Covering Spaces

1 Definitions

Let B be a topological space and G a discrete space. We say that the pair $(B \times F, \mathrm{pr}_1)$, is the trivial cover of B with fiber F, where $\mathrm{pr}_1 : B \times F \to B$ is the first projection.

A trivializable cover of B is a pair (E, p), where E is a topological space and p is a map from E to B such that there is a homeomorphism h from E to the product space $B \times F$ with F discrete, satisfying $\mathrm{pr}_1 \circ h = p$. h is then called a trivialization of (E, p).

A covering space of B is a pair (E, p), where E is a topological space and $p : E \to B$ is a continuous function satisfying the following condition: there is an open cover of B, such that for all U in the cover, the pair $(p^{-1}(U), p_U)$ is a trivializable cover of U, where $p_U : p^{-1}(U) \to U$ is induced by p. We say such an open set trivializes p, that is, the condition means that we can cover B by trivializing open sets.

We call B the base, E the covering space, and p the covering map. If $b \in B$, the topological space $p^{-1}(b)$ is discrete (take an open set containing b that trivializes p so that $p^{-1}(b)$ is isomorphic to $\{b\} \times F$ via h) and is called the fiber at b.

(a) Properties

Proposition 1.1. *If $p : E \to B$ is a covering map, p is an open map. If $y \in E$ and $x = p(y)$, there is a continuous section s of p defined on a neighborhood of x such that $y = s(x)$.*

Proof. Let $y \in E$ and $x = p(y)$. We need to prove that if V is a neighborhood of y, then $p(V)$ is a neighborhood of x. By shrinking V, we can suppose that V is contained in the inverse image of a trivializing neighborhood of x. The statement is clear in this reduced case of a trivial cover (by definition of pr_1 and the topology on $B \times F$). For the existence of a local section, we again consider a trivializing neighborhood of x, and suppose the inverse image of U via p is $U \times F$. Now it suffices to choose a continuous map s from U to F such that $s(x) = y$ (which is always possible). □

Proposition 1.2. *A continuous section s of a covering map, defined on a subset B' of B, is a homeomorphism from B' to $s(B')$.*

Proof. It suffices to note that s is injective. □

Proposition 1.3. *A covering map is a local homeomorphism.*

Proof. Follows from propositions 1.1 and 1.2. □

Proposition 1.4. *If U is a trivializing open set for p, that is, if the pair $(p^{-1}(U), p_U)$ is trivializable, $p^{-1}(U)$ is a disjoint union of open sets V_α such that, for each α, p restricts to a homeomorphism from V_α to U. If U is connected, the V_α are the connected components of $p^{-1}(U)$.*

Proof. The first part follows from the definition of a trivializable cover. For the second, note that $p^{-1}(U)$ is the disjoint union of connected sets V_α. □

Proposition 1.5. *If B' is a subset of B, $(p^{-1}(B'), p_{B'})$ is a cover of B' (the induced cover of B').*

Proposition 1.6. *The fiber cardinality is locally constant. If B is connected, the cardinality is constant.*

Proof. The fiber cardinality is constant on each trivializing open set. □

Proposition 1.7. *Suppose B is connected. Then, there exists a fixed discrete space F such that if U is a trivializing open set, $p^{-1}(U)$ is isomorphic to $U \times F$ (that is, each fiber is isomorphic to F).*

Proof. Let F be any discrete topological space. We show that the set B' of $b \in B$ such that the fiber $p^{-1}(b)$ is isomorphic to F is both open and closed. Suppose B' is non-empty. To show that B' is open, we consider $b \in B'$ and a trivializing open set U containing b. Then the fibers on points of U are all isomorphic (to F), and so U is contained in B'. To show B' is closed, we consider a limit point b of B'. Let U be a trivializing open set containing b; some point in U is in B', and the fiber over that point is isomorphic to F. So the fiber over each point in U is isomorphic to F, and hence U is contained in B'. So $b \in B'$. □

(b) Definitions

Definition 1. If $p : E \to B$ is a covering map, a sub-cover is a map (E', p') where E' is contained in E, and $p' = p|_{E'}$, which is a covering map.

Definition 2. We say a covering map is locally finite if each fiber is finite. If in addition B is connected, each fiber has the same number n of elements, and we say the covering map is finite and of degree n.

Definition 3. If (E, p) and (E', p') are two covers of B, the pair $(E \times_B E', p'')$ is a covering space of B, and is called the product of the two covers, where

$$E \times_B E' = \left\{ (x, y) \in E \times E' \;\middle|\; p(x) = p'(y) \right\}$$

(the fiber product of E and E' over B), and $p'' : (x, y) \mapsto p(x) = p'(y)$.

Proof. It suffices to treat the case where the covers are trivial: $E = B \times F$, $p = \mathrm{pr}_1$, $E' = B \times F'$, $p' = \mathrm{pr}_1$. We have $E \times_B E' = \left\{ (b, f, b', f') \in B \times F \times B \times F' \;\middle|\; b = b \right\}$ and this set is homeomorphic to $B \times F \times F'$. The pair $(E \times_B E', p'')$ and $(B \times F \times F', \mathrm{pr}_1)$ are then isomorphic. □

Definition 4 (Morphism of covers over a base B)**.** A morphism from (E, p) to (E', p') (two covers of the base B) is a continuous map $f : E \to E'$ such that $p' \circ f = p$.

Proposition 1.8. *Such a morphism is always an open map. If it is bijective, it is a homeomorphism.*

Proof. It suffices to treat the case where the covers are trivial: $p = \mathrm{pr}_1 : B \times F \to B$ and $p' = \mathrm{pr}_1 : B \times F' \to B$. The continuous map f is then of the form $(b, u) \mapsto (b, f'(b, u))$. Let (b, u) be an element of $B \times F$ and (b, u') the image via f and let V be a neighborhood of (b, u). There exists an open neighborhood V' of (b, u) contained in $B \times \{u\}$, $f^{-1}(B \times \{u'\})$ and V. Then V' can be written as $W \times \{u\}$, where W is an open set in B. We have $f(V') = W \times \{u'\}$ which is a neighborhood of (b, u'). This shows that f is open. □

2 Lifting continuous maps

Let (E, p) be a cover of the base B. Let X be a topological space and f a continuous map from X to B. We say a lift of f is a continuous map $g : X \to E$ such that $p \circ g = f$.

Proposition 2.1. *a) Two lifts which coincide at a point coincide in some neighborhood of the point.*

b) If X is connected, they coincide everywhere.

Proof. a) It suffices to treat the case where the cover is trivial: $p : B \times F \to B$. Let g_1 and g_2 be two lifts of f which coincide at x: we have by definition, $g_1(x) = g_2(x) = (f(x), u)$ where $u \in F$. $V = g_1^{-1}(B \times \{u\}) \cap g_2^{-1}(B \times \{u\})$ is a neighborhood of x. For $y \in V$, we have $g_1(y) = g_2(y) = (f(y), u)$ and hence $g_1 = g_2$ on V.

b) The set of points where g_1 and g_2 coincide is by hypothesis non-empty, and by above, open in X. It remains to show it is closed. We have the following:

a') If the two lifts g_1 and g_2 differ at a point $x \in X$, they differ at every point in some neighborhood of x (the proof is analogous to part a)). □

We then obtain the following results:

Corollary 2.2. *Two continuous sections s and s' of a cover $p : E \to B$ which coincide at a point, coincide in some neighborhood of the point. If B is connected, they coincide everywhere.*

Proof. Let $X = B$ and $f = \mathbb{1}_B$. □

Corollary 2.3. *Two morphisms f_1, f_2 of the covers (E, p) and (E', p') which coincide at a point of E, coincide everywhere when E is connected.*

Proof. The continuous maps f_1 and f_2 are two lifts of p. □

Corollary 2.4. *If E is a connected covering space of B, $\mathrm{Aut}_B\, E$ (the group of automorphisms of the cover (E, p)) acts freely on each fiber of the cover.*

Proof. If $f \in \mathrm{Aut}_B\, E$, we have $f \circ p = p$, and $\mathrm{Aut}_B\, E$ acts on the fibers of p. Let $f \in \mathrm{Aut}_B\, E$ fix an element of a fiber. Then f and $\mathbb{1}$ are two automorphisms of the connected cover E which coincide at a point, hence are equal. □

3 Criteria for a cover to be trivializable

Let (E, p) be a cover of B.

Proposition 3.1. *Suppose B is connected. Then (E, p) is trivializable if and only if there is a continuous section passing through each point $x \in E$ (for all $x \in E$, there exists a continuous section $s : B \to E$, such that $s(p(x)) = x$).*

Proof. Suppose that (E, p) trivial. We have $(E, p) = (B \times F, \mathrm{pr}_1)$. Then the section $y \mapsto (y, u)$ passes through $x = (b, u)$. So if (E, p) is trivializable, there is a section passing through each point of E.

Conversely, let F be the set of continuous sections for p, equipped with the discrete topology. Consider the map $f : B \times F \to E$ given by $f(b, s) = s(b)$. We have, by definition, $p \circ f = \mathrm{pr}_1$, where $\mathrm{pr}_1 : B \times F \to B$. First, f is a continuous map. To see this, let V be a neighborhood of $f(b, s) = s(b)$. Since s is continuous, there exists a neighborhood V_b of b in B such that $s(x) \in V$ for all $x \in V_b$. Then $V_b \times \{s\}$ is a neighborhood of (b, s) in $B \times F$ and $f(V_b \times \{s\})$ is contained in V. The map f is a surjection by hypothesis. f is injective, since if $f(b, s) = f(b', s')$, then we have $b = b'$ and $s(b) = s'(b)$, so $s = s'$, since B is connected. Then f is a morphism of covers which is a bijection, and hence is a homeomorphism, showing that (E, p) is trivializable. □

Proposition 3.2. *Suppose further that one of the following conditions is satisfied:*

a) the cover is finite;

b) B is locally connected.

Then to show that (E, p) is trivializable, it suffices to verify that there is a section passing through each point of a particular fiber.

Proof. Suppose that the cover is finite. Let $b \in B$, with $p^{-1}(b) = \{x_1, \dots x_n\}$ (the cover is of degree n). Let $s_1, \dots, s_n$ be sections passing through $x_1, \dots, x_n$ respectively. We have $s_i(p(x_i)) = x_i$, that is, $s_i(b) = x_i$, so that the s_i are different at each point of B (since the x_i are distinct). Now consider any element b' of B; the set $\{s_1(b'), \dots, s_n(b')\}$ is of cardinality n. Since $p \circ s_i = \mathbb{1}_B$, we must have $\{s_1(b'), \dots, s_n(b')\} = p^{-1}(b')$. If $x \in E$ is in the fiber on b', we must have $x = s_i(b')$ for some i, that is, $x = s_i(p(x))$. This shows that there is a section passing through x.

Suppose that B is locally connected. Define

$$U = \left\{ b' \in B \;\middle|\; \text{there are sections passing through each point of } p^{-1}(b') \right\}.$$

By assumption, U is non-empty. We show that U is open. Consider a point $b' \in U$. Let V be a connected neighborhood of b' on which p is trivializable. We verify that V is contained in U. A similar argument shows that the complement of U is open. □

Proposition 3.3. *Let $p : E \to B$ be a cover, with B connected. Let E' be a connected component of E. Then $(E', p|_{E'})$ is a cover if*

- *either E is a finite cover;*
- *or B is locally connected.*

Proof. Supposons B is locally connected. Let $b \in B$. Let V be a connected neighborhood on which p is trivializable. Then we have a homeomorphism h from $p^{-1}(V)$ to $V \times F$ such that $\mathrm{pr}_1 \circ h = p|_V$. Identify $p^{-1}(V)$ with $V \times F$ via h; if E' contains an element (v, u) of $V \times F$, it contains all of $V \times \{u\}$ (since $V \times \{u\}$ is a connected subset of $C \times F$). From this we deduce that $E' \cap (V \times F) = V \times F'$ for some discrete subspace F' of F. Then we have that $(E', p|_{E'})$ is a cover.

Now suppose that $E \to B$ is a finite cover. We will show that a connected component E' of E is a cover of B and is both open and closed in E (a connected component is always closed). We proceed by induction on n, the degree of the cover.

If $n = 0$, E is empty and the statement is clear. Again, if E is connected, the result is clear. Suppose E is not connected so that it can be written as the disjoint union of two non-empty open subsets U and V. We show that U and V are covers of B. Choose $b \in B$, W a neighborhood of b on which E is trivializable and $h : p^{-1}(W) \to F \times W$ a trivialization. Define $F_1 = \left\{ f \in F \,\middle|\, h^{-1}(f, b) \in U \right\}$ and $F_2 = \left\{ f \in F \,\middle|\, h^{-1}(f, b) \in V \right\}$. Then F_1 and F_2 partition F. Since F is finite and U and V are open, there exists a neighborhood W' of b, contained in W, such that $h^{-1}(F_1 \times W')$ is contained in U and $h^{-1}(F_2 \times W')$ is contained in V. Since U and V are disjoint, we must have $h^{-1}(F_1 \times W') = U \cap p^{-1}(W')$ and $h^{-1}(F_2 \times W') = V \cap p^{-1}(W')$. Now $U \to B$ and $V \to B$ are covers of degree $< n$, since each is non-empty. E', being connected, is contained in either U or V and hence is a connected component of either U or V. We are now done by the induction hypothesis. □

4 Pullback of a cover

Let us be given a cover (E, p) of a base B, a space B' and a continuous map f from B' to B. Then we can construct $B \times_B E$, the fiber product of B' and E over B and (natural) continuous maps p_1, f_1 such that the following diagram commutes:

$$\begin{array}{ccc} B' \times_B E & \xrightarrow{f_1} & E \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B \end{array}$$

By definition $B' \times_B E = \left\{ (b', e) \in B' \times E \mid f(b') = p(e) \right\}$, p_1 is the restriction of pr_1 to $B' \times_B E$ and f_1 is the restriction of pr_2 to $B' \times_B E$.

Proposition 4.1. *$(B' \times_B E, p_1)$ is a cover of B'.*

Definition 5. This cover is called the pullback of (E, p) by f and is denoted $f^* E$.

Proof. Let $b' \in B'$. Define $b = f(b')$. Choose an open set U in B containing b and a trivialization $h : p^{-1}(U) \to U \times F$. Let $U' = f^{-1}(U)$. Then $p_1^{-1}(U')$ is

$$\begin{aligned} \left\{ (x', e) \in U' \times E \,\middle|\, f(x') = p(e) \right\} &= \left\{ (x', e) \in U' \times p^{-1}(U) \,\middle|\, f(x') = p(e) \right\} \\ &\cong \left\{ (x', x, a) \in U' \times U \times F \,\middle|\, f(x') = x \right\} \\ &\cong U' \times F \end{aligned}$$

So on U' , p_1 is a trivializable cover. □

Remark. If $b' \in B'$, the fiber $p_1^{-1}(b')$ is canonically identified with the fiber $p^{-1}(b)$, where $b = f(b')$.

Remark. There is a canonical one-to-one correspondence between continuous sections of p_1 and continuous lifts of f to E.

More precisely, a continuous section of p_1 is a continuous map $s : B' \to B' \times_B E$ such that $p_1 \circ s = \mathbb{1}_{B'}$. In other words, it is a map from B' to $B' \times E$ of the form $b' \mapsto (b, g(b'))$ where g is continuous from B' to E with $f = p \circ g$.

5 Simply connected spaces

Definition 6. We say a topological space V is simply connected (in the sense of covers) if every cover of V is trivializable.

Remark. If B is simply connected, B is connected. Otherwise, there exist disjoint non-empty open sets U and V whose union is B and then the canonical injection $U \to B$ is a cover which is not trivializable.

Proposition 5.1. *An interval in $\mathbb{R}$ is simply connected.*

Proof. Let I be an interval in $\mathbb{R}$ and (E, p) a cover of I. We will prove that there is a section passing through every point of E, that is, if $x \in E$, there exists a continuous section $s : I \to E$ taking the value x at the point $b = p(x)$. It suffices to construct s separately on $(-\infty, b] \cap I$ and $[b, \infty) \cap I$. Let J be the set of points $b' \geq b$ in I such that there exists a continuous section of p defined on $[b, b']$ taking the value x at b. Let β be the supremum of J.

There exists a continuous section of p defined on $[b, \beta)$ (since if two continuous sections of p on $[b, b_1]$ and $[b, b_2]$ with $(b_1 \leq b_2)$ take the same value x at the point b, they coincide on $[b, b_1]$).

Now suppose that β is in I. We show that s extends continuously to β, and if β is not the maximum of I, to $[\beta, \beta + \epsilon)$ for some $\epsilon > 0$. There exists a neighborhood $(\beta - \epsilon, \beta + \epsilon) \cap I$ of β in I on which E is trivializable, that is, homeomorphic to $((\beta - \epsilon, \beta + \epsilon) \cap I) \times F$ with F discrete. Then s restricted to $(\beta - \epsilon, \beta + \epsilon) \cap I$ takes values in a fixed horizontal (that is, $\mathrm{pr}_2^{-1}(f)$ with $f \in F$ fixed). Hence s can be extended to $[\beta, \beta + \epsilon) \cap I$ as a continuous section. Since β is the supremum of J, $[b, \beta) = I \cap [b, \infty)$. □

6 Lifting of paths

A path in a topological space X is a continuous map $c : [0, 1] \to X$; $c(0)$ is called the initial or starting point of the path and $c(1)$ is called the terminal or end point of the path.

Proposition 6.1. *Let $p : E \to B$ be a cover, c a path in B starting at b and $x \in E$ such that $p(x) = b$. Then there exists a unique path c' in E starting at x which lifts c.*

Proof. Uniqueness: two continuous lifts of c which take the same value at 0 are equal since $[0, 1]$ is connected.

Existence: we consider the fiber product $c^*E = [0,1] \times_B E$. We have a diagram of covers:

$$\begin{array}{ccc} c^*E & \longrightarrow & E \\ \downarrow & & \downarrow p \\ [0,1] & \longrightarrow & B \end{array}$$

which is commutative. We need to prove that $c^*E \to [0,1]$ has a continuous section passing through the point $(0,x)$. But $[0,1]$ is simply connected, so c^*E is trivializable and there is a section passing through every point of c^*E. □

7 Sections and covers of $A \times B$

Let $p : E \to A \times B$ be a cover and s a section, not necessarily continuous, of p.

Proposition 7.1. *Suppose B is locally connected and connected. Let $b \in B$; suppose s is continuous on $A \times \{b\}$ and on $\{x\} \times B$, for every x in A. Then s is everywhere continuous.*

Proof. First we look at the case where E is trivial. We then have $E = A \times B \times F$ with F discrete and $p = pr_1$. On $A \times \{b\}$, s is of the form $(x,b) \mapsto (x,b,f(x))$, where $f : A \to F$ is a continuous map. Fix $x \in A$ and consider s on $\{x\} \times B$. It is of the form $(x,y) \mapsto (x,y,g_x(y))$, where $g_x : B \to F$ is a continuous map. Since B is connected and F is discrete, g_x is constant. So for all y in B, we have $g_x(y) = g_x(b) = f(x)$. Then s is the map $(x,y) \mapsto (x,y,f(x))$ which is continuous.

Now for the general case. Fix $a \in A$ and define U to be the set of $y \in B$ such that s is continuous in a neighborhood of (a,y) in $A \times B$. We will prove that U is non-empty, open and closed; since B is connected, we then must have $U = B$. Then it follows that s is continuous in a neighborhood of the point (a,y), for every $y \in B$ and every $a \in A$, and hence s is continuous everywhere.

The set U is open, by definition; to show U is non empty, we prove that $b \in U$. Choose a neighborhood V_a of a and a neighborhood V_b of b such that E is trivializable on $V_a \times V_b$. By applying the above on $p^{-1}(V_a \times V_b)$, s is continuous on $V_a \times V_b$.

Now we want to show U is closed. Let $y \in \overline{U}$. Let V_a be a neighborhood of a in A and V_y be a connected neighborhood of y in B such that E is trivializable on $V_a \times V_y$. There exists z in $U \cap V_y$; s is continuous on a neighborhood of (a,z), which contains $V'_a \times \{z\}$ for V'_a a neighborhood of a contained in V_a. Consider $p^{-1}(V'_a \times V_a)$. It is a trivializable cover of $V'_a \times V_y$; V_y is connected and s restricted to $V'_a \times V_y$ is continuous on $V'_a \times \{z\}$ and on $\{x\} \times V_y$ for every $x \in V'_a$. Again by the argument above, s is continuous on $V'_a \times V_y$ which is an neighborhood of (a,y). Thus y belongs to U and U is closed. □

Corollary 7.2. *If A and B are simply connected and if B is locally connected, $A \times B$ is simply connected.*

Proof. Let (E,p) be a cover of $A \times B$. Let $x \in E$ and $(a,b) = p(x)$. Since A is simply connected, there exists a continuous section s_1 of p defined on $A \times \{b\}$ such that $s_1(a,b) = x$ (since $A \times \{b\}$ is also simply connected). Since B is simply connected, for every $t \in A$, there exists a continuous section s_t of p defined on $\{t\} \times B$ such that $s_t(t,b) = s_1(t)$. Then, the map from $A \times B \to E$ defined by $(t,u) \mapsto s_t(u)$ is a section of E, continuous on $A \times \{b\}$ and on $\{t\} \times B$ for every

$t \in A$. Since B is locally connected, it is continuous everywhere (B is connected since it is simply connected). ☐

Lemma 7.3. *Let E, B be two topological spaces, (E, p) a cover of B and $f : X \times Y \to B$ a continuous map (X and Y two topological spaces) with Y connected and locally connected. Then every lift $g : X \times Y \to E$ of f which is continuous on $X \times \{y\}$ for some $y \in Y$ and on $\{x\} \times Y$ for every $x \in X$ is everywhere continuous.*

Proof. Let f^*E be the fiber product of $X \times Y$ and E over B. Then the following diagram commutes:

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \big\downarrow & & \big\downarrow p \\ X \times Y & \xrightarrow{f} & B \end{array}$$

We are thus reduced to the case of sections which we've already seen. ☐

Corollary 7.4. *Let E an cover of $A \times B$. Assume B to be locally connected and simply connected (and hence connected). Then E is isomorphic to a cover of the form $(p, \mathbb{1}_B) : E' \times B \to A \times B$ where $p : E' \to A$ is a cover of A.*

Proof. Choose $b \in B$. Consider the inverse image, $p^{-1}(A \times \{b\})$, of $A \times \{b\}$ as a cover of $A \times \{b\}$. Denote this by E'. We shall construct an isomorphism from $E' \times B$ to E. For every $a \in A$, the cover on $\{a\} \times B$ is trivializable (since B is simply connected). More precisely, it is isomorphic to $E'_a \times B$, where E'_a designates the fiber at the point a. We then define a map h from $E' \times B$ to E which is continuous on one horizontal and every vertical. By the lemma, h is continuous. It is bijective, hence it is an isomorphism. ☐

8 Consequences

Proposition 8.1. *Let (E, p) be a cover of the base B. Let f_1 and f_2 be two continuous maps from a topological space A to B. If f_1 and f_2 are homotopic, then the pullbacks over A, f_1^*E and f_2^*E, are isomorphic.*

Definition 7. We say that f_1 and f_2 are homotopic (denoted $f_1 \simeq f_2$) if there exists $H : A \times [0, 1] \to B$ continuous such that for every $a \in A$, $H(a, 0) = f_1(a)$ and $H(a, 1) = f_2(a)$. We say H is a homotopy between f_1 and f_2 (denoted $f_1 \simeq_H f_2$).

Proof. Let $H : A \times [0, 1] \to$ an be a homotopy between f_1 and f_2. Let $i_0 : A \to A \times [0, 1]$ be such that $i_0(a) = (a, 0)$ and $i_1 : A \to A \times [0, 1]$ such that $i_1(a) = (a, 1)$. We then have $i_0^*(H^*E) \approx (H \circ i_0)^*E \approx f_1^*E$. Similarly $i_1^*(H * E) = f_2^*E$. By above, H^*E is isomorphic to a cover $E' \times [0, 1]$, where E' is a cover of A. The pullbacks of such a cover via i_0 and i_1 are isomorphic to E'. ☐

Proposition 8.2. *A contractible topological space is simply connected.*

Definition 8. We say that a topological space X is contractible if $\mathbb{1}_X : X \to X$ is homotopic to a constant map.

Proof. Let E be a cover of X. By hypothesis, $\mathbb{1}_X$ is homotopic to a constant map $c : X \to X$, say one which takes all of X to x. Then, E is isomorphic to $(\mathbb{1}_X)^*E$ which is isomorphic to c^*E. The diagram

$$\begin{array}{ccccc} c^*E & \longrightarrow & i^*E & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{c} & \{x\} & \xrightarrow{i} & X \end{array}$$

is commutative; i^*E is trivializable (since the base is a single point), hence so is $c^*(i^*E) \cong c^*E$ and also E. □

Remark. A star convex subset A of $\mathbb{R}^n$ is contractible.

Proof. We may suppose that the distinguished point is 0. Then the map $A \times [0,1] \to A$ defined by $(a,t) \mapsto at$ is an homotopy between the zero map and $\mathbb{1}_A$. □

Proposition 8.3 (Lifts of homotopies). *Let (E,p) be a cover of B (a topological space). Let f_1 and f_2 be two continuous maps and $H : A \times [0,1]$ a homotopy between f_1 and f_2. Let g_1 be a lift of f_1. Then there exists a unique homotopy $H' : A \times [0,1] \to E$ lifting H, with $H'(a,0) = g_1(a)$ (for every $a \in A$). This is a homotopy between g_1 and a lift of f_2.*

Proof. Let $H'(a,0) = g_1(a)$. By lifting of paths, for every $a \in A$, there exists a unique continuous map on $\{a\} \times [0,1]$ which lifts H restricted to $\{a\} \times [0,1]$ and takes the value $g_1(a)$ at $(a,0)$: combining these paths gives us a lift H' of H, continuous on $\{a\} \times [0,1]$ for every $a \in A$, and coinciding with $(a,0) \mapsto g_1(a)$ on $A \times \{0\}$ (further, such a lift is unique). This H' is continuous by previous results. □

9 The fundamental (or Poincaré) group

Definition 9. Let X be a topological space. Let x be a point of X. We say two paths c and c' in X can be composed if $c(1) = c'(0)$. The composed path cc' is

$$t \mapsto \begin{cases} c(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ c'(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

Definition 10. Two paths c_1 and c_2 in X are path homotopic if there exists a continuous map $H : [0,1] \times [0,1] \to X$ satisfying the identities: $H(t,0) = c_1(t)$, $H(t,1) = c_2(t)$, $H(0,u) = c_1(0)$, $H(1,u) = c_2(1)$.

The relation of path homotopy is compatible with the composition of paths; so we have on the set of path homotopy classes, a partially defined law of composition.

For x and y in X, we denote by $\pi_1(X;x,y)$ the set of path homotopy classes of paths joining x to y. We have a law of composition

$$\pi_1(X;x,y) \times \pi_1(X;y,z) \to \pi_1(X;x,z)\,,$$

which given a class of paths from x to y, and a class of paths from y to z, gives us a class of paths from x to z.

This composition law is associative; there is for each x an identity element e_x which is the class of the constant path at the point x; further, the class of the path $t \mapsto c(1-t)$ in $\pi_1(X; y, x)$ is the (both-sided) inverse of the class of c in $\pi_1(X; x, y)$.

Now $\pi_1(X; x, y)$ equipped with this composition forms a groupoid with base X. It is called the fundamental (or Poincaré) groupoid of X.

(Let X be a set. A groupoid with base X is given by, for every $(x, y) \in X \times X$, a set $F(x, y)$ and for every triplet $(x, y, z) \in X \times X \times X$ a map $F(x, y) \times F(y, z) \to F(x, z)$ with: a) the associativity property, b) for each $F(x, x)$ the existence of an identity element e_x, c) the existence of an inverse map $F(x, y) \to F(y, x)$.)

Definition 11. Let X be a topological space; let $x \in X$. Then the set $\pi_1(X; x, x)$ i.e. the set of classes of loops at x is a group, called the fundamental group of X at x, and denoted $\pi_1(X, x)$.

Let x and y be two points in X contained in the same path component. Then

a) $\pi_1(X, x)$ and $\pi_1(X, y)$ are isomorphic;

b) the choice of a class in $\pi_1(X; x, y)$ defines a canonical isomorphism from $\pi_1(X, x)$ to $\pi_1(X, y)$.

c) these isomorphisms from $\pi_1(X, x)$ to $\pi_1(X, y)$ are the same, modulo interior automorphisms;

10 Action of the fundamental groupoid on the fibers of a cover

Let $p : E \to B$ an cover (arbitrary). We shall construct an action of the fundamental groupoid on the fibers of the cover, that is, if x and y are in B, we get an explicit map

$$\pi_1(X; x, y) \times p^{-1}(x) \to p^{-1}(y).$$

Let α be an element of $\pi_1(X; x, y)$ and $x' \in p^{-1}(x)$. We choose a representative f of α: f is a path joining x to y. There exists an unique lift of f starting at x' (it is a path in E starting at x' which lifts f). Let $y' \in p^{-1}(y)$ be its end point. The action $\pi_1(X; x, y) \times p^{-1}(x) \to p^{-1}(y)$ is defined by $(\alpha, x') \mapsto y'$.

In particular, the group $\pi_1(B, x)$ right acts on the fibre $p^{-1}(x)$: we choose a point x' in $p^{-1}(x)$ and α in $\pi_1(B, x)$. We choose a loop f in the class α and lift f to a path g in E, starting at x', and the endpoint y' of g is by definition $\alpha \cdot x'$.

Choose a point x' in $p^{-1}(x)$. We have a (natural) homomorphism of groups $u : \pi_1(E, x') \to \pi_1(B, x)$.

Proposition 10.1. *u is injective and the image in $\pi_1(B, x)$ is the stabilizer of the point x'.*

Proof. Let c be a loop at x' in E, whose class is in the kernel of u. This means that $p \circ c$ is path homotopic in B to the constant loop α_x at x.

Let H be a path homotopy between $p \circ c$ and α_x. Then H lifts to a homotopy $H' : [0,1] \times [0,1] \to E$ such that $H'(t,0) = c(t)$. Since $H(0,u) = x$, H' lifts H, and $H'(0,0) = x'$ we have $H'(0,u) = x'$ for all u. Similarly we have $H'(1,u) = x'$ for all u. Further, $H(t,1) = x$ for all t forcing $H'(t,1) = x'$ for all t. Then H' is a path homotopy between c and the constant loop at x'. So the class of c is trivial (in $\pi_1(E,x')$); this shows that u is injective.

Let f be at x and $[f]$ a class in $\pi_1(B,x)$; $[f]$ is contained in the stabilizer of x' if and only if the unique (continuous) lift f' of f starting at x' ends at x'. If f satisfies this property, then $[f'] \in \pi_1(B,x')$ and $u([f']) = [f]$. Conversely, if $[f]$ is contained in the image of u, then there exists a loop g at x' such that $u[g] = [f]$, i.e. $[p \circ g] = [f]$. Then, we have $[f].x' = \text{end point of } g = x'$; so that $[f] \in \text{Stab}(x')$. □

Remark. If E is path connected (which is the case if E is connected and B is locally path connected), the group $\pi_1(B,x)$ acts transitively on $p^{-1}(x)$.

Proof. Let x' and y' two points in $p^{-1}(x)$. There exists a path c in E which joins x' to y'. Then c is the lift of the loop $p \circ c$ at x' and thus $[p \circ c].x' = y'$. □

Category of covers over a given base

Let $p : E \to B$ be a cover. Let f_1 and f_2 two continuous maps from a topological space A to B. We have seen that f_1 is homotopic to f_2, then f_1^*E is isomorphic to f_2^*E. In fact, if we choose a homotopy H between f_1 and f_2, it corresponds to an isomorphism $u_H : f_1^*E \to f_2^*E$ which depends on H, .

Let $\mathbf{Cov}(B)$ be the category of covers of the base B. We have two functors

$$T_1 \text{ and } T_2 : \mathbf{Cov}(B) \to \mathbf{Cov}(A)$$

defined by $T_1 : E \mapsto f_1^*E$ and $T_2 : E \mapsto f_2^*E$. Fix H, a homotopy between f_1 and f_2; then to every cover E of B, there is a corresponding isomorphism $u_{H,E} : T_1(E) = f_1^*E \to T_2(E) = f_2^*(E)$.

This is a natural isomorphism from T_1 to T_2.

Definition 12. If X and Y are topological spaces, we call a continuous map $f : X \to Y$ a homotopy equivalence (or homeotopy) if there exists a continuous $g : Y \to X$ with $f \circ g$ homotopic to $\mathbb{1}_Y$ and $g \circ f$ homotopic to $\mathbb{1}_X$. We say g is a homotopy inverse to f. If there is such an f, we say X and Y are homotopy equivalent.

Example 10.1. The canonical injection $S^1 \to \mathbb{C} \setminus \{0\}$ is a homotopy equivalence.

Proposition 10.2. *If $f : X \to Y$ is a homotopy equivalence, then $T : \mathbf{Cov}(Y) \to \mathbf{Cov}(X)$ defined by $T(E) = f^*E$ is an equivalence of categories.*

Proof. Let $g : Y \to X$ be a homotopy inverse of f. Let $T' : \mathbf{Cov}(X) \to \mathbf{Cov}(Y)$ be the functor defined by $T'(E') = g^*E'$; $f \circ g$ is homotopic to $\mathbb{1}_y$, so $T' \circ T$ is naturally isomorphic to the identity functor $\mathbb{1}_{\mathbf{Cov}(Y)}$ of $\mathbf{Cov}(Y)$. Similarly $T \circ T'$ is naturally isomorphic to $\mathbb{1}_{\mathbf{Cov}(X)}$. □

11 Galois covers

Let B be a connected non-empty topological space.

Definition 13. A cover (E, p), E non-empty, of B is called Galois if the following two conditions are satisfied:

a) E is connected;

b) $G = \mathrm{Aut}(E, p)$ acts transitively on each fiber.

We say that G is the Galois group of the cover (E, p).

Remark. If B is locally connected or if the cover is finite, it suffices to verify that the action is transitive on a particular fiber.

Proof. First suppose that the cover is finite, of degree n. Since G freely on E, if $x \in B$, the following conditions are equivalent:

(i) G acts transitively on $p^{-1}(x)$;

(ii) $|G| = \left|p^{-1}(x)\right|$;

(iii) $|G| = n$.

Now if B is locally connected, let $x \in B$. Let U be a connected neighborhood in B containing x on which the cover is trivializable. We have $p^{-1}(U) \cong U \times F$ where F is discrete; every element $g \in G$ acts on $p^{-1}(U)$ (identified with $U \times F$) by a permutation $(y, f) \mapsto (y, \alpha_g(f))$, where $\alpha_g \in \mathfrak{S}_F$. Then, G acts transitively on $p^{-1}(x)$ if and only if α_g acts transitively on F, if and only if, for every $y \in U$, G acts transitively on $p^{-1}(y)$. Hence the result, since B is connected. □

Lemma 11.1. *In a Galois cover $p : E \to B$, with Galois group $G = \mathrm{Aut}(E, p)$, the fibers are isomorphic to G (with the discrete topology). Given a (local) section, the identifications are compatible with the action of G.*

Proof. Let $x \in B$ and U be a neighborhood of x in B on which p has a continuous section s. Then we have a homeomorphism $U \times G \to p^{-1}(U)$ defined by $(x, g) \mapsto gs(x)$ (G has the discrete topology). The continuity is clear. Since this is a map of covers, we just show that it is a bijection: injectivity holds since G acts freely and surjectivity holds since G acts transitively on fibers of p. This homeomorphism is clearly compatible with the left action of G on $U \times G$: $g_0.(x, g) = (x, g_0 g)$. □

Let (E, p) be a Galois cover of B and $G = \mathrm{Aut}(E, p)$ be the Galois group.

If F is a discrete right G-set (i.e. a set with the discrete topology on which G acts on the right), we construct a new cover of B, denoted $F \times_G B$, defined by

$$F \times_G E = {}^{F \times E}\!\big/_{(fg, x) \sim (f, gx)}$$

The projection to B is induced by $(f, x) \mapsto p(x)$ by passing to the quotient.

Lemma 11.2. *The map described above makes $F \times_G E$ into a covering space of B.*

Proof. To prove that $F \times_G E$ is a cover, it suffices (by choosing a local trivialization) to treat the case where $E = B \times G$, with G acting on the left on the second factor. We have

$$F \times_G E = F \times B \times G \Big/ (fg, b, h) \sim (f, b, gh)$$

Then $F \times_G E$ is homeomorphic to $F \times B$ by $(f, b) \mapsto [(f, b, 1)]$ and $[(f, b, h)] \mapsto (fh, b)$. $\square$

Definition 14. We say that a cover of B is associated to a Galois cover (E, p) if it is isomorphic to a cover $F \times_G E$ for some G-set F.

Proposition 11.3. *Let (E', p') be a cover of B. It is associated to (E, p) if and only if p^*E' is a trivializable cover of E.*

Proof. Suppose E' is associated to E. Up to isomorphism, we have $E' = F \times_G E$. By definition, we have $p^*E' = \{(x, y) \in E \times E' \mid p(x) = p'(y)\}$. p^*E' is thus isomorphic to

$$\{(x, f, z) \in E \times F \times E \mid p(x) = p(z)\} \Big/ (x, fg, z) \sim (x, f, gz) \cong E \times F$$

The latter isomorphism is given by: $(x, f) \mapsto [(x, f, x)]$ and $[(x, f, z)] \mapsto (x, fg)$ where $g \in G$ is such that $z = gx$. This is a trivial cover of E.

Conversely, suppose p^*E' trivializable. We consider the set F of continuous maps $f : E \to E'$ such that $p' \circ f = p$; G acts on F on the right: $(f \cdot g)(x) = f(gx)$. We have the map $F \times_G E \to E'$ defined by $(f, x) \mapsto f(x)$ since (fg, x), (f, gx) have the same image. We show that this is bijective.

Injectivity: suppose that $f(x) = f'(x')$ for some $x, x' \in E$, f and $f' \in F$. We have $p'(f(x)) = p'(f'(x'))$, so that $p(x) = p(x')$. Then there exists a $g \in G$ such that $x' = gx$. Hence $f(x) = f'(gx) = (f' \cdot g)(x)$ and so $f = f' \cdot g$ (since E is connected and the two lifts are coincident at a point). Thus $(f, x) = (f'g, x)$ and $(f', x') = (f', gx)$.

Surjectivity: pick an element $x' \in E'$. Choose $x \in E$ such that $p(x) = p'(x')$. There exists (since p^*E' is trivializable) a section of p^*E' which takes the value (x, x') at x. Such a section has as the second component a map $f : E \to E'$ which belongs to F and takes the value x' at x so that $(f, x) \mapsto f(x) = x'$. $\square$

Example 11.1. Let H be a subgroup of G. The quotient $H \backslash E$ is a cover of B associated to E.

Proof. $H \backslash E$ is canonically homeomorphic to $(H \backslash G) \times_G E$ via $[(Hg, x)] \mapsto H(gx)$ and $Hx \mapsto [(H, x)]$. $\square$

Equivalence of categories

Consider the category of right G-sets and the covers of B associated to E. We shall show they are equivalent. More precisely:

Proposition 11.4. *We have an equivalence of categories:*

$$\begin{array}{ccc} \textit{right } G\textit{-sets} & & \textit{covers of } B \textit{ associated to } E \\ F & \longmapsto & F \times_G E \\ \mathrm{Mor}(E, E') & \longleftarrow & E' \end{array}$$

given by the above functors.

Proof. Let F be a right G-set. We will first show that the map

$$\begin{aligned} F &\longrightarrow \text{Morphisms from } E \text{ to } F \times_G E \\ f &\longmapsto x \mapsto [(f,x)] \end{aligned}$$

is a bijection (the images are morphisms of covers of base B).

Suppose that we have $[(f,x)] = [(f',x)]$; by definition, there exists $g \in G$ such that $f = f'g$ and $x = gx$. Since G acts freely on E, we must have $g = 1$ and $f = f'$; this proves the injectivity. Now consider u, a morphism from E to $F \times_G E$. Choose a point x_0 in E and let $u(x_0) = [(f_1, x_1)]$. By definition (of morphisms of covers over the base B) x_0 and x_1 are in the same fiber; since the cover is Galois, there exists $g \in G$ such that $x_1 = gx_0$. Then $[(f_1, x_1)] = [(f_1 g, x_0)]$. We deduce that u is the morphism of covers mapped by $f_1 g$, since these two morphisms coincide at x_0. This proves the surjectivity.

Let F, F' be two right G-sets. We consider

$$\begin{aligned} \mathrm{Mor}(F,F') &\longrightarrow \mathrm{Mor}(F \times_G E, F' \times_G E) \\ \varphi &\longmapsto [(f,x)] \mapsto [(\varphi(f),x)] \end{aligned}$$

We need to show surjectivity. Let u be a morphism $F \times_G E \to F' \times_G E$. Fix $x_0 \in E$; we have $u([(f,x_0)]) = [(\varphi(f),x_0)]$ for a certain (unique) $\varphi(f) \in F'$ by the previous argument. Then $u([(f,x)]) = [(\varphi(f),x)]$ for all x. We now show that φ is compatible with G. For this, we note that we have $u([(fg,x_0)]) = u([(f,gx_0)])$ and $u([(fg,x_0)]) = [(\varphi(fg),x_0)]$ pour tout $g \in G$. Then $[(\varphi(f),gx_0)] = [(\varphi(f)g,x_0)]$ and we deduce that $\varphi(fg) = \varphi(f)g$. □

Example 11.2. Every connected cover of B associated to E is isomorphic to $H\backslash E$ where H is a subgroup of G. It is Galois if and only if H is normal in G. The covers $H\backslash E$ and $H\backslash E'$ are isomorphic if and only if H and H' are conjugates in G.

Proof. If F is an G-set on which G acts transitively on the right (and $F \times_G E$ is connected), F is isomorphic to $H\backslash G$ (take $f \in F$, call the stabilizer H, and send Hg to fg, this is a bijection between $H\backslash G$ and F compatible with G).

Then let $b \in B$, choose $x \in E$ (which is non empty) mapping to b. Then the fiber at b of the cover $F \times_G E$ is in bijection with F via $f \mapsto [(f,x)]$. The group of automorphisms of the G-set F is in bijection with the group of automorphisms of the cover $F \times_G E$; the cover is Galois if and only if the group of automorphisms of the G-set F acts transitively on F.

The automorphisms of the G-set $H\backslash G$: let u be such an automorphism and let $g \in G$ be such that $u(H) = Hg$. Then we have $u(Hg') = u(H)g' = Hgg'$ if $g' \in H$; now $gg'g^{-1} \in H$ if $g' \in H$ and $g \in N_G H$ (le normalisateur de H dans G).

The required automorphism group is $N_G H/H$. The orbit of the trivial coset in $H\backslash G$ under this group is $H\backslash N_G H$. This is equal to $H\backslash G$ if and only if $N_G H = G$ if and only if H is normal in G. □

We now look at the analogues that existent between the theory of fields and that of covers over a connected base B.

Connected base B	Field K
Finite non empty connected cover E of B	Finite extension K' of K
The cover is Galois	K'/K is Galois
The Galois group $G = \mathrm{Aut}_B E$	$\mathrm{Aut}_K(K')$
Connected cover associated to E (cf. *)	Finite separable extensions of K embeddable in K' (cf. *)
Up to isomorphism they correspond to sub-groups of G up to conjugation	They correspond to sub-groups of G up to conjugation
*: E_1 is associated to E if and only if p^*E_1 is trivializable i.e. $E_1 \times_B E \approx E \times F$ (F finite) where $F = \mathrm{Mor}(E_1, E)$	*: K_1 is embeddable in K' iff $K_1 \otimes_K K' \cong K'^F$ (F finite) via $a \otimes b \mapsto \sigma(a)b$; $\sigma \in F$, where $F = \{\text{embeddings of } K_1 \text{ in } K'\}$.

Proposition 11.5. *Let B be a connected topological space. If B has a cover E which is non empty and simply connected then the cover is Galois.*

Definition 15. We say such an E is a universal cover of B.

Proof. E is connected. Let x, y be two points of E in the same fiber $p^{-1}(b)$. Since E is simply connected, p^*E is trivializable, and by the dictionary between sections of p^*E and maps from E to E, compatible with p, we see that there exists a continuous $f : E \to E$ such that $p \circ f = p$ and $f(x) = y$; f is a map of covers from E to E. Similarly, there exists $f' : E \to E$ with $f'(y) = x$; then $f' \circ f(x) = x$. This implies $f' \circ f = \mathbb{1}_E$ and $f \circ f' = \mathbb{1}_E$; then $f \in \mathrm{Aut}(E)$ and E is Galois. □

Remark. If E is a universal cover of B, then every cover of B is associated to E.

Proof. Let E' be a cover of B; p^*E' is an cover of E which is trivializable since E is simply connected. □

We deduce that there is an equivalence of categories:

$$\{\text{Right } G\text{-sets where } G = \mathrm{Aut}_B E\} \longrightarrow \mathbf{Cov}(B)$$

$$F \longmapsto F \times_G E$$

Two universal covers E, E' of B are isomorphic. More precisely, if $x \in E$ and $y \in E'$ have the same image $b \in B$, then there exists an unique isomorphism $E \to E'$ which takes x to y.

Example 11.3 (of a universal cover). We consider

$$\mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\}$$

$$z \longmapsto \exp(2\pi i z)$$

This is a universal cover with Galois group $\mathbb{Z}$; the action on $\mathbb{C}$ is by translations $z \mapsto z + n$ where $n \in \mathbb{Z}$.

The non empty connected covers of $\mathbb{C} \setminus \{0\}$ up to isomorphism correspond bijectively to subgroups of $\mathbb{Z}$.

$$0\mathbb{Z} = \{0\} \longrightarrow \text{universal cover}$$

$$n\mathbb{Z}, n \geq 1 \longrightarrow \text{covers } z \mapsto z^n \text{ from } \mathbb{C} \setminus \{0\} \text{ to } \mathbb{C} \setminus \{0\}$$

We have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C}/n\mathbb{Z} & \xrightarrow{\cong : z \mapsto \exp(\frac{2\pi i z}{n})} & \mathbb{C} \setminus \{0\} \\ & {\scriptstyle z \mapsto \exp(2\pi i z)} \searrow \quad \swarrow {\scriptstyle t \mapsto t^n} & \\ & \mathbb{C} \setminus \{0\} & \end{array}$$

Definition 16. A topological space X is called semi-locally simply connected if it satisfies the following conditions:

a) it is connected and non empty;

b) it is locally path connected;

c) for every $x \in X$, there exists a neighborhood U of x such that natural homomorphism $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.

Example 11.4. If X is a connected non empty topological manifold, X is semi-locally simply connected.

Theorem 11.6. *A semi-locally simply connected topological space has an universal cover .*

Proof. Fix a point a in X. Let E be the set of path homotopy classes of paths starting at a: we have

$$E = \coprod \pi_1(X; a, x).$$

Let $p : E \to X$ be the map that to a path homotopy class of paths starting at a associates the common end point of the paths, i.e. p takes $\pi_1(X; a, x)$ to x.

We now equip the set E with a topology: let $c \in E$ with $p(c) = x$ ($c \in \pi_1(X; a, x)$). Let U be an open set of X containing x. Consider $c' \in E$ of the form $c.[d]$ where d is a path starting at x contained in U. Denote by $A_{c,U}$ the set of such c'.

Now note that for any U, U' neighborhoods of $p(c)$, there is some U'' such that $A_{c,U} \cap A_{c,U'}$ contains $A_{c,U''}$ (it suffices to pick $U'' = U \cap U'$). So for a given c, as U varies over all neighborhoods of $p(c)$, $A_{c,U}$ form a local basis at c .

We equip E with the topology generated by the sets $(A_{c,U})$. By definition, $E' \subset E$ is open if and only if for all $c \in E'$, there exists an open U containing $p(c)$ such that $A_{c,U}$ is contained in E'.

Now, let $c \in E$ and $x = p(c)$. Let U be a open neighborhood of x. Then $p^{-1}(U)$ contains $A_{c,U}$, which is an neighborhood of c. Thus, p is continuous.

Now we want to show that (E, p) is an cover de X. Let $x \in X$ and let U be a open path connected neighborhood of x such that the map $\pi_1(U, x) \to \pi_1(X, x)$ is trivial. We show that on such a U, p is a trivializable cover:

Lemma 11.7. *We have*

$$p^{-1}(U) = \coprod_{c \in p^{-1}(x)} A_{c,U}.$$

Proof. We show that the union is disjoint. Suppose $A_{c_1,U} \cap A_{c_2,U}$ is non empty (with c_1 and c_2 in $p^{-1}(x)$). Then there exists $y \in U$, d_1 and d_2 paths from x to y in U such that $c_1.[d_1] = c_2.[d_2]$ (as path homotopy classes in X). Then $c_2^{-1}.c_1.[d_1].[d_2]^{-1} = 0$ in $\pi_1(X, x)$. But $[d_1].[d_2]^{-1} \in \mathrm{Im}(\pi_1(U, x) \to \pi_1(X, x))$, hence is 0. So $c_2^{-1}.c_1 = O$ and $c_1 = c_2$. $\square$

It remains to prove that for a fixed c, the map $p_{c,U} = p|_{A_{c,U}} : A_{c,U} \to U$ is an homeomorphism.

We know that it is continuous, and surjective since U is path connected. To show that it is injective: let d_1 and d_2 be two paths in U from x to the same point y; $d_1 d_2^{-1}$ is from x to x in U. So the class of this loop in $\pi_1(X, x)$ is trivial. Then d_1 and d_2 are path homotopic in X, implying $c.[d_1] = c.[d_2]$.

Now we show that $p_{c,U})$ is an homeomorphism. Let c' be an element of $A_{c,U}$. A local basis at c' in $A_{c,U}$ is formed by the $A_{c',U'}$, with U' open path connected containing $p(c')$. The image under p of $A_{c',U'}$ is U'. Then the image under p of every neighborhood of c' is a neighborhood of $p(c')$.

The next proposition will show that E is simply connected. □

Remark. E has a natural point base e_a, the class of the constant loop at a.

Proposition 11.8. *E is simply connected, and hence is a universal cover of X.*

Proof. Let (E', p') be a cover of E and b a point in $p'^{-1}(a)$. We will construct a continuous section s of p' that takes the value b at e_a.

Let $c \in E$; c is the path homotopy class of a path γ (in X) starting at a; γ lifts to $\tilde{\gamma}$ starting at e_a in E. Concretely, if $t \in [0, 1]$, $\tilde{\gamma}(t)$ is the path homotopy class of $\gamma|_{[0,t]}$; $\tilde{\gamma}$ has a lift $\tilde{\tilde{\gamma}}$ in E' starting at b. Let y be its endpoint. This does not depend on the choice of γ (since the path homotopies lift). We have $p'(y) = c$ since

$$p'(y) = \tilde{\gamma}(1) = \text{homotopy class of } \gamma = c.$$

Now mapping c to y defines a section s of p', $s : E \to E'$. We must prove that this is continuous. Choose an open path connected set U in X containing $p(c)$ with $\pi_1(U, P(c)) \to \pi_1(X, p(c))$ trivial. Let U be small enough so that E' is trivializable on $A_{c,U}$. Our section s takes $A_{c,U}$ to a path component of $p^{-1}(A_{c,U})$ (and thus is continuous). Indeed let γ be as before; we choose a path d from $p(c)$ to $p(c')$ in U. We denote by $\tilde{d}$ the lift via the homeomorphism $A_{c,U} \to U$, and by $\tilde{\tilde{d}}$ the lift via the homeomorphism

$$\text{path component of } s(c) \text{ in } p'^{-1}(A_{c,U}) \to A_{c,U}.$$

Then, $\tilde{\tilde{\gamma}}\tilde{\tilde{d}}$ is the lift of γd and its end point lies in the same path component as $s(c)$. □

Remark. The Galois group of this particular universal cover is canonically isomorphic to $\pi_1(X, a)$. $\pi_1(X, a)$ acts on the left on E: $(g, c) \mapsto g \cdot c$ (where $g \in \pi_1(X, a)$ and $c \in E$).

Remark. The fiber at a is $\pi_1(X, a)$. On this fiber the action is left multiplication (which is transitive).

Remark. Let X be a semi-locally simply connected space and let (E, p) an arbitrary universal cover of X (not necessarily the one described). Let G be the Galois group. Fix a base point a of X and let $b \in p^{-1}(a)$. Then there is a group isomorphism $\alpha_b : \pi_1(X, a) \to G$, satisfying the equality:

$$b \cdot c = \alpha_b(c) \cdot b,$$

where $b.c$ designates the right action of $\pi_1(X, a)$ on the fiber at a and $\alpha_b(c).b$ the left action of G on E.

Proof. We show α_b is a homomorphism. Let c and c' be in $\pi_1(X,a)$. We have the following equalities: $\alpha_b(cc') \cdot b = b \cdot (cc') = (b \cdot c)c' = (\alpha_b(c) \cdot b) \cdot c' = \alpha_b(c) \cdot (b \cdot c')$ since the action of G is compatible with that of $\pi_1(X,a)$ on the fiber E_a at a (transfer of structure). And $\alpha_b(c) \cdot (b \cdot c') = (\alpha_b(c)\alpha_b(c')) \cdot b$.

To prove that this is an isomorphism (i.e. is bijective), we can replace (E,p) by an isomorphic cover i.e. by the canonical cover associated to the base point a, with b replaced by a. □

Let B be a semi-locally simply connected space; let $a \in B$; let E be a universal cover of B with Galois group G. We have an equivalence of categories between the covers of B, the right G-sets and the right $\pi_1(X,a)$-sets:

$$\begin{array}{ccc} \text{right } G\text{-sets} & & \text{right } \pi_1(X,a)\text{-sets} \\ F & \longmapsto & F \times E_a \big/ (f,gb) \sim (fg,b) \\ A \times \pi_1(X;a,a') \big/ (ug,y) \sim (u,gy) & \longleftarrow\!\!\shortmid & A \end{array}$$

where $u \in A$, $g \in \pi_1(X,a)$ and $y \in \pi_1(X;a,a')$.