Problem Set #4 – Capacitors

Solutions

PART A: CONCEPTUAL QUESTIONS

(A) All the processes involve a force that increases as the process proceeds. In (a), the force needed to push water into the vertical pipe has first to overcome the weight of the water already in the pipe. This weight increases as the height of water increases. In (b), the force needed to push more air in the tyre has to increase to overcome the pressure from the air already in the tyre. In (c) the force that the spring exerts increases with compression, and in (d), as negative charge accumulates on the metal plates it becomes harder to force more charge onto it. The dissimilarities of interest lie in (b). Although this example has many qualitative features that make it similar to the others, the fact that the forces involved are not conservative means that the analysis is very different.

(B) In (a), apart from the mechanical advantage provided by the pump (which is a constant and depends on the cross sectional area of the pipe and area of the pump diaphragm), it is the difference between the levels of water in the reservoir and in the vertical pipe, the density of water and the gravitational field strength.
In (b), again there is the mechanical advantage of the pump, and the difference in pressure between the air in the tyre and atmospheric pressure. The pressure in the tyre (at constant temperature) depends on the mass of air in the tyre and the volume of the tyre.
In (c), $F = kx$
In (d), the voltage source has to overcome the repulsive force of the electrons already on the plate. These will depend on the number of excess charges on the plate and the surface area of the plate

(C) In (a), the energy stored in the system is: $E_{(a)} = \frac{1}{2} mgh$, where m is the mass of water in the pipe, g is the gravitational field strength and h is the height of the water in the pipe. Since $m = \rho Ah$, (ρ is the density of water and A the cross sectional area of the pipe, Ah is the volume of water in the pipe), $E_{(a)} = \frac{1}{2}\rho Agh^2$ or $E_{(a)} = \frac{1}{2}\frac{g}{\rho A}m^2$. Both these equations are of the type $E = \frac{1}{2}(\text{constant})(\text{variable})^2$. $E_{(a)}$ is also the work required to pump the water into the pipe. [Note: this is an approximation since it is assumed that g does not change with height]

In (b), the situation is somewhat more complex. The energy in the system is the kinetic energy of the gas molecules $E_{kinetic} = \frac{1}{2}m\left(v^2\right)_{avg} N$, where N is the number of gas molecules, m is the mass of the gas molecules and $\left(v^2\right)_{avg}$ is the average of the square velocities of the molecules. It cannot be said that this energy is "stored" in the gas because it cannot all be retrieved, furthermore it cannot be said that it required an amount of work equal to E_{kintic} to raise the gas to that temperature and pressure since there are different ways by which the final state can be achieved and each way requires a different amount of work.

In (c), the energy stored in the spring is $E_{spring} = \frac{1}{2}kx^2$. This also the amount of work required to compress the spring.

In (d), the energy is stored in the electric field, it is equal to $E_{electric} = \frac{1}{2}\frac{Q^2}{C}$, where Q is the amount of excess charge on the plate and C is a constant. This is also the amount of work needed to place the excess charge on the plate.

(D) Since the plates have opposite charge, reducing the space between the plates makes it easier to put charge on the plates. The attractive force produced by the opposite plate lessens the force of repulsion from the charges on the same plate.

(E) Moving the plates apart increases the energy stored in the system because the net repulsive force between the charges on the same plate increases. This energy comes from the work done by the external agent in separating the plates against the force of attraction between the plates.

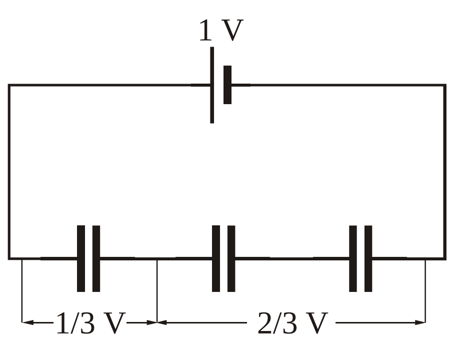
(F) When a dielectric substance is placed in the field between the plates, polarisation of charge takes place in the dielectric. The side of the dielectric nearest the positive plate becomes negative and the side of the dielectric nearest the negative plate becomes positive. The effect of this is similar to bringing the plates closer to each other, and more charge can be placed on the plates with the same battery.

(G) (d)

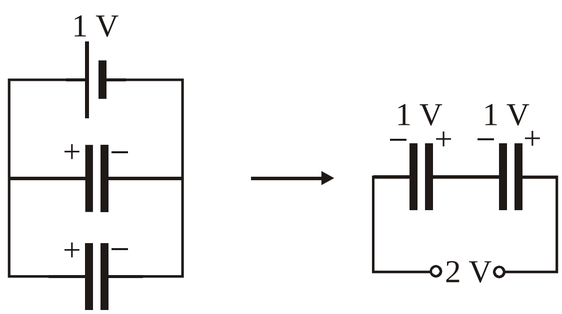
(H) (c)

(I) The answer is yes to all.

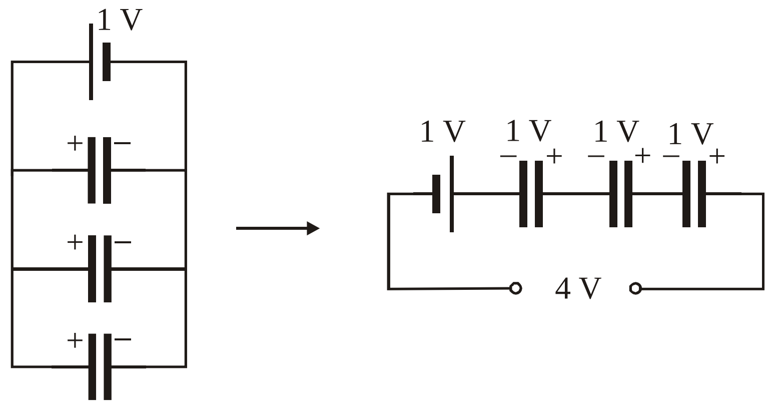
To get 1/3 and 2/3 V:



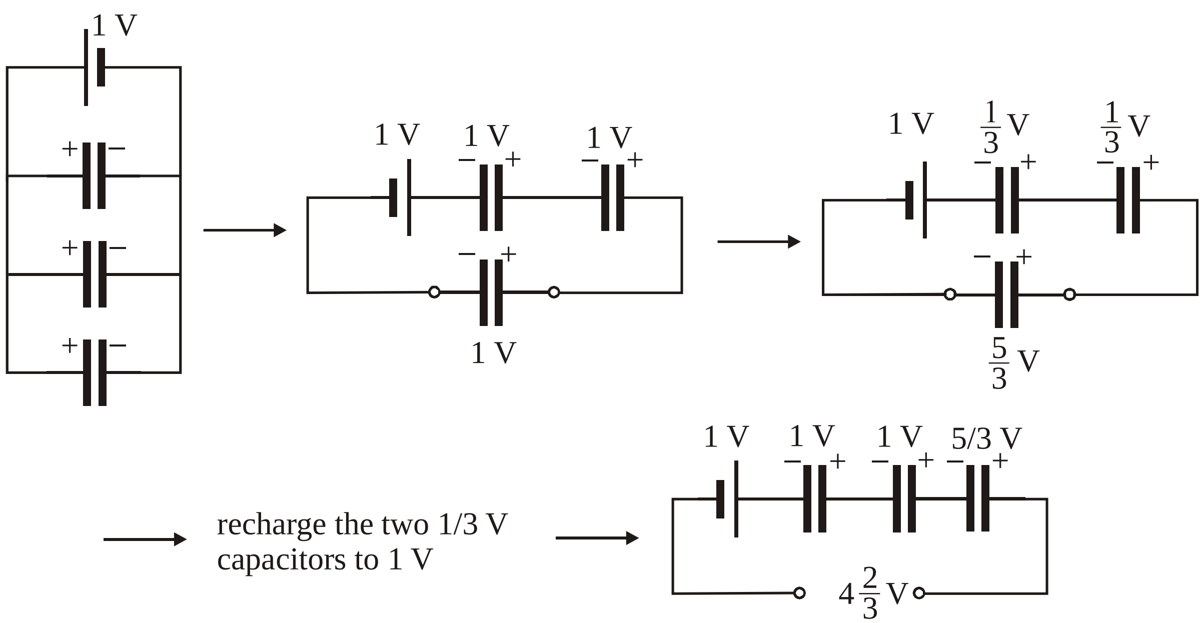
To get 2 V:



To get 4 V:



To get more than 4 V:



PART B: NUMERICAL QUESTIONS

QUESTION 1

$$Q = CV = 80\times10^{-9}\times500 = \boxed{4\times10^{-5}\text{ C}}$$

QUESTION 2
(a) Since the capacitor is disconnected from the charging source, its charge does not change when the dielectric is inserted into it.

$$\Rightarrow Q = V_0 C_0 = VC$$

$$\Rightarrow \frac{V_0}{V} = \frac{C}{C_0} = \frac{200}{50} = 4$$

But $\kappa = \frac{C}{C_0}$ $\Rightarrow \boxed{\kappa = 4}$

(b) The energy stored in the capacitor before the dielectric is inserted is:

$$U_i = \tfrac{1}{2} C_0 V_0^2 = \tfrac{1}{2} \times 2 \times 10^{-3} \times 200^2 = 40 \text{ J}$$

The energy stored in the capacitor after the dielectric is inserted is:

$$U_f = \tfrac{1}{2} CV^2 = \tfrac{1}{2} \kappa C_0 V^2 = \tfrac{1}{2} \times 4 \times 2 \times 10^{-3} \times 50^2 = 10 \text{ J}$$

Work done $= U_f - U_i = \boxed{-30 \text{ J}}$

Since the capacitor has lost energy, it is the electric field in the capacitor that has done the work. The work was done on the dielectric.

QUESTION 3
(a) In order for the electric field across the Pyrex to be its maximum value of 44×10^6 V/m, the space between the plates must be: $d = \frac{6000}{44 \times 10^6} = 1.364 \times 10^{-4}$ m

Since $C = \frac{\kappa \varepsilon_0 A}{d}$

$$A = \frac{Cd}{\kappa \varepsilon_0} = \frac{0.2 \times 10^{-6} \times 1.364 \times 10^{-4}}{5.6 \times 8.85 \times 10^{-12}} = \boxed{0.55 \text{ m}^2}$$

(b) $U = \tfrac{1}{2} CV^2 = \tfrac{1}{2} \times 0.2 \times 10^{-6} \times 6000^2 = \boxed{3.6 \text{ J}}$

QUESTION 4

(a)The equivalent capacitance of the series combination is $C_{eq,S} = \frac{C_2C_1}{C_2+C_1} = \frac{2\times 4}{2+4} = \frac{4}{3}\ \mu\text{F}$.

The charge on the combination is $Q_{eq,S} = Q_{2i} = Q_{1i} = C_{eq,S}V = \frac{4}{3}x10^{-6}x100 = \frac{4}{3}x10^{-4}C$.

[The subscript *i* indicates the initial condition and the subscript *f* indicates the final condition]

When the plates are reconnected positive plate to positive plate and negative plate to negative plate, they form a parallel combination in which the total charge is the sum of the individual charges, i.e. $Q_{eq,P} = \frac{4}{3}x10^{-4} + \frac{4}{3}x10^{-4} = \frac{8}{3}x10^{-4}C$

The equivalent capacitance of the parallel combination is $C_{eq,P} = 4+2 = 6\ \mu\text{F}$

The potential difference across the parallel combination is

$$V_P = V_{2f} = V_{1f} = \frac{Q_{eq,P}}{C_{eq,P}} = \frac{\frac{8}{3}\times 10^{-4}}{6\times 10^{-6}} = 44.4\ \text{V}$$

The charge on the 2 μF capacitor is therefore $Q_{2f} = V_{2f}C_2 = 44.4x2x10^{-6} = 8.9x10^{-5}C$

And the charge on the 4 μF capacitor is $Q_{1f} = V_{1f}C_1 = 44.4x4x10^{-6} = 1.78x10^{-4}C$

4. (b) The initial energy stored in C_1 is $\frac{1}{2}\frac{Q_{1i}^2}{C_1} = \frac{(1.33\times 10^{-4})^2}{2\times 4\times 10^{-6}} = \boxed{2.22\times 10^{-3}\ \text{J}}$

The initial energy stored in C_2 is $\frac{1}{2}\frac{Q_{2i}^2}{C_2} = \frac{(1.33\times 10^{-4})^2}{2\times 2\times 10^{-6}} = \boxed{4.44\times 10^{-3}\ \text{J}}$

The final energy stored in C_1 is $\frac{1}{2}\frac{Q_{1f}^2}{C_1} = \frac{(1.78\times 10^{-4})^2}{2\times 4\times 10^{-6}} = \boxed{3.95\times 10^{-3}\ \text{J}}$

The final energy stored in C_1 is $\frac{1}{2}\frac{Q_{2f}^2}{C_2} = \frac{(8.9\times 10^{-5})^2}{2\times 2\times 10^{-6}} = \boxed{1.98\times 10^{-3}\ \text{J}}$

QUESTION 5

(a) The equivalent capacitance of the 3-μF and 6-μF parallel combination is:

$$C_{eq,3,6} = 3 + 6 = 9\ \mu\text{F}$$

The equivalent capacitance of the 18-μF and 9-μF ($C_{eq,3,6}$) series combination is:

$$C_{eq,18,9} = \frac{18\times 9}{18+9} = 6\ \mu\text{F}$$

The equivalent capacitance of the 4-μF and 12-μF series combination is:

$$C_{eq,4,12} = \frac{4\times 12}{4+12} = 3\ \mu\text{F}$$

The equivalent capacitance of the $C_{eq,4,12}$ and $C_{eq,18,9}$ parallel combination, which is the equivalent capacitance of the circuit, is:

$$C_{eq} = 6 + 3 = \boxed{9\ \mu\text{F}}$$

(b) The 3-μF capacitor is part of the parallel combination $C_{eq,3,6}$. Since the voltage across that combination is 8.0 V, the charge on $C_{eq,3,6}$ is:

$$Q_{eq,3,6} = V_{eq,3,6} C_{eq,3,6} = 8\times 9\times 10^{-6} = 7.2\times 10^{-5}\ \text{C}$$

Since that combination is in series with the 18-μF capacitor, the charge on the 18-μF capacitor is also $7.2\times 10^{-5}\ \text{C}$. This means that the voltage across the 18-μF capacitor is:

$$V_{18} = \frac{7.2\times 10^{-5}}{18\times 10^{-6}} = 4.0\ \text{V}$$

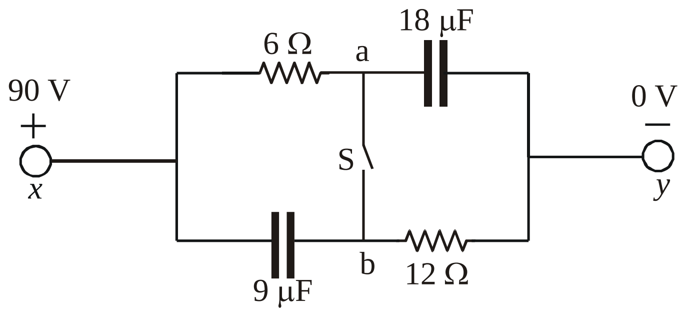
$$\Rightarrow V_{ab} = V_{18} + V_{eq,3,6} = 4.0 + 8.0 = 12.0\ \text{V}$$

The charge on the 4-μF and 12-μF is the same and is equal to the charge on $C_{eq,4,12}$.

$$\Rightarrow Q_4 = Q_{eq,4,12} = V_{ab} C_{eq,4,12} = 12.0\times 3\times 10^{-6} = \boxed{36\ \mu\text{C}}$$

QUESTION 6

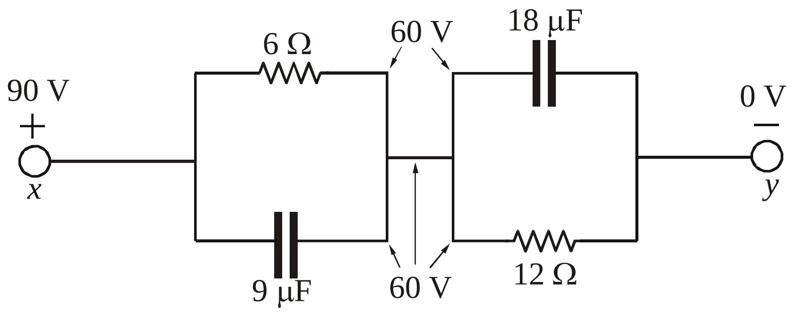
(a) With the switch open, there is zero current through the circuit once the capacitors have become charged. Since the current is zero, the voltage drop across the resistors is also zero, i.e.



$V_a = 90\text{ V},\ V_b = 0\text{ V}$

$\Rightarrow V_{xb} = V_x - V_b = 90 - 0 = \boxed{90\text{ V}}$

(b) With the switch closed, the circuit may be redrawn in the following way:



Current flows through the two resistors, which are now connected in series, and of course, no current flows through the capacitors. The voltage drop across the 6-Ω resistor is 30 V and the voltage drop across the 12-Ω resistor is 60 V. The potential difference across the 9-μF capacitor is $V_9 = 90 - 60 = \boxed{30\text{ V}}$

QUESTION 7

(a) Since the capacitors are in series the charge is the same on both of them and is the same as on the equivalent capacitance that represents them. The equivalent capacitance of the series combination is: $C_{eq} = \frac{4\times 2}{4+2} = \frac{4}{3}\ \mu\text{F}$

The time constant of the charging circuit is: $\tau_C = R_C C_{eq} = 5\times 10^6 \times \frac{4}{3}\times 10^{-6} = \frac{20}{3}\text{ s}$

$$Q(t) = C_{eq}V\left(1 - e^{-t/\tau_C}\right) = \frac{4}{3}\times 10^{-6}\times 20\times\left(1 - e^{-\frac{10}{20/3}}\right) = \boxed{2.07\times 10^{-5}\text{ C}}$$

The current after 10 s of charge is: $I(t) = \frac{V}{R_C}e^{-t/\tau_C} = \frac{20}{5\times 10^6}e^{-\frac{10}{20/3}} = \boxed{8.93\times 10^{-7}\text{ A}}$

(b) When the switch is flipped to B, the capacitors discharge through the 6-MΩ-3-MΩ parallel combination. The equivalent resistance of this combination is $R_D = \frac{3\times 6}{3+6} = 2\ \text{M}\Omega$

The time constant of the discharging circuit is: $\tau_D = R_D C_{eq} = 2\times 10^6 \times \frac{4}{3}\times 10^{-6} = \frac{8}{3}\ \text{s}$

The charge on the capacitors after 10 s of discharge is:

$$Q(t) = C_{eq} V e^{-t/\tau_D} = \frac{4}{3}\times 10^{-6} \times 20 \times e^{-\frac{10}{8/3}} = \boxed{6.27\times 10^{-7}\ \text{C}}$$

The voltage across the 4-μF capacitor at that time is: $V_4 = \frac{Q(10)}{C_4} = \frac{6.27\times 10^{-7}}{4\times 10^{-6}} = \boxed{0.157\ \text{V}}$

The current in the equivalent R after 10 s of discharge is:

$I(t) = \frac{V}{R_D} e^{-t/\tau_D} = \frac{20}{2\times 10^6} e^{\frac{10}{8/3}} = \boxed{2.35\times 10^{-7}\ \text{A}}$ Therefore, the voltage across the equivalent R is $V = RI = 2\times 10^6 (2.35\times 10^{-7}) = 4.7\times 10^{-1} V$ so the current in the 3MΩ resistor will then be: $I(t) = \frac{V}{R} = \frac{4.7\times 10^{-1}}{3\times 10^6} = 1.57\times 10^{-7} A$

QUESTION 8

Experiment 1

(a) Yes

(b) When C_1 is between the large plates it becomes charged. The potential difference between its plates is equal to the potential difference across 1 cm of the field between the large plates, i.e. $\frac{240\ \text{V}}{12\ \text{cm}} \times 1\ \text{cm} = 20\ \text{V}$

The initial current is therefore: $\frac{V}{R} = \frac{20}{4} = \boxed{5\ \text{A}}$

The situation is identical for C_2 so the current is also $\boxed{5\ \text{A}}$

Experiment 2

(a) In the first circuit the capacitors are connected in series, so the initial voltage across the bulbs is twice the voltage of a single capacitor. The bulbs are connected in parallel and therefore each receives that voltage. The bulbs therefore shine more brightly than in experiment 1. However, the capacitors discharge more quickly because the current in the circuit is greater. In the second circuit the voltage supplied by the capacitors in parallel is the same as in experiment 1 but since the bulbs are in series, the current is smaller and so the bulbs shine less brightly. The capacitors discharge less quickly because their combination has more available charge and because the current is smaller.

(b) The bulbs will shine with equal brightness when the current through them is the same. In the first circuit the equivalent capacitance is $C/2 = 0.5\text{ F}$, and the equivalent resistance is $R/2 = 2\ \Omega$. The time constant of the circuit is $\tau_1 = R_1C_1 = 2 \times 0.5 = 1\text{ s}$.
The current through each bulb, as a function of time, in this circuit is given by:
$I_1(t) = \dfrac{V_{01}}{R} e^{-t/\tau_1} = \dfrac{40}{4} e^{-t/1}$.

In the second circuit the equivalent capacitance is $2C = 2\text{ F}$, and the equivalent resistance is $2R = 8\ \Omega$. The time constant of the circuit is $\tau_2 = R_2C_2 = 8 \times 2 = 16\text{ s}$.
The current as a function of time in this circuit is given by: $I_2(t) = \dfrac{V_{02}}{R_2} e^{-t/\tau_2} = \dfrac{20}{8} e^{-t/16}$.

$I_1 = I_2$ when $\dfrac{40}{4} e^{-t/1} = \dfrac{20}{8} e^{-t/16}$

$$\Rightarrow \frac{320}{80} = \frac{e^{-t/16}}{e^{-t}} = e^{\frac{15t}{16}} \qquad \Rightarrow \ln 4 = \frac{15t}{16} \qquad \Rightarrow t = \frac{16 \ln 4}{15} = \boxed{1.48\text{ s}}$$