On the entropy of radiation reaction

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Energy balance

Power radiated by an accelerating charged particle:

$$P = m\tau|\boldsymbol{a}|^2 \quad \text{where} \quad \tau = \frac{q^2}{6\pi\varepsilon_0 mc^3}$$

$(\tau = 2r_e/3c \sim 10^{-23}\,\text{s for an electron})$

Work done on the particle = energy gained by the particle:

$$\int \boldsymbol{f} \cdot d\boldsymbol{x} = -\int P\,dt$$

Energy balance

$$
\begin{aligned}
\int_C \boldsymbol{f} \cdot d\boldsymbol{x} &= \int_{t_1}^{t_2} \boldsymbol{f} \cdot \boldsymbol{v}\, dt \\
&= -m\tau \int_{t_1}^{t_2} \dot{\boldsymbol{v}}^2 dt \\
&= \; - \; m\tau \left[\dot{\boldsymbol{v}} \cdot \boldsymbol{v} \right]_{t=t_1}^{t=t_2} + m\tau \int_{t_1}^{t_2} \ddot{\boldsymbol{v}} \cdot \boldsymbol{v}\, dt
\end{aligned}
$$

$$
\boxed{\implies \; \boldsymbol{f} = m\tau \dot{\boldsymbol{a}}}
$$

...subject to the appropriate boundary conditions.

Lorentz-Abraham equation

Abraham (1905), Lorentz (1909):

$$m\boldsymbol{a} = \boldsymbol{F} + m\tau\dot{\boldsymbol{a}} \quad \text{with} \quad \boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are background (i.e. applied) fields.

Contentious!

- $\boldsymbol{a} = \tau\dot{\boldsymbol{a}}$ when $\boldsymbol{F} = 0$ so $\boldsymbol{a} = \boldsymbol{a}_0 \exp(t/\tau)$ ... runaway solution!
- Introduce integrating factor: $\boldsymbol{a} = -\frac{e^{t/\tau}}{m\tau} \int_{t_0}^{t} \boldsymbol{F}(t')e^{-t'/\tau}\, dt'$ Constant $\boldsymbol{F} \implies m\boldsymbol{a} = \boldsymbol{F}\left(1 - \exp\frac{t-t_0}{\tau}\right)$ so need $t_0 = \infty$ to agree with Newtonian physics ... but $t < \infty$ so the particle must be prescient!

Side-step

Ignore the pathologies! Follow Landau and Lifshitz (1951) and soldier on.

$$m\boldsymbol{a} = \boldsymbol{F} + m\tau\dot{\boldsymbol{a}} \quad \text{with} \quad \boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Reduction of order:

$$\begin{aligned} m\boldsymbol{a} &= \boldsymbol{F} + \tau\dot{\boldsymbol{F}} + \mathcal{O}(\tau^2) \\ &= q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) + \tau\left[q(\dot{\boldsymbol{E}} + \boldsymbol{v} \times \dot{\boldsymbol{B}}) + \frac{q}{m}\boldsymbol{F} \times \boldsymbol{B}\right] + \mathcal{O}(\tau^2) \end{aligned}$$

where $\dot{\boldsymbol{F}} = \partial_t \boldsymbol{F} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{F}$, etc. Then drop $\mathcal{O}(\tau^2)$.

DAB, A Noble“Aspects of electromagnetic radiation reaction in strong fields” Contemp. Phys. 55(2) 110 (2014)

Isolated bunch of identical particles

Focus on salient features : isolated spherical bunch of non-relativistic identical particles. No externally applied fields.

Single particle :

$$m\frac{d^2\boldsymbol{x}}{dt^2} = q\boldsymbol{E}(\boldsymbol{x},t) + q\tau\left[\partial_t\boldsymbol{E}(\boldsymbol{x},t) + \left(\frac{d\boldsymbol{x}}{dt}\cdot\boldsymbol{\nabla}\right)\boldsymbol{E}(\boldsymbol{x},t)\right]$$

Introduce $\boldsymbol{x}(t) = \langle\boldsymbol{x}\rangle(t) + \boldsymbol{\xi}(t)$

- $\langle\boldsymbol{x}\rangle$ denotes the ensemble average of $\boldsymbol{x}$ on the space of initial conditions on the particle.
- $\boldsymbol{\xi}$ is a fluctuation. N.B. $\langle\boldsymbol{\xi}\rangle = 0$.

Isolated bunch of identical particles

$$m\frac{d^2\langle \boldsymbol{x}\rangle}{dt^2} = q\boldsymbol{E}(\langle \boldsymbol{x}\rangle, t) + q\tau\left[\partial_t \boldsymbol{E}(\langle \boldsymbol{x}\rangle, t) + \left(\frac{d\langle \boldsymbol{x}\rangle}{dt}\cdot\boldsymbol{\nabla}\right)\boldsymbol{E}(\langle \boldsymbol{x}\rangle, t)\right]$$

$$\frac{d}{dt}\left(\frac{1}{2}m\langle \dot{\boldsymbol{\xi}}\cdot\dot{\boldsymbol{\xi}}\rangle\right) = \left\{q\langle \dot{\xi}^\mu \xi^\nu\rangle \partial_\nu E_\mu + q\tau[\langle \dot{\xi}^\mu \xi^\nu\rangle \partial_\nu \partial_t E_\mu + \langle \dot{\xi}^\mu \dot{\xi}^\nu\rangle \partial_\nu E_\mu + \langle \dot{x}^\nu\rangle \partial_\omega \partial_\nu E_\mu \langle \dot{\xi}^\mu \xi^\omega\rangle]\right\}\Big|_{\boldsymbol{x}=\langle \boldsymbol{x}\rangle}$$

Initial behaviour

Simple choices :

- Initial velocity and position uncorrelated : $\langle \dot{\xi}^\mu \xi^\nu \rangle|_{t=0} = 0$
- No preferred initial direction : $\langle \dot{\xi}^\mu \dot{\xi}^\nu \rangle|_{t=0} = \delta^{\mu\nu} \langle \dot{\boldsymbol{\xi}} \cdot \dot{\boldsymbol{\xi}} \rangle / 3$

$$\frac{d}{dt}\left(\frac{1}{2} m \langle \dot{\boldsymbol{\xi}} \cdot \dot{\boldsymbol{\xi}} \rangle\right)\bigg|_{t=0} = \left[q\tau \frac{1}{3} \langle \dot{\boldsymbol{\xi}} \cdot \dot{\boldsymbol{\xi}} \rangle \boldsymbol{\nabla} \cdot \boldsymbol{E} \right]\bigg|_{\boldsymbol{x}=\langle \boldsymbol{x} \rangle,\, t=0}$$

Radiation reaction is non-conservative, so a non-zero right-hand side is expected.

Temperature of the bunch

- Suppose that the initial distribution is Maxwell-Boltzmann with temperature T
- Thermal kinetic energy of N particles : $U = N\frac{1}{2}m\langle\dot{\boldsymbol{\xi}}\cdot\dot{\boldsymbol{\xi}}\rangle = 3Nk_BT/2$
- Charge density $\rho(\langle \boldsymbol{x}\rangle, t) = qN/V$ where $\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho/\varepsilon_0$ and V is the volume of the element representing the N particles

$$\left.\frac{dU}{dt}\right|_{t=0} = \left.\left[\tau\frac{k_BT}{m\epsilon_0}\rho^2 V\right]\right|_{\boldsymbol{x}=\langle\boldsymbol{x}\rangle,\, t=0}$$

$$\implies \left.\frac{dT}{dt}\right|_{t=0} = \left.\left[\frac{2\tau q\rho}{3m\epsilon_0}T\right]\right|_{\boldsymbol{x}=\langle\boldsymbol{x}\rangle,\, t=0}$$

The bunch (initially) heats up!

Entropy of an isolated bunch

- Appeal to the first law of thermodynamics : dU = T dS - p dV
- $dV/dt|_{t=0} = 0$ because $V \propto \langle \boldsymbol{\xi} \cdot \boldsymbol{\xi} \rangle^{3/2}$ and $\langle \dot{\boldsymbol{\xi}} \cdot \boldsymbol{\xi} \rangle|_{t=0} = 0$

$$\left.\frac{dS}{dt}\right|_{t=0} = \left.\left[\tau \frac{k_B}{m\epsilon_0}\rho^2 V\right]\right|_{\boldsymbol{x}=\langle \boldsymbol{x} \rangle,\, t=0}$$

The total entropy of a closed system never decreases, so the above seems ok.
...but is it really ok?

Relativistic considerations

Entropy is described by a 4-vector s^a where $\varrho = -s^a u_a$ is the entropy density measured by an observer with 4-velocity u^a.

The entropy of an isolated system should not decrease.

Local form of the entropy principle : $\partial_a s^a \geq 0$

Entropy of an isolated bunch

(From now on use units in which $c = \epsilon_0 = \mu_0 = 1$)
The Vlasov-Maxwell system induced from the Landau-Lifshitz equation yields :

$$\partial_a s^a = -\tau \frac{k_B}{m} \left(\underbrace{J_a J^a}_{\approx -\rho^2} + \underbrace{4 \frac{q^2}{m^2} T_{ab} S^{ab}}_{\text{rel. corr.}} \right)$$

- J^a is the electric 4-current of the bunch
- S^{ab} is the stress-energy-momentum tensor of the bunch
- T^{ab} is the stress-energy-momentum tensor of the electromagnetic field of the bunch

$$\partial_a s^a \geq 0 \implies \boxed{J_a J^a + 4 \frac{q^2}{m^2} T_{ab} S^{ab} \leq 0}$$

Spherical "cold" bunch

- "Cold" bunch : $S^{ab} = mnV^aV^b$
  - V^a is the 4-velocity of the bunch ($V_aV^a = -1$)
  - n is the proper number density of the bunch ($J^a = qnV^a$)
- $J_aJ^a + 4\frac{q^2}{m^2}T_{ab}S^{ab} \leq 0 \implies \boxed{\mathcal{E} \leq mn/4}$

where $\mathcal{E} = T_{ab}V^aV^b$ is the bunch's electromagnetic energy density.

Work in the bunch frame.

- $\mathcal{E} = \boldsymbol{E}^2/2$ with $|\boldsymbol{E}| = |q|nr/3$ for a homogeneous spherical bunch (radius R, charge Q, mass M) with r the distance from the centre of the bunch

$$\boxed{\frac{Q^2}{6\pi M} \leq R}$$

c.f. $\tau = q^2/6\pi\varepsilon_0 mc^3$. Curious!

Physical example

An approximately spherical bunch exiting a laser-plasma wakefield accelerator is an extremely long ellipsoid in its rest frame.

$$J_a J^a + 4\frac{q^2}{m^2} T_{ab} S^{ab} \leq 0 \implies N \lesssim \frac{L}{2 r_e}$$

Extreme example : $L = 0.26\,\mathrm{mm}$ in rest frame ($\sim 1\,\mu$m in lab frame) yields

$$N \lesssim 4.6 \times 10^{10} \quad (\text{so } Q \lesssim 7.4\,\mathrm{nC})$$

which is merely within an order of magnitude greater than design values!

$$\boxed{\partial_a s^a < 0\,????}$$

A RESOLUTION

Closed system

Total stress-energy-momentum is conserved :

$$\partial_a T^{ab}_{\text{total}} = 0$$

Entropy principle satisfied by *total* entropy 4-current :

$$\partial_a s^a_{\text{total}} \geq 0$$

Conclusion : The stress-energy-momentum tensor $S^{ab} + T^{ab}$ does *not* adequately encode the total stress-energy-momentum of the bunch and its electromagnetic field.

System of *extended* particles

Appeal to the stress-energy-momentum balance of an *extended* charged particle

$$\partial_a (s^{ab} + t^{ab}) = 0$$

- s^{ab} is the particle's stress-energy-momentum tensor
- t^{ab} is the *total* electromagnetic stress-energy-momentum tensor (includes the particle's own contribution to the electromagnetic field)

Split off the purely self-induced contribution to t^{ab} :

$$\partial_a (s^{ab} + t^{ab}_{\text{self}}) = -f^{bc}_{\text{ext}}\, j_c$$

Smooth out the fields of a large number of extended particles :

$$\partial_a (S^{ab} + \underbrace{\Pi^{ab}}_{\text{remnant}}) = -\underbrace{F^{bc} J_c}_{\text{Lorentz}}$$

Charged fluid with radiation reaction

Use $\partial_a T^{ab} = F^{ab} J_b$ to give

$$\partial_a(\underbrace{S^{ab} + \Pi^{ab} + T^{ab}}_{\text{total SEM tensor}}) = 0$$

Resolution :

- The remnant Π^{ab} must be accompanied by a contribution σ^a to the entropy 4-current where $\partial_a(s^a + \sigma^a) \geq 0$
- σ^a must capture a flavour of the structure of the near-zone fields of the extended particles

The details of σ^a are currently unknown.

DAB, A Noble "On the entropy of radiation reaction"
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