Microeconomic Theory I

General Equilibrium

Andrea Canidio

CEU

Fall 2014

General Equilibrium: demand equal supply in all markets at the same time

Closing the circle: endogenous prices, endogenous wealth

- Agents and firms maximize taking prices as given, but where do these prices come from?
  - Partial equilibrium: the equilibrium price in one market is found by equating supply and demand in that market keeping what happen on all other markets constant.
  - General equilibrium: the equilibrium price vector is such that demand equal supply in all markets simultaneously (more or less, a more rigorous definition later)
- Agents maximize utility given wealth. Where does this wealth come from?

Endowement

Assumption:

wealth is the value of an endowment vector ω (i.e. what an agent owns at the beginning of life)

$$w \equiv p\omega$$

- Budget line:

$$\begin{aligned} p_1x_1 + p_2x_2 &= p_1\omega_1 + p_2\omega_2 \\ px &= p\omega \end{aligned}$$

- Demand is $x(p, p\omega)$

Endowment

- We can define income and substitution effect also in this case.
- The Slutsky equation for this setting:

$$x_i(p, \underbrace{p\omega}_{w})$$

$$\frac{\partial x_i(p,p\omega)}{\partial p_j} = \underbrace{\left.\frac{\partial x_i(p,p\omega)}{\partial p_j}\right|_{p\omega=w}}_{(1)} + \frac{\partial x_i(p,p\omega)}{\partial w}\omega_j$$

$$\text{old Slutsky for (1)}: \frac{\partial x_i(p,w)}{\partial p_j} = \frac{\partial h_i(p,\bar{U})}{\partial p_j} - \frac{\partial x_i(p,w)}{\partial w}x_j(p,w)$$

$$\implies \text{New Slutsky: } \frac{\partial x_i(p,p\omega)}{\partial p_j} = \frac{\partial h_i(p,\bar{U})}{\partial p_j} + \frac{\partial x_i(p,p\omega)}{\partial w}(\omega_j - x_j(p,p\omega$$

Pure Exchange Economy: the only economic activity is trade

Pure exchange economy

Definition

The **total initial endowment** of an economy is $\omega = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_L \end{bmatrix}$ where ω_l is the total amount of good l available.

Definition

An **allocation** is an assignment of commodities to consumers:

$$\langle x_1, \ldots, x_I \rangle$$

where $x_1 = \begin{bmatrix} x_{11} \\ \vdots \\ x_{L1} \end{bmatrix}$ (L goods, I consumers)

Pure exchange economy

Definition

An **initial endowment** is an allocation in the form $\langle \omega_1, \ldots, \omega_I \rangle$ such that $\sum_{i=1}^{I} \omega_i = \omega$.

Definition

An allocation $\langle x_1, \ldots, x_I \rangle$ is **feasible** if $\sum_i x_i \leq \omega$.

Pareto Optimality: welfare property of an allocation

Definition

The **utility profile** induced by an allocation $\langle x_1, \ldots, x_I \rangle$ is $\langle U_1(x_1), \ldots, U_I(x_I) \rangle$.

Definition

An allocation $\langle x_1, \ldots, x_I \rangle$ is said to **Pareto dominate** (PD) another allocation $\langle x_1', \ldots, x_I' \rangle$ if $\langle U_1(x_1), \ldots, U_I(x_I) \rangle > \langle U_1(x_1'), \ldots, U_I(x_I') \rangle$ (i.e. $U_i(x_i) \geq U_i(x_i')$ for all i, with $U_i(x_i) > U_i(x_i')$ for some i).

Definition

A feasible allocation is said to be **Pareto optimal** (PO) if it is not Pareto dominated by any other feasible allocation. The set of PO allocations is called the **contract curve**.

Equilibrium Prices

Definitions

In a pure exchange economy, $x_i(\cdot)$ is the **demand** of household i with endowment ω_i.

$$x_i(p) = \arg\max \{ U_i(x_i) \mid px_i \leq p\omega_i \}$$
$$px_i(p) = p\omega_i \quad \text{(assuming Walras' Law holds)}$$

$z_i(p) = x_i(p) - \omega_i$ is **excess demand**.

Definition

An allocation $\langle x_1, \ldots, x_I \rangle$ is **supported by** price p if $x_i = x_i(p) \quad \forall i$.

Pure exchange economy

Definition

A price p is **market clearing** if

$$\sum_i z_{li}(p) = 0 \quad \text{if } p_l > 0$$

$$\sum_i z_{li}(p) \leq 0 \quad \text{if } p_l = 0$$

Definition

Given an initial endowment $\langle \omega_1, \ldots, \omega_I \rangle$ a **competitive equilibrium (CE) of a pure exchange economy** is a p^* and a $\langle x_1^*, \ldots, x_I^* \rangle$ such that

- $\langle x_1^*, \ldots, x_I^* \rangle$ is supported by p^*
- p^* is market clearing.

Exercise

- two agents with utility functions

$$u_1(x_1, x_2) = x_1 + 2x_2$$

$$u_1(x_1, x_2) = x_1 + \frac{1}{2}x_2$$

- endowments $\omega_{11} + \omega_{12} = 2$ and $\omega_{21} + \omega_{22} = 1$
- let's normalize $p_1 = 1$ and let's call the price of $x_2 = p$

Exercise

Demand of good 1

$$x_{11}(p) = \begin{cases} 0 & \text{if } p < 2 \\ \tau_1 & \text{if } p = 2 \\ \omega_{11} + p\omega_{21} & \text{if } p > 2 \end{cases}$$

with $\tau_1 \in [0, \omega_{11} + p\omega_{21}]$

$$x_{12}(p) = \begin{cases} 0 & \text{if } p < \frac{1}{2} \\ \tau_2 & \text{if } p = \frac{1}{2} \\ \omega_{12} + p\omega_{22} & \text{if } p > \frac{1}{2} \end{cases}$$

with $\tau_2 \in [0, \omega_{12} + p\omega_{22}]$

Exercise

Demand of good 2

$$x_{21}(p) = \begin{cases} \frac{\omega_{11}}{p} + \omega_{21} & \text{if } p < 2 \\ \frac{\omega_{11} - \tau_1}{p} + \omega_{21} & \text{if } p = 2 \\ 0 & \text{if } p > 2 \end{cases}$$

$$x_{22}(p) = \begin{cases} \frac{\omega_{12}}{p} + \omega_{22} & \text{if } p < \frac{1}{2} \\ \frac{\omega_{12} - \tau_2}{p} + \omega_{22} & \text{if } p = \frac{1}{2} \\ 0 & \text{if } p > \frac{1}{2} \end{cases}$$

Exercise

Excess demand for good 1

$$z_{11}(p) = \begin{cases} -\omega_{11} & \text{if } p < 2 \\ \tau_1 - \omega_{11} & \text{if } p = 2 \\ p\omega_{21} & \text{if } p > 2 \end{cases}$$

$$z_{12}(p) = \begin{cases} -\omega_{12} & \text{if } p < \frac{1}{2} \\ \tau_2 - \omega_{12} & \text{if } p = \frac{1}{2} \\ p\omega_{22} & \text{if } p > \frac{1}{2} \end{cases}$$

Exercise

Excess demand for good 2

$$z_{21}(p) = \begin{cases} \frac{\omega_{11}}{p} & \text{if } p < 2 \\ \frac{\omega_{11} - \tau_1}{p} & \text{if } p = 2 \\ -\omega_{21} & \text{if } p > 2 \end{cases}$$

$$x_{22}(p) = \begin{cases} \frac{\omega_{12}}{p} & \text{if } p < \frac{1}{2} \\ \frac{\omega_{12} - \tau_2}{p} & \text{if } p = \frac{1}{2} \\ -\omega_{22} & \text{if } p > \frac{1}{2} \end{cases}$$

Exercise

Equilibrium, Assume first that $\frac{1}{2} < p^\star < 2$. This price that clears the market for good 1 is:

$$z_{11}(p^\star) + z_{12}(p^\star) = 0$$

$$p^\star \omega_{22} - \omega_{11} = 0$$

$$p^\star = \frac{\omega_{11}}{\omega_{22}}$$

hence, $\frac{1}{2} < p^\star < 2$. is an equilibrium price if only if $\frac{1}{2} < \frac{\omega_{11}}{\omega_{22}} < 2$.

(you should check that also the market for good 2 clears)

Exercise

Assume now that $p^\star = \frac{1}{2}$. This price is market clearing if $\exists \tau_2 \in [0, \omega_{12} + p\omega_{22}]$ such that:

$$\tau_2 - \omega_{12} - \omega_{11} = 0$$

in other words, if $\omega_{11} \in [-\omega_{12}, p^\star \omega_{22}]$. Hence, $p^\star = \frac{1}{2}$ is an equilibrium price if only if $\frac{\omega_{11}}{\omega_{22}} \leq \frac{1}{2}$.

(you should check that also the market for good 2 clears)

Exercise

Assume now that $p^\star = 2$. This price is market clearing if $\exists \tau_1 \in [0, \omega_{11} + p\omega_{21}]$ such that:

$$\tau_1 - \omega_{11} + p^\star \omega_{22} = 0$$

in other words, if $-p^\star \omega_{22} \in [-\omega_{11}, p^\star \omega_{21}]$, Hence, $p^\star = 2$ is an equilibrium price if only if $\frac{\omega_{11}}{\omega_{22}} \geq 2$.

(you should check that also the market for good 2 clears)

Conclusion:

For every possible endowment distribution, there exist an equilibrium price and equilibrium allocation (the equilibrium allocation is ...)

Examples of CE in pure exchanges economy

- Both consumers have linear indifference curves.
- Both consumers have utility $U = \min\{\alpha x, \beta y\}$
- Utilities are: $u_1(x_1, x_2) = |x_1 - \alpha x_2|$, $u_2(x_1, x_2) = |x_1 - \beta x_2|$
- A cares about good 2 only, B has C-D utility function.

General Equilibrium with Firms

General equilibrium with production

- J firms, with production possibilities frontier Y_j
- $y_j(p)$: optimal production plan for firm j

$$y_j(p) = \arg\max_{y} \{py \mid y \in Y_j\}$$

- θ_{ij}: share of firm j owned by consumer i $\left(\sum_i \theta_{ij} = 1\right)$

Definition

An **allocation** is an assignment of goods to consumers and production plans to firms

$$\langle x_1, \ldots, x_I, y_1, \ldots y_J \rangle$$

General equilibrium with production

Definitions

An allocation is **feasible** if $\sum_i x_i \leq \omega + \sum_j y_j \text{ and } y_j \in Y_j \quad \forall j$

An allocation $\langle x_1, \ldots, x_I, y_1, \ldots y_J \rangle$ is **supported by** price p if

$$x_i = x_i(p)$$
$$y_j = y_j(p) \; .$$

A price p is **market clearing** if

$$z_l(p) \equiv \sum_i x_{il}(p) - \sum_i \omega_{il} - \sum_j y_{jl}(p) \begin{cases} = 0 & \text{if } p_l > 0 \\ \leq 0 & \text{if } p_l = 0 \; . \end{cases}$$

where $z_l(p)$ is the aggregate excess demand for good l

General equilibrium with production

- In this case the utility maximization problem is

$$x_i(p) = \arg\max_x \left\{ U_i(x) \mid px \leq p\omega_i + \sum_j \theta_{ij} \underbrace{py_j(p)}_{\pi_j(p)} \right\}$$

- If Walra's law holds, $pz(p) = 0 \quad \forall p$, i.e. the **value** of the aggregate excess demand is zero at any price.
  - individual demand satisfies

$$px_i(p) = p\omega_i + \sum_j \theta_{ij} py_j(p)$$

  aggregate over all consumers

$$p\left(\sum_i [x_i(p) - \omega_i] - \sum_j y_j(p)\right) = 0$$

General equilibrium with production

Definitions

Given an initial endowment $\langle \omega_1, \ldots, \omega_I \rangle$ and shares $\{\theta_{1,1}, \theta_{21}, ...\theta_{I,J}\}$, a **competitive equilibrium (CE)** is an allocation $\langle x_1^*, \ldots, x_I^*, y_1^*, \ldots y_J^* \rangle$ and a price p^* such that

- $\langle x_1^*, \ldots, x_I^* \rangle$ is supported by p^*
- p^* is market clearing.

General equilibrium with production

Walras' Law for markets

Suppose Walras' Law (for demand) holds. Then if $p \gg 0$ and $L - 1$ markets clear, also the Lth market clears.

General equilibrium with production

Proof.

- Walras' Law for demands implies that, for consumer i:

$$
\begin{aligned}
px_i &= p\omega_i + \sum_j \theta_{ij} p y_j \\
\sum_{l<L} p_l x_{il} + p_L x_{iL} &= \sum_{l<L} p_l \left(\omega_{il} + \sum_j \theta_{ij} y_{jl} \right) + p_L \left(\omega_{iL} + \sum_j \theta_{ij} y_{jL} \right)
\end{aligned}
$$

why? Because

$$
\begin{aligned}
p\omega_i + \sum_j \theta_{ij} p y_j = p\omega + \sum_l p_l \sum_j \theta_{ij} y_{jl} = \\
= \sum_l p_l \left(\omega_{il} + \sum_j \theta_{ij} y_{jl} \right)
\end{aligned}
$$

General equilibrium with production

$$\forall i \quad \sum_{l<L} p_l \left(x_{il} - \omega_{il} - \sum_j \theta_{ij} y_{jl} \right) = - p_L \left(x_{iL} - \omega_{iL} - \sum_j \theta_{ij} y_{jL} \right)$$

Sum over all consumers:

$$\sum_{l<L} p_l \left(\sum_i (x_{il} - \omega_{il}) - \sum_j y_{jl} \right) = - p_L \left(\sum_i (x_{iL} - \omega_{iL}) - \sum_j y_{jL} \right)$$

$$\sum_i (x_{il} - \omega_{il}) - \sum_j y_{jl} = 0 \quad (0 \ \forall l < L)$$

$$\sum_i (x_{iL} - \omega_{iL}) - \sum_j y_{jL} = 0 \implies \text{the } L\text{th market also clears}$$

□

****************** Math Aside: fix point theorem **************

Definition

$\hat{x}$ is a **fixed point** if $f(\hat{x}) = \hat{x}$.

Brouwer fixed-point theorem

For X compact and convex, $f : X \to X$, f continuous, $\exists x^* \in X$ such that $f(x^*) = x^*$.

************************ end of math aside ***************

Existence of General Equilibrium

Proposition

Suppose that $z(p)$ is defined for every $p \in \mathbb{R}^L_+$ $(p \neq 0)$, continuous, O^0, and satisfies Walras' Law. Then

$$\exists p^* \text{ such that } z(p^*) \leq 0 \text{ with } \begin{array}{l} z_l(p^*) = 0 \text{ if } p^*_l > 0 \\ z_l(p^*) \leq 0 \text{ if } p^*_l = 0 \,. \end{array}$$

- This is the simplest GE existence proof there is!
- Very restrictive assumptions in terms of underlying preferences. Why?

Existence of General Equilibrium

Proof.

(1)

- Define:

$$S = \left\{ p \mid p \in \mathbb{R}^L_+, \ \sum_l p_l = 1 \right\} \quad \text{(unit simplex)}$$

- Define a normalization function

$$n(p) = \frac{1}{\sum_l p_l} p$$

i.e. $n(p)$ is a price vector that is element of the unit simplex.

- Remember that:

$$z(n(p)) = z(p) \text{ by } O^0 \text{ of } z(p)$$

Existence of General Equilibrium

(2)

$$\forall l \text{ define } z_l^+(p) = \begin{cases} z_l(p) & \text{for } z_l(p) > 0 \\ 0 & \text{for } z_l(p) \leq 0 \end{cases}$$

$$f(p) \equiv n\left(p + z^+(p)\right) \quad f: S \rightarrow S$$

- Start from a price on the simplex
- increase p_l if excess demand for l is positive
- normalize so that the final price is an element of the simplex
- HINT: we are looking for a CE price that is an element of the simplex.

Existence of General Equilibrium

- $z_l(p)$ continuous $\Rightarrow$ $z_l^+(p)$ continuous $\Rightarrow$ f continuous
- By Brouwer fixed-point theorem,

$$\exists p^* \text{ s.t. } f(p^*) = n\left(p^* + z^+(p^*)\right) = p^*$$

$$p^* = n\left(p^* + z^+(p^*)\right) = \lambda^{-1}\left(p^* + z^+(p^*)\right)$$
$$z^+(p^*) = (\lambda - 1)p^*$$
$$\text{with } \lambda \equiv \sum_l \left(p_l^* + z_l^+(p^*)\right)$$

Existence of General Equilibrium

(3)
We want to show that $p^\star$ is a competitive equilibrium, i.e.

$$z_l(p^*) = 0 \quad \text{if} \quad p_l^* > 0$$
$$z_l(p^*) \leq 0 \quad \text{if} \quad p_l^* = 0$$

Existence of General Equilibrium

- We just established $z^{+}(p^*) = (\lambda - 1)p^*$
- Hence $\lambda - 1 \geq 0$ (by def. of $z^{+}(p^*)$)
- Suppose $\lambda - 1 > 0 \;\Rightarrow\; z_l^{+}(p^*) > 0$ whenever $p_l > 0$
  - In this case, $z_l(p^*) = z_l^{+}(p^*) > 0$
    $\Longrightarrow\; p^* z(p^*) = p^* z^{+}(p^*) = (\lambda - 1)(p^* p^*) > 0$
  - But $p^* z(p^*) = 0$ by Walras' Law $\Longrightarrow$ contradiction
  - we cannot have $\lambda - 1 > 0$

Existence of General Equilibrium

- We know that $\lambda - 1 = 0$, which implies $z^{+}(p^{*}) = 0$, which implies $z(p^{\star}) \leq 0$
- remember that $p^{\star} z(p^{\star}) = 0$ (by walra's law)
  - either $z_l(p^{*}) = 0$
  - or $z_l(p^{*}) \leq 0$ and $p_l^{\star} = 0$
- $\implies$ p^{*} is a CE

□

How does Pareto Optimality relate with General Equilibrium?

General equilibrium and welfare

Proposition (1st fundamental welfare theorem)

Suppose that preferences are LNS. Then any CE is PO.

General equilibrium and welfare

Proof.

- Suppose it is not true, i.e. $\exists \langle x_1', \dots, x_I' \rangle$ PD $\langle x_1(p^*), \dots, x_I(p^*) \rangle$
- $\Rightarrow$ $x_i' \succeq x_i(p^*)$ $\forall i$, with $\succ$ for some i
- Then, by definition of demand, $p^* x_i' \geq p^* x_i(p^*) = p^* \omega_i$ for every i, with $>$ for some agent i
    - the agents who strictly prefer the allocation $\langle x_1', \dots, x_I' \rangle$ to the CE allocation must find the allocation $\langle x_1', \dots, x_I' \rangle$ unaffordable at prices $p^\star$
    - all the other agents must value the allocation $\langle x_1', \dots, x_I' \rangle$ at least as much as the CE. By LNS, they must be on their budget line.

General equilibrium and welfare

- Define the aggregate quantities:

$$x' = \sum_i x_i' \;\; x(p^*) = \sum_i x_i(p^*) \quad \omega \equiv \sum_i \omega_i$$

- Sum up across all consumers:

$$p^* x' > p^* x(p^*) = p^* \omega \;\Rightarrow \sum_l p_l^* x_l' > \sum_l p_l^* \omega_l$$

$$\text{for at least one } l, \quad p_l^* x_l' > p_l^* \omega_l$$

$$\Longrightarrow x_l' > \omega_l$$

$$\langle x_1', \ldots, x_I' \rangle \text{ is not feasible}$$

- Any allocation *PD* the CE allocation is not feasible $\Longrightarrow$ the CE allocation is PO

General equilibrium and welfare

Proposition (2nd fundamental theorem of welfare economics)

If the utility functions are continuous, $\succeq_i$ strictly convex and monotonic, then any PO allocation can be implemented as a CE for some vector of initial endowment (ω).

Intuition:

- by redistributing endowment we can reach any PO allocation we may want.
- If preferences are strictly convex, the set of PO allocation is the set of tangency among indifference curves
- by choosing a ω corresponding to the PO allocation to implement, this allocation is also a CE (i.e. prices will be such that nobody wants to trade)

General Equilibrium and Uncertainty

General equilibrium under uncertainty: intuitively.

- We can include **uncertainty** and **time** in the framework just developed by introducing **state-** and **time-contingent commodities**.
  - The idea is that a pizza tomorrow if it rains is a different good compared to a pizza in a week if it snows.
- For simplicity, we only look at the case a single period (i.e. no time) with uncertainty.
- We represent uncertainty as a set of possible ***states of the world:***
  - in period 0, the future is uncertain, in the sense that S states of the world are possible in period 1.
  - in period 1, a state of the world $s \in S$ is realized.
    - in period 1 the economy can be in only one state of the world.
    - once a state of the world is realized there is no more uncertainty.
- We show here that, if consumers can trade state-contingent goods (i.e. promises to deliver) in period 0, then the problem is identical to the one already studied.

General equilibrium under uncertainty: more formally.

- L goods, S states of the world, I consumers with preferences $\succsim_i$ defined over $\mathbb{R}^{LS}$.
- A *state-contingent* allocation is an assignment of goods to consumers in each possible state of the world

$$\langle x_{11}, \dots, x_{I1}, \dots, x_{1S}, \dots, x_{IS} \rangle \in \mathbb{R}^{LS}$$

- A *state-contingent* endowment is an allocation

$$\langle \omega_{11}, \dots, \omega_{I1}, \dots, \omega_{1S}, \dots, \omega_{IS} \rangle \in \mathbb{R}^{LS}$$

- A *state-contingent* allocation is feasible if

$$\sum_{i=1}^{I} x_{i,l,s} \leq \sum_{i=1}^{I} \omega_{i,l,s} \, \forall 1 \leq l \leq L, \, 1 \leq s \leq S$$

General equilibrium under uncertainty: more formally.

- In period 0, $S \cdot L$ *forward markets* open. Each consumer can buy/sell promises to deliver each good in each future state.
- For given price vector $p \in \mathbb{R}^{LS}$, the consumer's problem is

$$\max_{x_i} U_i(x_i)$$

$$s.t.\ px_i \leq p\omega_i$$

where $\omega_i \in \mathbb{R}^{LS}$ is consumer i state-contingent endowment vector, $x_i \in \mathbb{R}^{LS}$ is consumer i state-contingent consumption vector

General equilibrium under uncertainty: the equilibrium.

Definition

A *state-contingent* allocation $\langle x^\star_{11}, \ldots, x^\star_{I1}, ..., x^\star_{1S}, \ldots, x^\star_{IS} \rangle$ and a price vector $p^\star \in \mathbb{R}^{LS}$ constitute an *Arrow-Debreu equilibrium* if

- $\langle x^\star_{11}, \ldots, x^\star_{I1}, ..., x^\star_{1S}, \ldots, x^\star_{IS} \rangle$ is supported by price $p^\star$.
- $p^\star$ is market clearing.

- Note: if in period 0 there exist a market for every possible state-contingent commodity, then the problem is identical to the problem discussed earlier.
  - only difference: instead of L goods we now have LS goods.
- For existence of the equilibrium, welfare properties, ... see the previous part.
- PROBLEM: typically, there are no markets for state contingent commodities (or only a few of them).

Sequential Trade

- Consider the same situation, but where in period 0 there is only one contingent-commodity market.
  - ex. in period 0, consumers exchange promises to deliver good 1 in period 1 in different states of the world.
- Consumer i's problem is now

$$\max_{x_i} U(x_i)$$

$$\begin{aligned} s.t. \sum_{s=1}^{S} q_s z_{is} &\leq 0 \\ p_s x_{is} &\leq p_s \omega_{is} + p_{1s} z_{is} \, \forall s \end{aligned}$$

where z_{is} is the amount of good 1 in state s purchased on period 0, and q_s is its price.

Rational expectations

- This is a sequential problem. First decided how much to purchase of good 1 in each state, then maximize utility contingent on the state.
  - Note that, to simplify, we assumed no consumption and no endowment in period 0.
  - Market clearing in period 0 implies $\sum_i z_{is} \leq 0$ for all s.
- To solve this problem the consumer needs to forecast the price vector in each state of the world.

Assumption: Rational Expectations

All consumers have correct expectations regarding the equilibrium prices in case a given state of the world is realized.

Radner equilibrium

Definition

A state-contingent allocation $\langle x^{\star}_{11}, \ldots, x^{\star}_{I1}, ..., x^{\star}_{1S}, \ldots, x^{\star}_{IS} \rangle$, a period-0 price vector $q^{\star} \in \mathbb{R}^{S}$, and a state-contingent price vector $p^{\star} \in \mathbb{R}^{LS}$ constitute a *Radner equilibrium* if

- $\langle x^{\star}_{11}, \ldots, x^{\star}_{I1}, ..., x^{\star}_{1S}, \ldots, x^{\star}_{IS} \rangle$ is supported by $q^{\star}$ and $p^{\star}$ (i.e. solves the utility maximization problem for each consumer at $q^{\star}$ and $p^{\star}$).
- $q^{\star}$ and $p^{\star}$ are market clearing.

Radner equilibrium

Proposition:

Suppose expectations are rational.

1. If $x^\star \in \mathbb{R}^{ILS}$ and contingent prices $(p_1, ...p_s) \in \mathbb{R}^{LS}$ constitute an Arrow-Debreu equilibrium, then there are prices $q \in \mathbb{R}^S$ for contingent good 1, and consumption plans $z^\star = (z_1^\star, ..., z_I^\star) \in \mathbb{R}^{IS}$ such that $x^\star$, $z^\star$, q and p constitute a Radner equilibrium.
2. If $x^\star \in \mathbb{R}^{ILS}$, $z^\star \in \mathbb{R}^{IS}$, $q \in \mathbb{R}^S$ and $(p_1, ...p_s) \in \mathbb{R}^{LS}$ are a Radner equilibrium, there are multipliers $(\mu_1, ...\mu_s) \in \mathbb{R}^S$ such that the allocation $x^\star$ and the price vector $(\mu_1 p_1, ...\mu_s p_s) \in \mathbb{R}^{LS}$ constitute an Arrow-Debreu equilibrium.

- Equivalence of Radner equilibria and Arrow-Debreu equilibria.
- Therefore, all the theorems/results we derived previously can be applied here as well.

Equivalence of Radner equilibria and Arrow-Debreu equilibria: proof.

(1) Let $q_s = p_{1s}$ for every s. We want to show that, for every consumer i, the budget set of the Arrow-Debreu problem

$$B_i^{AD} = \left\{ (x_{i1}, ..., x_{iS}) \in \mathbb{R}^{LS} : \sum_s p_s (x_{is} - \omega_{is}) \leq 0 \right\}$$

is identical to the budget set of the Radner problem

$$\begin{aligned} B_i^R \; = \; & \Big\{ (x_{i1}, ..., x_{iS}) \in \mathbb{R}^{LS} : \text{ there are} (z_{i1}, ..., z_{iS}) \text{ such that} \\ & \sum_s q_s z_{is} \leq 0 \text{ and } p_s (x_{is} - \omega_{is}) \leq p_1 z_{is} \;\; \text{for all} s \Big\} \end{aligned}$$

Equivalence of Radner equilibria and Arrow-Debreu equilibria: proof.

- Take an $x_i \in B_i^{AD}$. For every s, denote $z_{si} = (1/p_{si})p_s(x_{is} - \omega_{is})$. Then $\sum_s q_s z_{is} = \sum_s p_s z_{is} = \sum_s p_s(x_{is} - \omega_{is}) \leq 0$ and $p_{is} z_{is} = p_s(x_{is} - \omega_{is})$for all s. Hence $x_i \in B_i^R$.
- Take an $x_i \in B_i^R$. It follows that for some $(z_{i1}, ... z_{iS})$ we have $\sum_s q_s z_{is} \leq 0$ and $p_s(x_{is} - \omega_{is}) \leq p_{is} z_{is}$ for every s. Summing over s gives $\sum_s p_s(x_{is} - \omega_{is}) \leq \sum_s p_s z_{is} = \sum_s q_s z_{is} \leq 0$. Hence $x_i \in B_i^{AD}$.

- Therefore, an Arrow-Debreu equilibrium allocation is also a Radner equilibrium allocation supported by $q \in \mathbb{R}^S$, spot market prices $(p_1, ... p_S)$, and contingent trades $(z_{i1}^\star, ..., z_{iS}^\star) \in \mathbb{R}^S$ defined by $z_{si}^\star = (1/p_{si})p_s(x_{is} - \omega_{is})$, $q_s = p_{1s}$.
- Note that the spot markets clear. Why?

Equivalence of Radner equilibria and Arrow-Debreu equilibria: proof.

(2) Choose μ_s so that $\mu_s p_{1s} = q_s$. We can rewrite the Radner budget set as

$$B_i^R = \Big\{(x_{i1}, ..., x_{iS}) \in \mathbb{R}^{LS} : \text{ there are} (z_{i1}, ..., z_{iS}) \text{ such that} \\ \sum_s q_s z_{is} \leq 0 \text{ and } \mu_s p_s (x_{is} - \omega_{is}) \leq q_s z_{is} \ \text{ for all} s \Big\}$$

- Again, summing over all s we get the Arrow-Debreu budget constraint:

$$B_i^R = B_i^{AD} = \left\{ (x_{i1}, ..., x_{iS}) \in \mathbb{R}^{LS} : \sum_s p_s (x_{is} - \omega_{is}) \leq 0 \right\}$$

- Hence, if a consumption plans $x^\star$ maximize the utility of consumer i under the B_i^R constraint, it also maximized the utility of consumer i under the B_i^{AD} constraint (which is true for all consumers). Therefore the price vector $(\mu_1 p_1, ..., \mu_s p_s) \in \mathbb{R}^{LS}$ is an equilibrium price supporting the allocation $x^\star$, and is market clearing.

Financial markets

- The promise to pay one unit of a specific good in a specific state of the world is an example of financial asset.
- More in general, one unit of an **asset** (or **security**) is a title to receive an amount r_s f good 1 if state s occurs.
    - hence, each asset can be fully described by its return vector $r = (r_1, ..., r_s) \in \mathbb{R}^S$.
- Consumers can be long or short on a given asset:
    - can either promise to pay (short) or expect to receive (long) r_s in state s.
- We assume that there are K assets in the economy.
    - it is the set of promises to pay that can be exchanged (and enforced) in a given economy.

Financial markets

Definition

A portfolio plans $z_i^\star = (z_{1i,}^\star ... z_{Ki}) \in \mathbb{R}^K$ for all i, a state-contingent allocation $x_i^\star$ for all i, a period-0 price vector $q^\star \in \mathbb{R}^K$, and a state-contingent price vector $p^\star \in \mathbb{R}^{LS}$ constitute a *Radner equilibrium* if

- $x_i^\star$, $z_i^\star$ solve

$$\max_{x_i} U(x)$$

$$\begin{aligned} s.t. \sum_k q_k z_{ki} &\leq 0 \\ p_s x_{is} &\leq p_s \omega_{is} + \sum_k p_{1s} z_{ik} r_{sk} \; \forall s \end{aligned}$$

  at $q^\star$ and $p^\star$ for all i.

- $q^\star$ and $p^\star$ are market clearing.

- NOTE: this is a generalization of our previous definition.

Financial markets

- We call the *return matrix*

$$R = \begin{bmatrix} r_{11} & \dots & r_{1K} \\ \dots & \dots & \dots \\ r_{S1} & \dots & r_{SK} \end{bmatrix}$$

  where each column represents one asset.
- An asset structure is *complete* if the return matrix R has rank S.
  - that is: there is some subset S of assets with linearly independent returns.

Financial markets

Proposition:

Suppose the asset structure is complete.

1. If $x^\star \in \mathbb{R}^{ILS}$ and contingent prices $(p_1, ...p_s) \in \mathbb{R}^{LS}$ constitute an Arrow-Debreu equilibrium, then there are asset prices $q \in \mathbb{R}^K$ for contingent good 1, and portfolio plans $z^\star = (z_1^\star, ..., z_I^\star) \in \mathbb{R}^{IK}$ such that $x^\star$, $z^\star$, q and p constitute a Radner equilibrium.
2. If $x^\star \in \mathbb{R}^{ILS}$, $z^\star \in \mathbb{R}^{IK}$, $q \in \mathbb{R}^K$ and $(p_1, ...p_s) \in \mathbb{R}^{LS}$ are a Radner equilibrium, there are multipliers $(\mu_1, ...\mu_s) \in \mathbb{R}^S$ such that the allocation $x^\star$ and the price vector $(\mu_1 p_1, ...\mu_s p_s) \in \mathbb{R}^{LS}$ constitute an Arrow-Debreu equilibrium.

Financial markets

- Equivalence of Radner equilibria and Arrow-Debreu equilibria only if the asset structure is complete.
- Therefore, all the theorems/results we derived previously can be applied only if the asset structure is complete.
- If the asset structure is not complete, then the welfare theorems may fail.
  - why should the asset structure not be complete?