Artin groups related to reflections on manifolds (joint work with Priyavrat Deshpande)

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Outline

1. What are Artin groups ?
2. Topology of Artin groups
3. Reflections on manifolds
4. Future Work

Outline

Coxeter groups

Definition (Coxeter system)

A pair (W, S) consisting of a finite set S and an abstract group

$$W = \langle S \mid s^2 = 1,\ (st)^{m_{st}} = 1 \text{ for } s \neq t \rangle$$

where m_{st} are integers ≥ 2 or ∞. If $m_{st} = \infty$, the relation is omitted.

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where m_{st} are integers ≥ 2 or ∞. If $m_{st} = \infty$, the relation is omitted.

- The order of st is exactly m_{st}.
- Then the relations $(st)^{m_{st}} = 1$ can be rewritten as:

$$\underbrace{sts\cdots}_{m_{st}} = \underbrace{tst\cdots}_{m_{st}},$$

which are called braid-like relations.

Artin groups

Definition (Artin System)

A pair (G_W, Σ) corresponding to a Coxeter system (W, S) where $\Sigma = \{\sigma_s \mid s \in S\}$ is a set indexed by S and G_W is the group

$$G_W = \langle \Sigma \mid \underbrace{\sigma_s\sigma_t\sigma_s\cdots}_{m_{st}} = \underbrace{\sigma_t\sigma_s\sigma_t\cdots}_{m_{st}} \rangle.$$

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$$G_W = \langle \Sigma \mid \underbrace{\sigma_s \sigma_t \sigma_s \cdots}_{m_{st}} = \underbrace{\sigma_t \sigma_s \sigma_t \cdots}_{m_{st}} \rangle.$$

- G_W is the Artin group corresponding to W.
- The mapping $\sigma_s \mapsto s$ gives us an exact sequence

$$1 \to PG_W \hookrightarrow G_W \twoheadrightarrow W \to 1$$

  and there is a set-theoretic section from W to G_W.
- The kernel PG_W is called the pure Artin group.

Examples

The order two group

$$W = \langle s \mid s^2 = 1 \rangle = \mathbb{Z}/2$$
$$G_W = \langle \sigma_s \rangle = \mathbb{Z}$$
$$PG_W = \langle 2\sigma_s \rangle = 2\mathbb{Z}$$

The infinite dihedral group

$$W = \langle s, t \mid s^2 = t^2 = 1 \rangle = \mathbb{Z}/2 * \mathbb{Z}/2$$
$$G_W = \langle \sigma_s, \sigma_t \rangle = \mathbb{Z} * \mathbb{Z}$$
$$PG_W = \text{free group on infinitely many generators}$$

More examples: braid and symmetric groups

The symmetric group S_n

Let s_i denote the transposition $(i, i+1)$ for $1 \leq i \leq n-1$. Then

$$S_n = \langle s_1, \ldots, s_{n-1} \mid s_i^2, (s_i s_{i+1})^3, (s_i s_j)^2 \text{ for } |i-j| \geq 2 \rangle.$$

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The braid group on n-strands

The Artin group corresponding to S_n
$$G_{S_n} = \left\langle \sigma_1, \ldots, \sigma_{n-1} \mid \begin{cases} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq 2 \end{cases} \right\rangle.$$

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The pure or colored braid group

The kernel, or the pure Artin group corresponding to S_n.

Outline

Coxeter groups are Reflection groups

Theorem (Bourbaki '68, Tits '68, Vinberg '71)

For every Coxeter group W there exists a finite-dimensional vector space V over $\mathbb{R}$ such that

$$W \hookrightarrow \mathrm{GL}(V)$$

as a discrete group generated by reflections.

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For every Coxeter group W there exists a finite-dimensional vector space V over $\mathbb{R}$ such that

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as a discrete group generated by reflections.

- This is the reflection representation of the Coxeter group.
- W acts properly discontinuously on a certain nonempty, W-stable, open convex cone I.
- If W is finite then $I = V$.
- Let $R = \{wsw^{-1} \mid w \in W, s \in S\}$ – the set of all reflections.
- Each $r \in R$ fixes a hyperplane H_r in V.

The reflection arrangements

Definition

The reflection arrangement corresponding to W is the collection

$$\mathcal{A}_W = \{H_r \mid r \in R\}$$

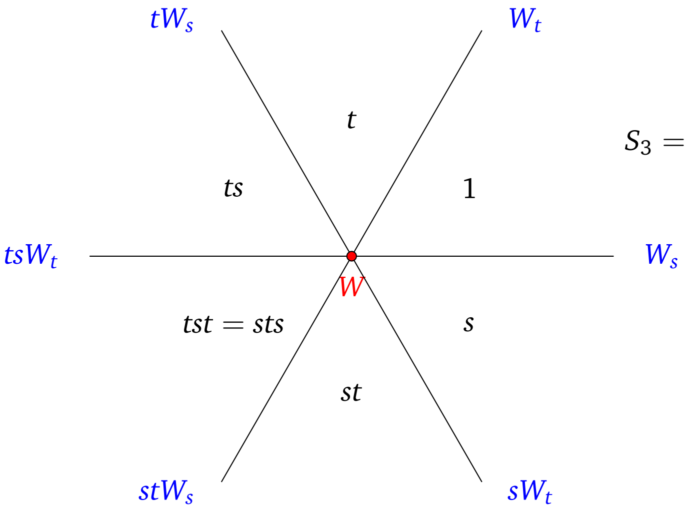
of reflecting hyperplanes in V.

- $\mathcal{A}_W$ stratifies V into cones of varying dimensions.
- The codimension-0 strata are known as chambers.
- Chambers in I are indexed by elements of W.
- Other strata are indexed by parabolic subgroups wW_T, for $w \in W$ and $T \subseteq S$.

Definition

- A *standard parabolic subgroup* of W, given by $T \subseteq S$ and denoted W_T, is the subgroup generated by $\{\sigma_t \mid t \in T\}$.
- A parabolic subgroup is a coset of a standard parabolic subgroup.

Example (The symmetric group S_3)

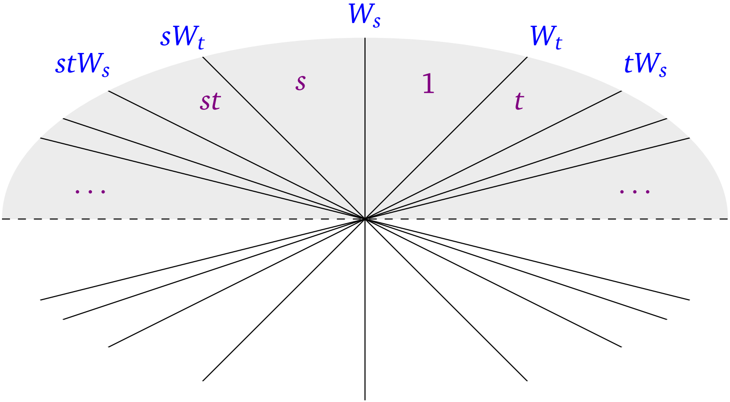


$$V = I = \mathbb{R}^2$$
$$S_3 = \{1, s, t, st, ts, sts\}$$

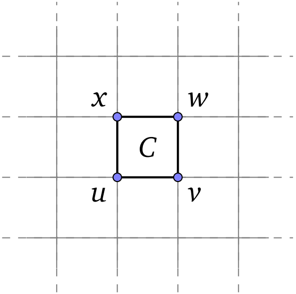
— H_s
/ H_t
\ H_{sts}

Example (The infinite dihedral group D_∞)

$D_\infty = \langle s = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \; t = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \rangle$. I is the upper half plane.



Stratification of V and the face poset



Faces of $\mathcal{A}_W$, intersected with an affine subspace

Faces and chambers

- Each strata F formed by $\mathcal{A}$ is called a face.
- A codimension-0 face is a chamber.
- $\mathcal{F}(\mathcal{A})$ is the set of all faces and $\mathcal{C}(\mathcal{A})$ is the set of chambers.
- Faces are ordered by topological inclusion, i.e. $F \leq G$ if $F \subseteq \overline{G}$.
- On the left, $u \leq uv, u \leq ux, \ldots$ and $uv \leq C, vw \leq C, \ldots$

Complexified complement

The complexified complement of $\mathcal{A}_W$

$$M_W = (V \otimes \mathbb{C}) \setminus \bigcup_{r \in R} (H_r \otimes \mathbb{C}).$$

There is a covering space action of W on M_W. Consequently, we have the quotient

$$N_W = M_W / W.$$

For infinite W, we need to restrict to $I \times I \subseteq V \times V \cong V \otimes \mathbb{C}$.

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Theorem (Arnol'd '68, Brieskorn '71, Van der Lek '83)

1. $\pi_1(M_W) = PG_W$;
2. $\pi_1(N_W) = G_W$.

The short exact sequence associated with the regular covering $M_W \to N_W$ is

$$1 \to PG_W \hookrightarrow G_W \twoheadrightarrow W \to 1.$$

The Salvetti complex

Two constructions

Given $\mathcal{A}_W$, there are two equivalent ways of constructing a regular cell complex $\mathrm{Sal}(\mathcal{A}_W)$, called the Salvetti complex:

1. from the stratification of I;
2. from the group theoretic relations between parabolic subgroups.

Salvetti complex from the face poset

Definition (Salvetti poset)

The elements of the poset are the pairs

$$\{(F,C) \mid F \in \mathcal{F}(\mathcal{A}), C \in \mathcal{C}(\mathcal{A}), F \leq C\},$$

the partial order is given by

$$(F,C) \preceq (G,D) \iff F \leq G, C_G = D_G$$

where the second condition means that no H_r which contains G separates C and D.

Definition (Salvetti complex)

The geometric realization of the Salvetti poset. Denoted by $\mathrm{Sal}(\mathcal{A}_W)$.

Salvetti complex from the group data

Definition (Salvetti poset)

The elements of the poset are the pairs

$$\{(w,T) \mid w \in W, T \subset S, |W_T| < \infty\},$$

the partial order is given by

$$(w',T') \preceq (w,T) \iff \begin{cases} w^{-1}w' \in W_{T'}, \\ \ell(w^{-1}w') < \ell(w^{-1}w't), \ \forall t \in T \end{cases}$$

where ℓ is the length function given by S.

Definition (Salvetti complex)

The geometric realization of the Salvetti poset. Denoted by $\mathrm{Sal}(\mathcal{A}_W)$.

Homotopy equivalence

Theorem (Salvetti '87, Salvetti '94, Charney-Davis '95)

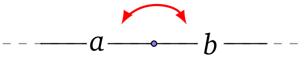
There exists a W-equivariant homotopy equivalence

$$f\colon \mathrm{Sal}(\mathcal{A}_W) \to M_W$$

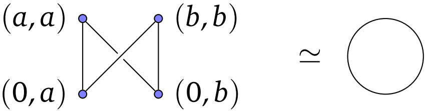
which induces a homotopy equivalence on the quotients.

Example

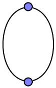
$W = \mathbb{Z}/2$, $M_W = \mathbb{C} \setminus \mathbf{0}$.



The Salvetti poset and its geometric realization $\mathrm{Sal}(\mathcal{A}_W)$:



$\mathrm{Sal}(\mathcal{A}_W)/W$:



Outline

The Setting

- X - a smooth, (connected) real l-manifold.
- $\mathrm{Diff}(X)$ - group of self-diffeomorphims.
- If $r \in \mathrm{Diff}(X)$, then X_r is the set of fixed points of r.

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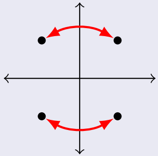
Definition

A reflection on X is $r \in \mathrm{Diff}(X)$ such that

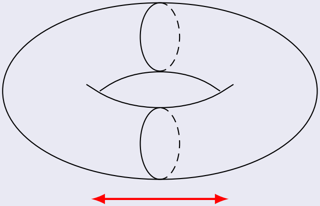
- $r^2 = 1$;
- the complement $X \setminus X_r$ has 2 connected components;
- the map $r_* : T_x(X) \to T_x(X)$ is a usual reflection $\forall x \in X_r$.

Reflections and not reflections

Reflections



$$(x, y) \mapsto (-x, y)$$



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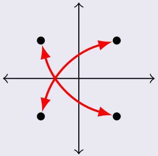

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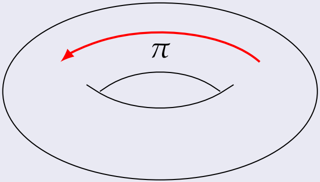

The Setting (contd.)

- A reflection r interchanges the two components of $X \setminus X_r$
- X_r is a codimension-1 submanifold.
- $W \leq \mathrm{Diff}(X)$, a discrete subgroup generated by a finite set S of reflections.
- $R = \{wsw^{-1} \mid w \in W, s \in S\}$ is again the set of reflections in W.
- The map $W \times X \to X \times X$ given by $(w,x) \mapsto (wx,x)$ is smooth and proper, and hence:
  1. Isotropy subgroups W_x are finite.
  2. If $H \leq W$ is finite then $\mathrm{Fix}(H)$ is a locally flat submanifold.
- k-fold intersections of X_r are either empty or of codimension k.

Arrangement of Submanifolds

Theorem (Störm '81, Vinberg '83, Davis '83, Gutkin '87)

If W is a discrete group of diffeomorphisms generated by finitely many reflections which acts on the manifold smoothly and properly then W is a Coxeter group.

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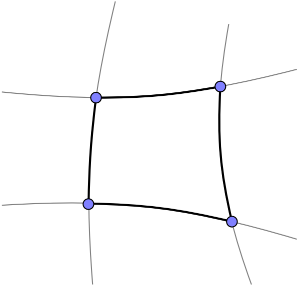
Definition

Reflection arrangement of submanifolds is the finite collection

$$\mathcal{A}_W = \{X_r \mid r \in R\}$$

of codimension-1 submanifolds fixed by reflections in W.

Stratification of X and the face poset



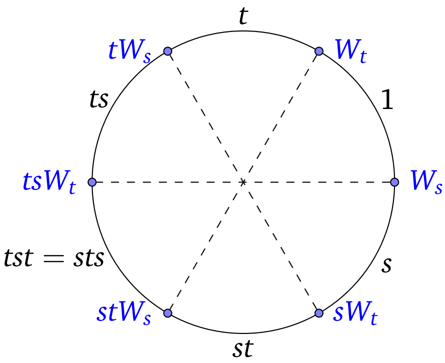
A 2-dimensional chamber.
Face orderings are as before.

Faces and chambers

- $\mathcal{A}_W$ stratifies X.
- Each strata F formed by $\mathcal{A}$ is called a face.
- A codimension-0 face is a chamber.
- $\mathcal{F}(\mathcal{A})$ is the set of all faces and $\mathcal{C}(\mathcal{A})$ is the set of chambers.
- Faces are ordered by topological inclusion, i.e. $F \leq G$ if $F \subseteq \overline{G}$.

Example: S_3 acting on S^1

Restrict the action of S_3 on $\mathbb{R}^2$ to S^1.



Chambers are "nice"

It is known that the chambers are nice manifolds with corners.

Definition

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Theorem (Wiemeler ’11)

If two nice manifolds with corners of the same dimension have isomorphic face posets, then they are diffeomorphic.

Goal and further assumptions

Our goal is to prove Salvetti's theorem in this context. The definition of the Salvetti complex needs to be appropriately adapted.
We require the following assumptions:

- The stratification on X is a regular cell complex structure. That is, there is no identification on the boundary of a chamber.
- The chambers have the face poset of a simple convex polytope P. By Wiemeler's theorem, it is then diffeomorphic to P. This induces diffeomorphisms from each face in $\mathcal{F}(\mathcal{A}_W)$ to a face of P.

Recovering the manifold

Conversely, the manifold X with W-action can be reconstructed from the closed chamber C using the universal construction of Vinberg.

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- Define an equivalence relation on $W \times C$:

$$(g,x) \sim (h,y) \iff x = y,\ gx = hx\,.$$

- $\mathcal{U}(W,C)$ is the quotient space.
- W acts on $\mathcal{U}(W,C)$ by $g \cdot [h,x] = [hg^{-1}, gx]$.

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Theorem (Vinberg, Davis)

The W-equivariant map $\mathcal{U}(W,C) \to X$ induced by the inclusion $C \hookrightarrow X$ is a homeomorphism.

The tangent bundle complement

Recall,

$$\begin{aligned} M_W &= (V \otimes \mathbb{C}) \setminus \Big(\bigcup_{r \in R} H_r \otimes \mathbb{C}\Big) \\ &= TV \setminus \bigcup_{r \in R} TH_r. \end{aligned}$$

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Definition

Given a smooth manifold X and a reflection group W the tangent bundle complement of the corresponding reflection arrangement is defined as follows

$$M_W = TX \setminus \bigcup_{r \in R} TX_r.$$

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W acts properly and fixed-point freely on M_W and hence we have the quotient

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S_3 acting on S^1 (contd.)

- M_W is TS^1 (a cylinder) with 6 punctures.
- It has the homotopy type of $\bigvee_7 S^1$.
- N_W has the homotopy type of $S^1 \vee S^1$.
- For the covering $M_W \to N_W$, we have the following exact sequence on π_1:

$$1 \to F_7 \hookrightarrow F_2 \twoheadrightarrow S_3 \to 1$$

The Salvetti complex

Two constructions

Both the constructions of the Salvetti poset work in the manifold setting:

1. $\mathcal{A}_W$ stratifies X and the construction via the face poset is identical.
2. For the group theoretic construction, need to restrict to acceptable subsets of S.

Definition

A subset T of S is acceptable if $|W_T| < \infty$ and $\bigcap_{s \in T} X_s$ is non-empty.

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Theorem (D.-Deshpande)

The two constructions above produce isomorphic posets, denoted by $\mathrm{Sal}_0(\mathcal{A}_W)$, with the same W-action.

Main theorem

Theorem (D.-Deshpande)

Denote by $\mathrm{Sal}(\mathcal{A}_W)$ *the geometric realization of* $\mathrm{Sal}_0(\mathcal{A}_W)$. *Then there exists a* W*-equivariant homotopy equivalence*

$$f\colon \mathrm{Sal}(\mathcal{A}_W) \to M_W$$

inducing a homotopy equivalence

$$\bar{f}\colon \mathrm{Sal}(\mathcal{A}_W)/W \to N_W$$

on the quotients.

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Proof idea

1. Construct a W-stable good open cover $\mathcal{N}$ of M_W.
2. The non-empty intersections of members of $\mathcal{N}$ equivariantly correspond to chains in $\mathrm{Sal}_0(\mathcal{A}_W)$.
3. Conclude by an equivariant form of nerve lemma.

Proof

Definition

An open cover U_α of a topological space Z is good if each U_α, and non-empty finite intersections of U_α are contractible.

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Given a cover $\mathcal{N}$, the non-empty intersections form an abstract simplicial complex. The geometric realization is the nerve of the cover.

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Given a cover $\mathcal{N}$, the non-empty intersections form an abstract simplicial complex. The geometric realization is the nerve of the cover.

Theorem (nerve lemma)

If $\mathcal{N}$ is a good open cover of a topological space Z, then the nerve of $\mathcal{N}$ is homotopy equivalent to Z.

If a group G acts on Z such that for $g \neq 1$, $U \in \mathcal{N}$, $gU \in \mathcal{N}$ and $U \cap gU = \varnothing$, then G also acts on the nerve, and the homotopy equivalence can be chosen to be G-equivariant.

Proof (contd.)

We want a good cover $\mathcal{N}$ of M_W such that the nerve is (W-equivariantly) isomorphic to $\mathtt{Sal}(\mathcal{A}_W)$. We construct $\mathcal{N} = \{U(F,C) \mid (F,C) \in \mathtt{Sal}_0(\mathcal{A}_W)\}$ such that

1. $\mathcal{N}$ is an open cover of M_W.
2. The assignment of $U(F,C)$ to (F,C) is one-to-one.
3. For $w \in W$, $wU(F,C) = U(wF,wC)$.
4. For $(F_0,C_0),(F_1,C_1),\ldots,(F_p,C_p) \in \mathtt{Sal}_0(\mathcal{A}_W)$, the intersection
$$U(F_0,C_0) \cap U(F_1,C_1) \cap \cdots \cap U(F_p,C_p)$$
is non-empty if and only if, up to permutation, we have a chain
$$(F_0,C_0) \preceq (F_1,C_1) \preceq \cdots \preceq (F_p,C_p)\,.$$
5. For such a chain, the intersection of $U(F_i,C_i)$ is contractible.
6. If $w \in W \setminus \{1\}$, then $wU(F,C) \cap U(F,C) = \varnothing$.

Outline

Work in progress

1. If X is aspherical then so is M_W.
2. Cohomology of M_W with group ring coefficients.
3. M_W is a duality space.
4. The word problem for $\pi_1(M_W/W)$ is solvable.
5. These groups are bi-automatic.

Vanishing cohomology

Theorem (D.-Deshpande)

Let M be the tangent bundle complement of the reflection arrangement in an l-sphere and let $\pi = \pi_1(M)$. Then

$$H^i(M;\mathbb{Z}[\pi]) = \begin{cases} 0, & i \neq n, \\ \bigoplus_{|W|} \mathbb{Z} & i = n. \end{cases}$$

Here $\mathbb{Z}[\pi]$ is a system of local coefficients on M. Hence M is an example of a non-aspherical duality space.

Proof

We construct a nerve of M such that each open set is geodesically convex. Then apply Mayer-Vietoris spectral sequence to compute the cohomology.

The work of Deligne

Definition

A space X is a $K(G,1)$ (an aspherical) space for a discrete group G if $\pi_1(X) = G$ and the universal cover of X is contractible.

Theorem (Deligne 1972)

If G_W is a finite type Artin group then N_W is a $K(G_W,1)$ space.

The $K(\pi,1)$ conjecture

Let (W,S) be any Coxeter system. Then N_W is a $K(G_W,1)$ space for G_W.

Algorithmic properties

Definition

The word problem for a finitely generated group is the problem of deciding whether two words in the generators represent the same element.
The conjugacy problem for a group G with a given presentation is the problem of determining, given two words x and y, whether or not they represent conjugate elements of G.

1. Deligne 1972 : The word problem is solvable for finite type Artin groups.
2. Charney 1992 : Finite type Artin groups are bi-automatic :
    1. the word problem is solvable in quadratic time,
    2. conjugacy problem is solvable,
    3. growth function is computable,
    4. isoperimetric inequalities hold.

Thank You