is $\{(\varnothing, w), (\varnothing, u)\}$ such that $u^{-1}w \in W_s$. The Salvetti complex inherits the free W-action on the tangent bundle complement. The orbit complex consists of exactly one 0-cell and two 1-cells with both their end points joined at the 0-cell. It has the homotopy type of wedge of two circles. Hence we get the following exact sequence of groups:

$$1 \to F_{2m+1} \hookrightarrow F_2 \twoheadrightarrow D_{2m} \to 1$$

If we denote the generators of F_2 by r and s then F_{2m+1} has the following generators:

$$\langle r^2, s^2, (rs)^n, rs^2r, rsr^2sr, \ldots, \underbrace{rsr\cdots}_{m-1}\epsilon^2\underbrace{srs\cdots}_{m-1}, sr^2s, \ldots, \underbrace{srs\cdots}_{m-1}\epsilon^2\underbrace{rsr\cdots}_{m-1}\rangle \ ,$$

where ϵ is r or s depending on the parity of m.

5.3 The non-essential action of the symmetric group

Let $\widetilde{\mathcal{A}}$ be the non-essential arrangement of type $\mathbf{A}_{l-1}$, i.e., the action of the symmetric group $\mathfrak{S}_l$ on $\mathbb{R}^l$ by permuting coordinates. The intersection of this arrangement with the unit sphere gives a reflection arrangement $\mathcal{A}$ on S^{l-1} as per Definition 4.1. In fact, the fundamental chamber can be recognized as the set of points $(x_1, x_2, \ldots, x_l) \in S^{l-1}$ with $x_1 \geq x_2 \geq \cdots \geq x_l$ (Section 2.1).

The action is non-essential (in the sense that the whole group fixes a one dimensional subspace, as noted below), but the intersection of all the reflecting sub-spheres is S^0. This implies that the induced cell-structure is regular, and the arrangement is mirrored so that Theorem 5.1 is applicable. Under the action of $\mathfrak{S}_l$, the line $\{x_1 = \cdots = x_l\}$ is fixed. Further, the action and the arrangement is invariant under the reflection across the hyperplane $H = x_1 + \cdots + x_l = 0$ (the orthogonal complement of the above line). Considering $H \cap S^{l-1}$ to be the equator, let $\mathcal{A}^+$ and $\mathcal{A}^-$ be the arrangements given by restricting $\mathcal{A}$ to the two hemispheres. Then $\mathcal{A}^+$ and $\mathcal{A}^-$ are combinatorially isomorphic to the essential braid arrangement of the same type (the Vinberg system discussed in Section 2.1), and in particular, have $l!$ chambers.

Setting $M = M(\mathcal{A})$, and using Theorem 5.1 we have

$$M \simeq \mathrm{Sal}(\mathcal{A}^+) \vee \bigvee_{l!} S^{l-1}\,, \tag{5.1}$$

where $\mathrm{Sal}(\mathcal{A}^+)$ has the homotopy type of the pure braid space on l strings. In cohomology, we have the corresponding isomorphism:

$$H^*(M) \cong H^*(\mathrm{Sal}(\mathcal{A}^+)) \oplus \bigoplus_{l!} \mathbb{Z}^{[l-1]}\,, \tag{5.2}$$

where $\mathbb{Z}^{[l-1]} \cong \tilde{H}^*(S^{l-1})$ is $\mathbb{Z}$ in degree $l-1$, and 0 everywhere else.

In [2], Arnol'd computed the cohomology ring of the pure braid space on l strings. Using the same result, and the above isomorphism, we have that $H^*(M)$ is torsion free and its Poincaré polynomial is given by

$$(1+t)(1+2t)\cdots(1+(l-1)t) + l!\,t^{l-1}\,.$$

This completely determines $H^*(M)$.

5.4 Essential actions

Let W be a Coxeter group of rank l acting essentially on $\mathbb{R}^l$ as origin-fixing isometries. That is, the only point of $\mathbb{R}^l$ fixed by W is the origin. Define $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ as above. Note that codimension-k faces of $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ correspond, except for the 0-dimensional face of $\widetilde{\mathcal{A}}$ given by $\{0\}$. Further note that $\mathrm{Sal}(\widetilde{\mathcal{A}})$ is a $K(\pi, 1)$ model for the pure Artin group of type W. Now using the cellular description of the Salvetti complex (Section 5.1), it is evident that $\mathrm{Sal}(\mathcal{A})$ is the $(l-1)$-skeleton of $\mathrm{Sal}(\widetilde{\mathcal{A}})$. Moreover, defining $\mathcal{A}^+$ like above, we have the following analogue to (5.1):

$$\mathrm{Sal}(\mathcal{A}) \simeq \mathrm{Sal}(\mathcal{A}^+) \vee \bigvee_{|\mathcal{C}(\mathcal{A}^+)|} S^{l-1}$$

Consequently the three Salvetti complexes, $\mathrm{Sal}(\mathcal{A})$, $\mathrm{Sal}(\widetilde{\mathcal{A}})$ and $\mathrm{Sal}(\mathcal{A}^+)$, all have isomorphic cohomology in degrees 0 to $l-2$. Recall that the finite reflection groups are also characterized by the fact that the algebras of invariant polynomials of these groups possess algebraically independent systems of generators [3, 21]. For example, in the $\mathfrak{S}_n$ case, these are the elementary symmetric polynomials. Let $m_1+1,\ldots,m_l+1$ be the degrees of the generators of the invariants of W; the numbers $m_1,\ldots,m_l$ are called the *exponents* of W. By the work of Brieskorn [5] generalizing the earlier result of Arnol'd, $M(\widetilde{\mathcal{A}})$ is has torsion-free cohomology, and the following Poincaré polynomial:

$$(1+m_1t)(1+m_2t)\cdots(1+m_lt)$$

This determines the cohomology of $\mathrm{Sal}(\mathcal{A})$ in degrees 0 through $l-2$.

Chapter 6

Future directions

Our work suggests several directions for further research, some regarding continuation of the theme and some about applications to other areas. We end the thesis with a description of 3 problems which we think would attract more attention.

Duality spaces. A connected l-space X with fundamental group π is said to be a *duality space* if the twisted cohomology $H^i(X; \mathbb{Z}\pi)$ is concentrated in dimension $i = l$, where it is free abelian. It has been shown by Davis et al [13] that the complexified complement of a hyperplane arrangement is a duality space. Moreover, under a suitable definition of torus arrangements Davis and Settepanella [12] show that the associated complement is also a duality space. The importance and relevance of duality spaces has been explained in the recent work of Denham et al [15]. In particular they show that for duality spaces the associated resonance varieties (these are certain deeper invariants associated to a space) 'propagate'.

We observe that the arguments for these results, involving Mayer-Vietoris spectral sequence, and a notion of 'small' convex covers, can be adapted to the sphere case. The aim is to compute analogous twisted cohomology of tangent bundle complements for sphere arrangements, and show that these are also duality spaces. It would also be interesting to find other smooth manifolds and Coxeter transformation groups for which the associated tangent bundle complement is a duality space.

Microbundle complements. To generalize the definition of complexified complements to a manifold X, we noted that $\mathbb{C}^n$ can be identified with the tangent bundle of $\mathbb{R}^n$, and looked at TX. Similarly, we can also identify $\mathbb{C}^n$ with $\mathbb{R}^n \times \mathbb{R}^n$, so that the analogous space to form a complement in would be $X \times X$. The natural copy of X in $X \times X$ is the diagonal embedding, and we can remove the subspaces $X_r \times X_r$ for reflections $r \in \mathcal{R}$, to get a free action. In the case of the dihedral group D_{2n} acting on S^1, each reflection fixes 2 points on S^1 and hence 4 points on $\mathbb{T}^2$. Hence the resulting complement space is the two torus with $4n$ punctures, which is homotopy equivalent to the wedge of $4n + 1$ circles. However the tangent bundle complement in this case has the homotopy type of $2n + 1$ circles. So this produces new spaces with W-actions related to the manifold arrangement, and we would like to know what analogues of above results apply in this context.

The diagonal embedding is essential to the above construction, and a general approach to deal with this example could be the theory of microbundles developed by Milnor. The crucial observation is that the local picture at a point on the diagonal is that of the tangent bundle (or for that matter, any rank n vector bundle). Further, the diagonal embedding is a microbundle for topological manifolds, which gives us an obvious generalization to look at. We can also consider more general examples of microbundles, other than the diagonal embedding $X \to X \times X$. Since pull-backs of microbundles exist and are well-behaved, the action of W on X would extend to the total space, and one can form complement spaces, expecting to capture some properties of the reflection action.

Asphericity. The $K(\pi, 1)$ conjecture (Conjecture 2.4) says that the complexified complement is aspherical. However, in the manifold context, at least in sufficiently 'nice' cases, X can be shown to be homeomorphic to a retract of M_W. Hence if X is not aspherical, we cannot expect M_W to be aspherical. When X is aspherical, this is not an obstruction, and a hopeful extension of the $K(\pi, 1)$ conjecture would be to generalize to this situation.

For instance, when $X = S^1$, the action is that of a dihedral group and M_W is always aspherical. The next simplest candidate space would be $\mathbb{T}^2 = S^1 \times S^1$. However, it is not clear what all the possible reflection arrangements on $\mathbb{T}^2$ would be, or how to systematically deal with all of them.

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