Grand Prix of Moscow Workshop Editorial

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Mikhail Tikhomirov & Gleb Evstropov

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A. Centroid Tree

Problem idea, developer: Konstantin Semenov
Editorial: Mikhail Tikhomirov

Distinguish a tree generated by a randomized Kruskal from a *centroid* tree of a random tree with very high precision.

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The idea behind the solution is that a centroid tree has a very special structure that is very unlikely to appear in a random tree, provided the number of vertices is not too small.

A. Centroid Tree

Here is one successful approach for checking if a tree is a centroid tree:

- Note that the minimal-index centroid v of a tree T must stay a centroid of $C(T)$. Try all options for the original centroid v (there are most two).

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- Note that the minimal-index centroid v of a tree T must stay a centroid of $C(T)$. Try all options for the original centroid v (there are most two).
- For each of the subtrees T_i its root u_i (adjacent to v) must be a centroid of T_i. Subsequently, all chlidren of u_i must be centroids of their corresponding trees, and so on. This results in an $O(n)$ DFS algorithm (assuming precomputed subtree sizes).

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- For each of the subtrees T_i its root u_i (adjacent to v) must be a centroid of T_i. Subsequently, all chlidren of u_i must be centroids of their corresponding trees, and so on. This results in an $O(n)$ DFS algorithm (assuming precomputed subtree sizes).

We tried this solution on 10^6 random trees with 60 vertices each, and it never made a mistake!

B. Completely Multiplicative Function

Problem idea, developer, editorial: Mikhail Tikhomirov

Construct a completely multiplicative function (*c.m.f*) such that its first 10^6 prefix sums are within 20 by absolute value.

B. Completely Multiplicative Function

Note that each c.m.f. is defined unambigiously by its values on prime numbers (provided $f(1) \neq 0$). After deciding $f(p)$ for all prime p one can construct all the rest values of f in linear (or almost linear) time using the sieve.

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Choosing random values for $f(p)$ doesn't work (the balance goes out of control). However, it was possible to obtain a suitable set of values using sophisticated greedy approach or local optimization. Further, the source size limit allowed to cram the answer into the source code if it took too long to produce.

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Still, that wasn't an intended solution.

B. Completely Multiplicative Function

Let us put

$$f(p) = \begin{cases} -1, & \text{if } p = 3k + 2, \\ +1, & \text{if } p = 3 \text{ or } p = 3k + 1, \end{cases}$$

and ensure that the n-balance of f is $O(\log n)$.

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One can obtain a similar construction with assigning $f(p) = -1$ iff p mod m is a square non-residue modulo (not too large) m.

C. Even and Odd

Problem idea: Mikhail Tikhomirov
Developer: Ivan Smirnov
Editorial: Mikhail Tikhomirov

Count the number of pairs of vertices v, u in a connected undirected graph such that each $v - u$ path has even/odd number of edges.

C. Even and Odd

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Recall that a *biconnected component* (or a *block*) of a graph is a maximal subset of equivalent edges under equivalency relation "edges e_1 and e_2 share a simple cycle".

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Recall that a *biconnected component* (or a *block*) of a graph is a maximal subset of equivalent edges under equivalency relation "edges e_1 and e_2 share a simple cycle".

Articulation points separate the graph into a *block-cut tree*. Using the block-cut tree, one can determine for each pair of vertices v, u which blocks any $v - u$ path must pass through.

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If all blocks between v and u are bipartite, it follows that $v - u$ paths have fixed parity.

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Otherwise, suppose that at least one non-bipartite block occurs between v and u. First, we assert that

Claim

If v and u are distinct vertices of a non-bipartite block, there is an even-length $v - u$ path and an odd-length $v - u$ path.

The claim above can be proven with case analysis.

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The claim above can be proven with case analysis.

Now consider any path passing through a non-bipartite block B. Due to the claim above, we may exchange the subpath inside B to a subpath of different parity. It follows that no fixed-parity pairs are on different sides of B, or partially inside B.

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Subsequently, for the purpose of the problem we may delete all edges of a non-bipartite block (we don't include thusly disconnected vertices in the answer).

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To sum up: we produce the block decomposition of the graph, erase all edges of non-bipartite blocks, and find the answer for the remaining bipartite graph.

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To sum up: we produce the block decomposition of the graph, erase all edges of non-bipartite blocks, and find the answer for the remaining bipartite graph.

Note that we don't even actually need the explicit block-cut tree. The blocks can be found with simple DFS (modification of articulation points finding algorithm with a stack). The complexity is $O(n + m)$.

D. Great Again

Problem idea: Mikhail Tikhomirov
Developer: Alexander Ostanin
Editorial: Mikhail Tikhomirov

Divide the segment region with elements taking values in $\{-1, 0, 1\}$ so that to maximize (the number of positive-sum segments) - (the number of negative-sum segments).

D. Great Again

There is a straightforward, but slow DP solution. Let ans_k be the answer for the first k elements, and p_k be the sum of the first k elements. Then we must have

$$dp_k = \max_{j=k-r}^{k-l} dp_j + \operatorname{sgn}(p_k - p_j)$$

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To avoid slow linear computation of dp_k we use some data structures. For particular k and b let us put

$$val_{k,b} = \max_{k-r \leqslant j \leqslant k-l, p_j = b} dp_j$$

(note that the maximum on the right-hand side may be taken over an empty set, by a common convention we then put $val_{k,b} = -\infty$).

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We now have

$$dp_k = \max\left(\max_{b<p_k}(val_{k,b}-1), \max_{b=p_k} val_{k,b}, \max_{b>p_k}(val_{k,b}+1)\right).$$

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How do we update $val_{k,b}$ when k changes? Note that at most element enters the range $[k-r, k-l]$, and at most one element leaves. Let us store a queue of relevant positions for each value of b (note that b can be negative), now $val_{k,b}$ is equal to maximum in the b-th queue.

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To push x, popback everyone at most x from the deque, and push x to the back. To pop an element, check if it is at the front of the deque and possibly popfront it. This approach has amortized linear complexity.

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RMQ is the hardest part, thus the complexity is thus $O(n \log n)$.

E. Jumping is Fun

Problem idea: Gleb Evstropov
Developer: Ivan Smirnov
Editorial: Gleb Evstropov

Given a sequence $a_1, a_2, \ldots, a_n$ we define a directed graph $G = (V, E)$, where $|V| = n$ and there is an arc from i to j iff $|j - i| \leqslant a_i$. Find a pair (i, j), such that $\min(\rho(i, j), \rho(j, i))$ is maximum possible. Here, $\rho(i, j)$ denotes the number of arcs on the shortest path from i to j.

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Observation: the set of nodes that can be reached from node i in k steps is a consecutive segment. Denote as $l_k(i)$ its leftmost node and as $r_k(i)$ its rightmost node. Knowing l_k and r_k we can compute

$$l_{k+1}(i) = \max(1, \min_{l_k(i) \leqslant j \leqslant r_k(i)} j - a_j)$$

and

$$r_{k+1}(i) = \min(n, \max_{l_k(i) \leqslant j \leqslant r_k(i)} j + a_j)$$

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To compute and keep all the above functions $l_k(i)$ we need up to $O(n^2)$ time and memory. Instead, we can use the idea similar to sparse tables and binary jumps for LCA, and compute $l_k(i)$ and $r_k(i)$ for all k that are powers of 2. Indeed,

$$l_{2k}(i) = \max(1, \min_{l_k(i) \leqslant j \leqslant r_k(i)} l_k(j)$$

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Using the segment tree to query minimum we can run the above computations in $O(n \log^2 n)$ time. Now, to process the query of minimum distance from i to j (consider $i < j$) we have to find minimum k, such that $r_k(i) \geqslant j$. This can be done in $O(\log^2)$ similar to binary jumps algorithm for LCA and LA problems.

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Now we would like to solve the problem for a fixed value of distance k. Does there exists a pair (i, j), such that $\min(\rho(i, j), \rho(j, i)) \geqslant k$?

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Consider triples $(l_k(i), i, r_k(i))$. We are looking for a pair of triples $(l_k(i), i, r_k(i))$ and $(l_k(j), j, r_k(j))$ such that $i < l_k(j)$ and $r_k(i) < j$. For each j we can query minimum value of $r_k(i)$ on the prefix from 1 to $l_k(j) - 1$. This can be done in linear time after all triples are built.

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The above solution requires $\log n$ iterations of binary search and $O(n \log^2 n)$ time to compute triples, thus $O(n \log^3 n)$ time in total. Note, that we can avoid recomputing triples from scratch every time using the fact that we actually go from largest powers of two to lowest (same as in LCA or LA). Thus, the running time will be reduced to $O(n \log^2 n)$.

F. Online LCS

Problem idea: Gleb Evstropov
Developer: Konstantin Semyonov
Editorial: Gleb Evstropov

Given two initially empty strings s and t you should process online q queries. Each query is to add '$0\widehat{}$' or '$1\widehat{}$' at the end of one of these strings and return the current length of the longest common substring of s and t.

F. Online LCS

Note, that if the current length of the longest common substring is k and we add a new character to the string s, the answer can increase only by 1, i.e. become equal to $k + 1$ and this might happen only if the suffix of string s of length $k + 1$ will be a substring of t.

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First we would like to introduce the solution to the following problem: given a static string t and some p that is know to be its substing, be able to add character c at the beginning of p or at the end of p if $c + p$ or $p + c$ (respectively) is still a substring of t. Also, we would like to be able to remove first character of p.

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Build a suffix automaton of string t. For string p we only need to keep the pointer to the current state v of automaton corresponding to p and $m = |p|$.

F. Online LCS

If the new character c is added at the end of p we should check whether there exists the corresponding arc from state v. If the character c is added at the beginning of p, we should first check whether p is the longest string corresponding to state v, i.e. whether $m = maxlen(v)$.

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If $m < maxlen(v)$ simply check that c is equal to character at corresponding position, otherwise we should find the suffix link going to v using character c. To check this we need to maintain two graphs, one of arcs of suffix automaton and the other of reversed suffix links of automaton.

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If $m < maxlen(v)$ simply check that c is equal to character at corresponding position, otherwise we should find the suffix link going to v using character c. To check this we need to maintain two graphs, one of arcs of suffix automaton and the other of reversed suffix links of automaton.

Finally, to remove a single character from the beginning of p we do nothing or rollback using suffix link. Note that all operations are performed in $O(1)$ time.

F. Online LCS

Moreover, we can add a new operation to this subproblem: add a new character at the end of the string t, as the algorithm of suffix automaton in online itself. The only thing that might change about v is that the state v can be split in two (clone operation in automaton), then we should pick the right one assuming length of p.

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Now, the final step is to be able to adopt the above algorithm to check the suffix of string s of length $k+1$ to be a substring of t. We would keep as a string p from the previous slide any substring $s(i,j)$, such that $|s|-k \leqslant i \leqslant j \leqslant |s|$, $s(i,j)$ is a substring of t, $i = |s|-k$ or $s(i-1,j)$ is not a substring of t, and $j = |s|$ or $s(i,j+1)$ is not a substring of t. Informal, $s(i,j)$ is maximal in some sense, i.e. it cannot be greedily expanded.

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Obviously, if $s(|s| - k, |s|)$ is a substring of t, any its substring is a substring of t, thus it would be the only valid maximal substring and we will detect the case of answer increment.

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From the other hand, the length of a valid substring $s(i, j)$ never decreases by more than 1 in one operation. This only happens if $i = |s| - k$ and a new character is appended to s but the answer doesn't change. Thus, the amorhtized running time of one operation is $O(1)$ and the total running time is $O(q)$.

G. Brawling

Problem idea: Mikhail Tikhomirov
Developer: Nikita Uvarov
Editorial: Mikhail Tikhomirov

There are n people in a row, two people facing each other may fight which results in pushing one of them out of the row. What is the smallest number of people that can be left after a sequence of fights?

G. Brawling

Obviously, if several leftmost people are facing left, they cannot take part in fights, hence they will stay there forever; same for the rightmost people facing right. If there are no more people other than mentioned above, the answer is n.

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Otherwise, there is a group of people in the middle, so that the situation looks like:

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It is easy to see that the middle group can not be cleared, but is possible to leave only one person there: kill all R's except for the leftmost one, then kill all L's with the leftmost one. The answer is then the total size of the extreme groups plus one.

H. I Spy

Author, developer: Alexander Golovanov
Editorial: Mikhail Tikhomirov

This is an interactive problem.

n points $p_1, \ldots, p_n$ with integer coordinates are placed in the plane. We can ask queries "how many points are within distance $\sqrt{R}$ of the point (x, y)?" with integer x, y, R. Guess all n points with a sufficiently small number of queries.

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Let $q = (x, y)$ be any point. Let us determine the unordered (multi)set of (squared) distances $||q - p_1||^2, \ldots, ||q - p_n||^2$.

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Let us do a binary search to find minimal R so that k of the given initial points are within $\sqrt{R}$. Now do this for all $k = 1, \ldots, n$.

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How many queries would we need? Each of n times we do an integer binary search within a range $[0, 4 \cdot 10^{12}]$ (the maximal value of square of a distance).

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How many queries would we need? Each of n times we do an integer binary search within a range $[0, 4 \cdot 10^{12}]$ (the maximal value of square of a distance).

Hence, the number of operations is roughly Kn, where $K = \log_2(4 \cdot 10^{12}) < 42$.

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Intended solution (unoptimized): choose three random points q_1, q_2, q_3 and find unordered distances from each of them to each initial point.

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There are now n circles centered at each of q_1, q_2, q_3. Any point p_i must lie on a triple intersection of some of these circles. We can explicitly find each intersection r of circles centered at q_1 and q_2, and check that:

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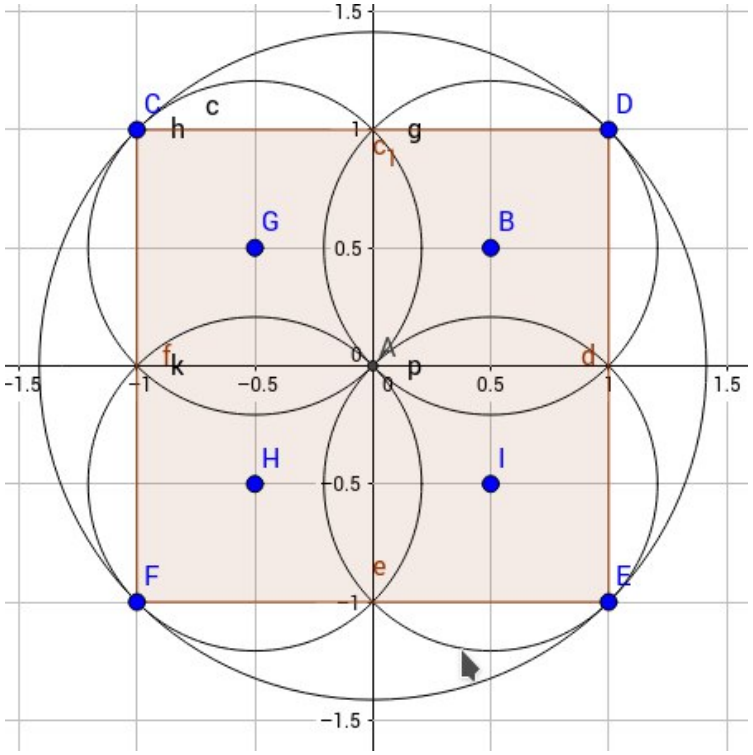
Finally, one may verify that all resulting points are among initial points by making a $R = 0$ query (this is hardly needed since coincidences are highly unlikely if q_1, q_2, q_3 are random). The number of queries is bounded by $3Kn + n < 13500$.

H. I Spy

Better solution (by Nikita Uvarov): perform a quad-tree like descent, with circles covering corresponding squares. Stop descending once the current circle contains no points.



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Note that at each step of the descent the radius of a circle is halved. It is convenient to start with $r = 2^m > \sqrt{2} \cdot 10^6$ so that the initial circle covers all points. Once the radius reaches $r = 2$, try all points inside the circle explicitly.

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Substituting $m \sim 20$, we obtain a solution that makes roughly 8000 queries (not counting $9n \leqslant 900$ queries at the bottom layer).

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Can we estimate the number of queries this solution makes?

Not really, but it is small and fits within the limit nicely.

I. Rage Minimum Query

Problem idea: Gleb Evstropov
Developer: Ivan Smirnov
Editorial: Gleb Evstropov

You are given an array of size n. Initially, all elements are set to ∞. Then m consecutive queries take place. Each query is of form "set element i to be equal to x". After each query we have to report the current minimum value in the array. This problem is classic and an $O(m \log n)$ solution is known. However, in this problem we are guaranteed that queries are random and equiprobable (all valid i and all valid x) and we are asked to find an $O(n + m)$ solution.

I. Rage Minimum Query

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Assuming i is selected equiprobably among $1, 2, \ldots, n$, the probability that $i = p$ is $\frac{1}{n}$. If that happens, we can simply select the minimum in linear time. Thus, the expected running time of a single operation is $\frac{n-1}{n} \cdot O(1) + \frac{1}{n} \cdot O(n) = O(1)$, and the overall expected running time is $O(m)$. Note, that we even didn't use that value of x is also selected at random.

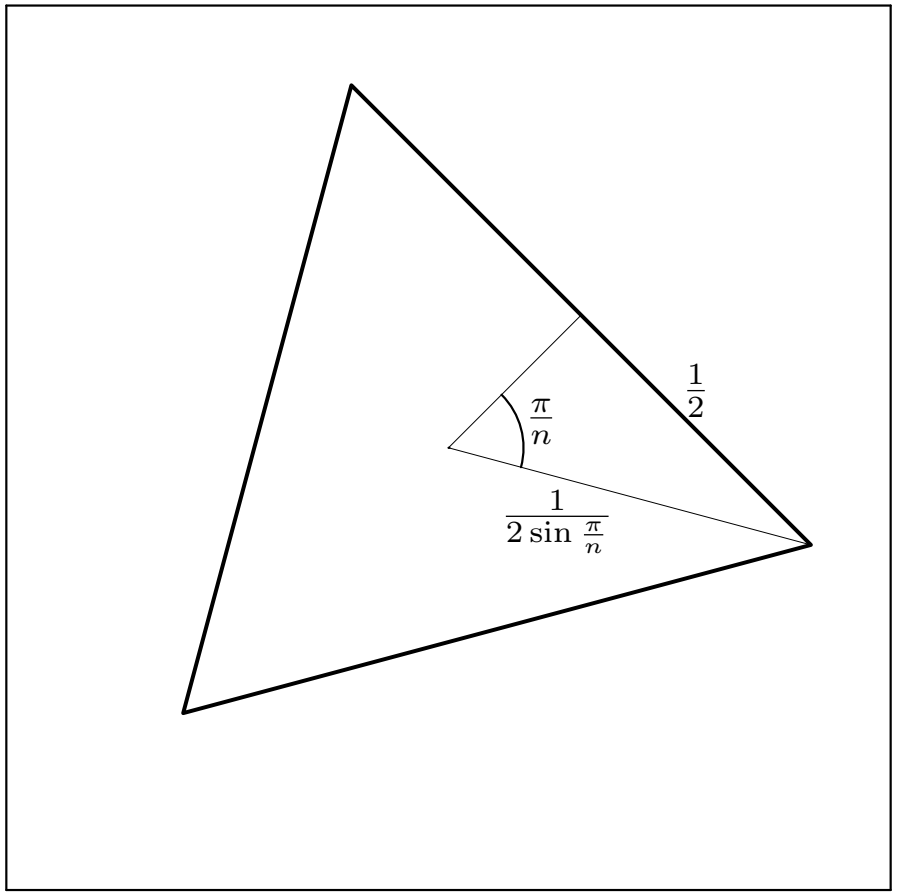
J. Regular Cake

Problem idea: Mikhail Tikhomirov
Developer: Alexander Golovanov
Editorial: Mikhail Tikhomirov

There is a regular n-gon with side length 1. Find the smallest side length of a regular m-gon with coinciding center and containing the n-gone.

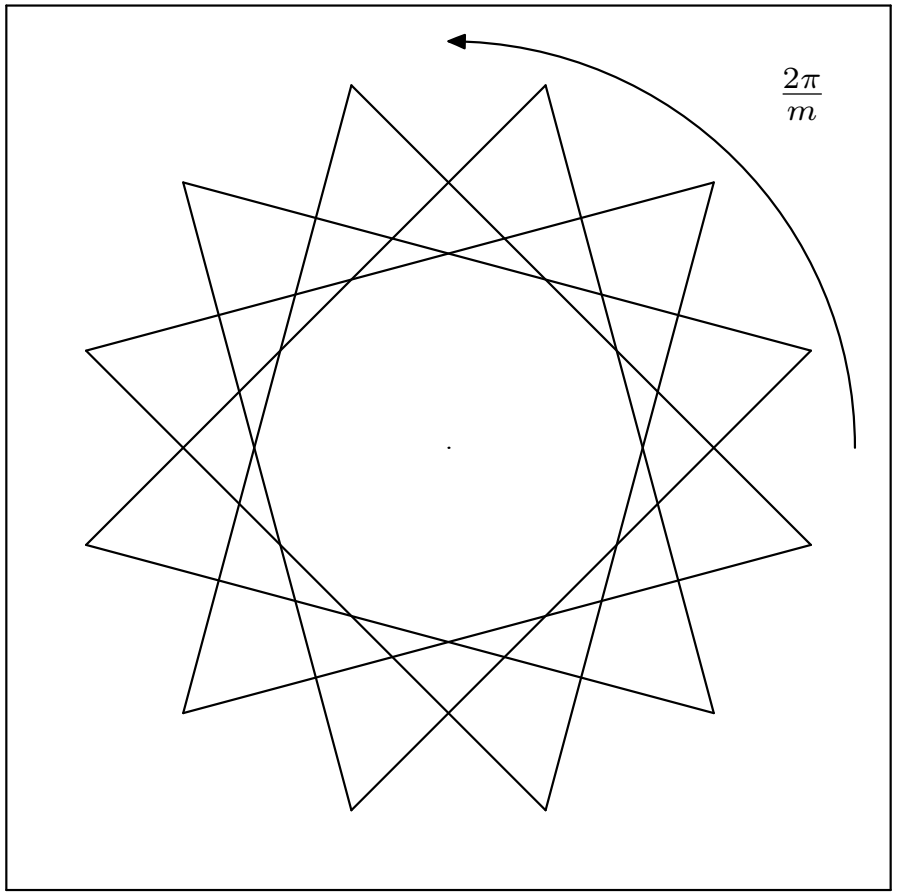
J. Regular Cake

First, compute the radius of the circumscribed circle of the inner polygon.



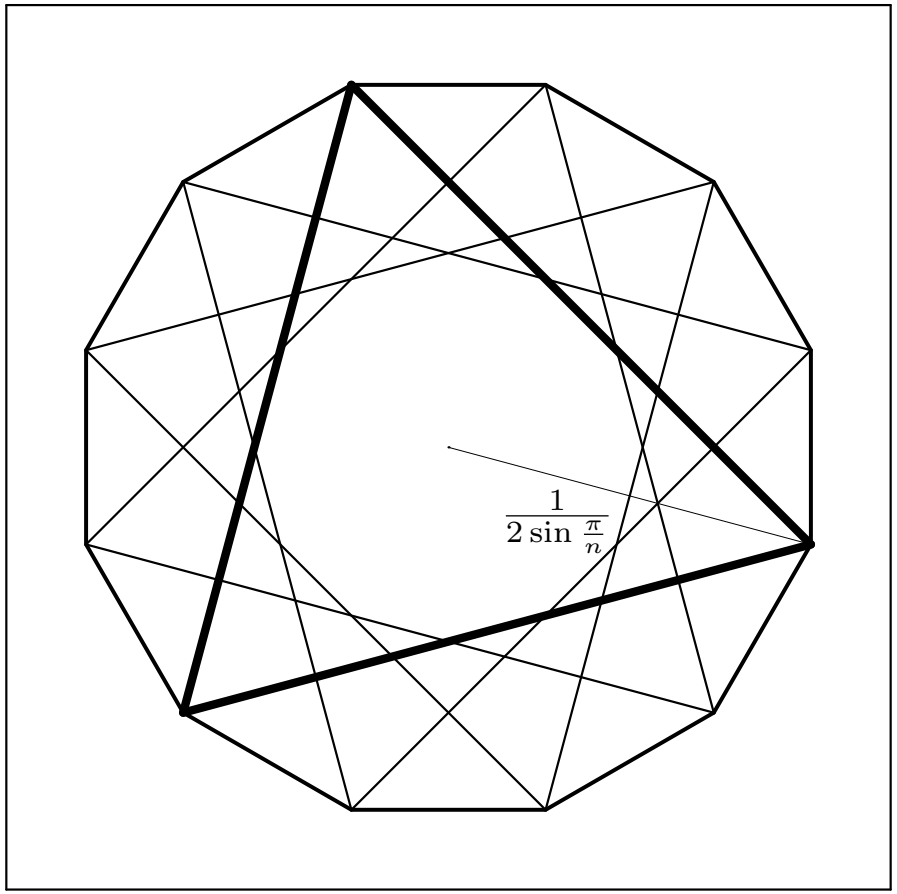
J. Regular Cake

Note that if the whole picture is rotated by $2\pi/m$ any number of times, the outer polygon stays the same.



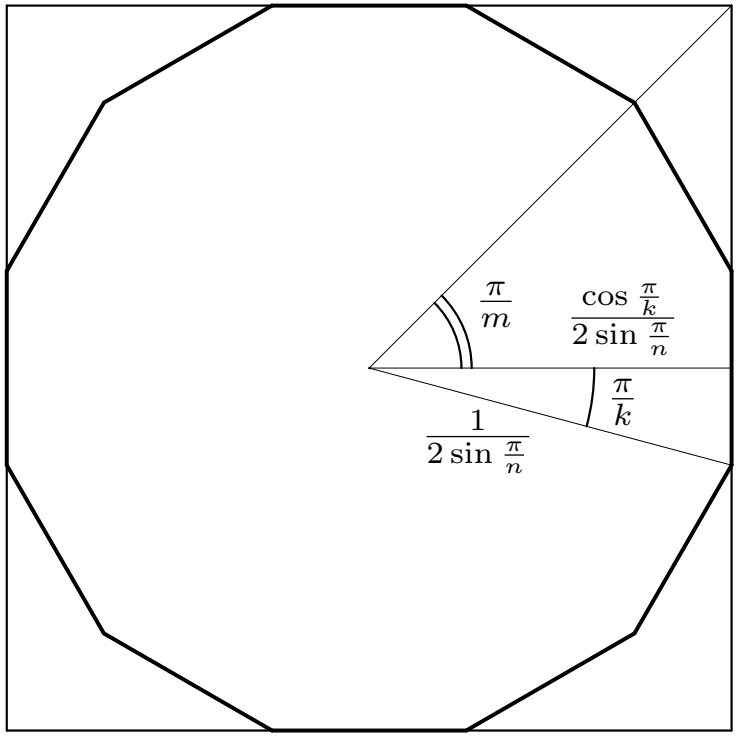
J. Regular Cake

However, the vertices of the inner polygon range equally spaced over the circumscribed circle. It is easy to see that the resulting shape is a regular k-gon, where $k = LCM(n, m)$.



J. Regular Cake

The inscribed circle of the m-gon cannot be smaller than of k-gon, but can be made equal by aligning sides since m divides k. The rest is straightforward trigonometry calculation.



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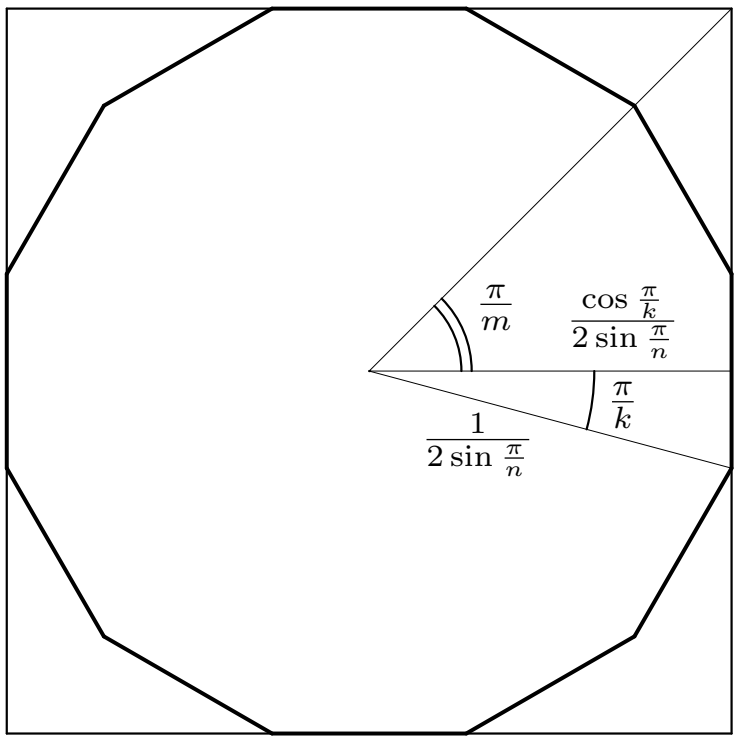

$\frac{\cos\frac{\pi}{k}}{2\sin\frac{\pi}{n}}\tan\frac{\pi}{m}$

The answer is $\frac{\cos(\pi/LCM(n,m))\tan(\pi/m)}{\sin(\pi/n)}$.

K. Piecemaking

Problem idea, developer: Alexander Ostanin
Editorial: Mikhail Tikhomirov

There is a tree with n vertices and $n - 1$ weighted edges. Some of the vertices are colored with black or white. Remove a set of edges with smallest total weight so that no black vertex is reachable from a white vertex.

K. Piecemaking

We will solve the problem via subtree DP. Let $dp_{v,f}$ be the smallest weight of removed edges in the subtree of v so that no conflicts occur, and, additionally:

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- $f = 0$: no colored vertices are reachable from v in the subtree,
- $f = 1$: only white vertices are reachable from v;
- $f = 2$: only black vertices are reachable from v.

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To recalculate $dp_{v,f}$ we process children one by one. Initially $dp_{v,0} = 0$, $dp_{v,1} = dp_{v,2} = \infty$ if v is uncolored, otherwise one of $dp_{v,1}$ and $dp_{v,2}$ is equal to 0.

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The answer is $\min_f dp_{r,f}$, where r is the root of the tree. In total, we obtain an $O(n)$ solution.

Thank you for reading!

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