Public Economics
ECN 741

Roozbeh Hosseini

This version: March 11, 2015

Contents

Acknowledgements

In preparing these notes I have benefited from: V.V. Chari and Larry Jone's first year Macro lecture notes Chari, Narayana Kocherlakota and Mike Golosov's lectures on Public Economics.

These notes are prepared for the Public Economics course at the W.P. Carey School of Business at Arizona State University. They may contain errors. Please use with caution and do not circulate. Comments are welcome.

Part I

Syllabus

TOPICS IN PUBLIC ECONOMICS

Instructor: Roozbeh Hosseini (http://www.public.asu.edu/~rhossein)

Office: BAC 638

Phone: (480)-727-7933

Time and place of meeting: Wednesdays 8:35 AM - 11:35 AM, BA 390

Office hours: By appointment

Introduction

In this course we study macroeconomics approach to public finance and issues in dynamic optimal taxation. We first review basic results in the "Ramsey approach to optimal taxation". This approach studies the problem of choosing optimal taxes given that only distortionary tax instruments are available (typically linear taxes). In this approach, lump sum taxes are excluded and more importantly it is assumed that all activities of agents are observable. We next turn to the study of "Mirrleesian approach to optimal taxation". This approach allows for arbitrary nonlinear taxes (including lump-sum). The key feature of Mirrleesian approach is that activities of individuals are unobservable. This imposes an endogenous restriction on the set of fiscal policy instruments as they should provide the correct incentive for individuals to exert effort according to their (unobserved) abilities. Because of these features, Mirrleesian models provide a rich environment suitable to study issues like: efficient redistribution, inequalities, intergenerational wealth transfers, provision of insurance over life cycle, etc.

We use the technical tools we develop in the course to study the insurance market imperfections and public vs. private provision of insurance.

Requirements

Students are expected to read the papers before the class and actively participate in class discussions. The papers that will be discussed in lectures are marked by '★'. Every student will choose a paper from the reading list and present it in class. Preference is given to the papers marked by '⋇'. In addition, each student will write a referee report on the paper that they present. I will also assign one or two computational assignments. The grade will be based on presentation, referee report and the computational assignments.

Tentative Plan

This is a very preliminary (and ambitious) plan for the course.[1]

Lecture	Topic	Papers covered in class
1	**Static Ramsey**	Chari-Kehoe(1999)
2-3	**Dynamic Ramsey**	Chamley(1986), Judd(1985) Atkeson-Chari-Kehoe(1999) Erosa-Gervais (2002),Conesa-Kitao-Krueger (2008) Werning (2007)
4	**Static Mirrlees**	Mirrlees(1971), Seade(1977) Saez(2001)
5-6	**Inverse Euler Equation**	Golosov-Kocherlakota-Tsyvinski (2003) Grochulski-Kocherlakota (2007) Farhi-Werning (2008)
6	**Long Run Properties**	Farhi-Werning (2007)
7-8	**Optimal Wealth Taxes**	Kocherlakota (2005), Farhi-Werning (2008) Grochulski-Kocherlakota (2007)
9-10	**Market vs. Government**	Hosseini (2008),Golosov-Tsyvinski (2006) Ales-Maziero (2008), Krueger-Perri (2009) Fukushima (2010) Prescott-Townsend(1984), Bisin-Gottardi(2004)
11-13	**Risk Sharing across Generations**	Gottardi-Kubler(2007) Henriksen-Spear(2006) Krueger-Kubler (2005)
13-15	**Student Presentations**	

Useful books

1. Kocherlakota, N. R (2010), *The New Dynamic Public Finance* (Princeton University Press)

2. Atkinson, A. and J. Stiglitz (1980), *Lectures in Public Economics* (New York: McGraw Hill).

3. Auerbach, A. and M. Feldstein, *Handbook of Public Economics: Volumes 1 & 2 & 3 & 4* (Amsterdam: North Holland).

[1] We will split each class into two lectures.

4. Myerson, Roger B. (1997), *Game Theory: Analysis of Conflict* (Harvard University Press).

5. Sargent, T. and Lars Ljungqvist (2004), *Recursive Macroeconomic Theory, 2nd edition*, (Cambridge: MIT Press).

6. Tuomala, M. (1990), *Optimal Income Tax and Redistribution* (Oxford: Clarendon Press).

Extended (yet incomplete) Reading List

Papers that are marked by '★' will be discussed in lectures. Please read them ahead of time. At the end of each class I will announce what paper(s) will be discussed in the following lecture. Papers marked with '⋇' are recommended for class presentation.

If you are interested to do research in this field, the following list should be a good starting point. However, this is by no means a complete list. You should view this as a sample of what is out there on each of the following topics.

1 Ramsey Approach to Optimal Taxation

1.1 Static results

1. ★ Atkinson, A. and J. Stiglitz (1972), The structure of indirect taxation and economic efficiency, *Journal of Public Economics* 1, 97–119.

2. ★ Diamond, P. and J. Mirrlees (1971), Optimal taxation and public production I: production efficiency, *American Economic Review* 61, 8–27.

3. ★ Diamond, P. and J. Mirrlees (1971), Optimal Taxation and Public Production II: Tax Rules, *American Economic Review* 61, 261-278.

4. Diamond, P. (1975), A Many-Person Ramsey Tax Rule, *Journal of Public Economics* 4, 335-342.

5. Ramsey, F. P. (1927), A Contribution to the Theory of Taxation, *Economics Journal* 37:47-61.

1.2 Capital and labor income taxation in dynamic environment (infinite horizon models)

1. Albanesi, S. and R. Armenter (2008), Understanding Capital Taxation in Ramsey Models, Working Paper, Columbia University.

2. Albanesi, S. and R. Armenter (2008), Intertemporal Distortions in the Second Best, Working Paper, Columbia University.

3. ★ Atkeson, A., V.V. Chari, and P. Kehoe (1999), Taxing Capital Income: A Bad Idea, Federal Reserve Bank of Minneapolis Quarterly Review 23, 3-18.

4. Barro, R. (1979), On the Determination of the Public Debt, *Journal of Political Economy*, 87, 940–71.

5. Bassetto, Marco and Narayana Kocherlakota (2004) ,On the irrelevance of government debt when taxes are distortionary, *Journal of Monetary Economics*, Elsevier, vol. 51(2), pages 299-304, March.

6. ★ Chari, V.V. and Patrick J. Kehoe (1999), Optimal fiscal and monetary policy, in: J. B. Taylor & M. Woodford (ed.), *Handbook of Macroeconomics, edition 1, volume 1*, chapter 26, pages 1671-1745 Elsevier (Also available as Minneapolis Fed Staff Report, SR 251).

7. Chamley, C. (1986), Optimal Taxation of Capital Income in General Equilibrium with Infinite Lives, *Econometrica* 54, 607–22.

8. ★ Judd, Kenneth L. (1985), Redistributive Taxation in a Simple Perfect Foresight Model, *Journal of Public Economics* 28, 59–83.

9. Jones, L., R. Manuelli and P. Rossi (1997), On the Optimal Taxation of Capital Income, *Journal of Economic Theory* 73, 93–117.

10. Lucas, R. and N. Stokey (1983), Optimal Fiscal and Monetary Policy in an Economy without Capital, *Journal of Monetary Economics*, 12, 55–93.

11. ⋇ Saez, E. (2002), Optimal Progressive Capital Income Taxation in the Infinite Horizon Model, NBER Working Paper 9046.

12. ⋇ Werning, Iván (2007), Optimal Fiscal Policy with Redistribution, *Quarterly Journal of Economics*.

1.2.1 Life cycle models (finite horizon)

1. Atkinson, A. B. (1971), Capital Taxes, the Redistribution of Wealth and Individual Savings, *Review of Economic Studies*, Blackwell Publishing, vol. 38(114), pages 209-227, April.
2. Atkinson A. and A. Sandmo (1980), Welfare Implications of the Taxation of Savings, *Economic Journal* 90, 529-549.
3. Auerbach, Alan J. (1985), The Theory of Excess Burden and Optimal Taxation, Handbook of Public Economics, in: A. J. Auerbach & M. Feldstein (ed.), *Handbook of Public Economics, edition 1, volume 1*, chapter 2, pages 61-127 Elsevier (Also available as NBER working paper 1025).
4. ⋇ A. Erosa and M. Gervais (2002), Optimal Taxation in Life-Cycle Economies, *Journal of Economic Theory* 105, 338-369.
5. ⋇ Conesa, J. C., Sagiri Kitao and Dirk Krueger (2007), Taxing Capital? Not a Bad Idea After All!, NBER Working Papers 12880, National Bureau of Economic Research, Inc.
6. Stiglitz, Joseph E. (1987), Pareto Efficient and Optimal Taxation and the New New Welfare Economics, Handbook of Public Economics, in: A. J. Auerbach & M. Feldstein (ed.), *Handbook of Public Economics, edition 1, volume 2*, chapter 15, pages 991-1042 Elsevier. (also available as NBER working paper 2189).

1.3 Other topics (not covered in this course)

1.3.1 Stochastic dynamic environment

1. Chari, V.V., Christiano, L. and P. Kehoe (1994) Optimal fiscal policy in a business cycle model, *Journal of Political Economy* 102, 617–52.
2. Chari, V.V., Christiano, L. and P. Kehoe (1996), Optimality of the Friedman rule in economies with distorting taxes. *Journal of Monetary Economics* 37, 203–23.

3. Correia, I., J.-P. Nicolini and P. Teles (2002), Optimal fiscal and monetary policy: equivalence results. Working Paper No. WP-02-16, Federal Reserve Bank of Chicago.

1.3.2 Optimal taxation with exogenously constraint debt market

1. Aiyagari, S.R., A. Marcet, T. Sargent and J. Seppala (2002), Optimal taxation without state-contingent debt. *Journal of Political Economy*, 110, 1220-1254.
2. ⋇ Angeletos, M. (2002), Fiscal Policy with Non-Contingent Debt and the Optimal Maturity Structure, *Quarterly Journal of Economics* 117:2
3. Farhi, E. (2006), Capital Taxation and Ownership when Markets are Incomplete, Working paper, Harvard University.
4. ⋇ Sleet, C. (2004), Optimal Taxation with Private Government Information, *Review of Economic Studies*, 71:1217-1239.
5. ⋇ Sleet, C. and S. Yeltekin (2006), Optimal Taxation with Endogenously Incomplete Debt Markets, *Journal of Economic Theory*, 127: 36-73.

1.3.3 Optimal taxation with incomplete markets and heterogeneous agents

1. Aiyagari S. Rao (1995), Optimal Capital Income Taxation with Incomplete Markets, Borrowing Constraints, and Constant Discounting, *Journal of Political Economy* 103 (1995), 1158-1175.
2. Aiyagari, S. Rao and Ellen McGrattan (1998), The Optimum Quantity of Debt, *Journal of Monetary Economics*, 42:447-469.
3. ⋇ Krueger D. (2006), Public Insurance against Idiosyncratic and Aggregate Risk: The Case of Social Security and Progressive Income Taxation, CESifo Economic Studies, Oxford University Press, vol. 52(4), pages 587-620, December.
4. ⋇ Conesa, Juan Carlos and Krueger, D. (2006), On the Optimal Progressivity of the Income Tax Code, *Journal of Monetary Economics*, Elsevier, vol. 53(7), pages 1425-1450, October.

1.3.4 Credible government policy, Time inconsistency, Lack of commitment

1. ⋇ Chari, V V and P. J. Kehoe (1990), Sustainable Plans, *Journal of Political Economy*, University of Chicago Press, vol. 98(4), pages 783-802, August.

2. Chari, V. V. and P. J. Kehoe (1993), Sustainable Plans and Debt, *Journal of Economic Theory*, Elsevier, vol. 61(2), pages 230-261, December.

3. Chari, V V and P. J. Kehoe (1993), Sustainable Plans and Mutual Default, *Review of Economic Studies*, Blackwell Publishing, vol. 60(1), pages 175-95, January.

4. Kydland, Finn E and E. C. Prescott (1977), Rules Rather Than Discretion: The Inconsistency of Optimal Plans, *Journal of Political Economy*, University of Chicago Press, vol. 85(3), pages 473-91, June.

5. Kydland, Finn E and E. C. Prescott (1980), Dynamic optimal taxation, rational expectations and optimal control, *Journal of Economic Dynamics and Control*, Elsevier, vol. 2(1), pages 79-91, May.

6. ⋇ Christopher Phelan and E. Stacchetti (2001), Sequential Equilibria in a Ramsey Tax Model, *Econometrica*, Econometric Society, vol. 69(6), pages 1491-1518, November.

7. ⋇ Yared, P. (2008), Politicians, Taxes and Debt, Working paper, Columbia Business School.

2 Mirrleesian Approach to Optimal Taxation

2.1 Static models

1. Diamond, P. (1988), Optimal Income Taxation: An Example with a U-Shaped Pattern of Optimal Marginal Tax Rates, *American Economic Review* 88, 83-95.

2. Grochulski, B. (2007), Optimal Nonlinear Income Taxation with Costly Tax Avoidance, *Federal Reserve Bank of Richmond Economic Quarterly*, Volume 93, Number 1, Pages 77–109.

3. ⋇ Luttmer, Erzo F.P. and Richard Zeckhauser (2008), Schedule Selection by Agents: from Price Plans to Tax Tables, NBER Working Paper No. 13808.

4. ★ Mirrlees, J. (1971), An Exploration in the Theory of Optimum Income Taxation, *Review of Economic Studies*, 38,175–208.

5. ★ Mirrlees, J. (1976), Optimal Tax Theory : A Synthesis, *Journal of Public Economics*, Elsevier, vol. 6(4), pages 327-358, November.

6. ★ Saez, E. 2001. Using Elasticities to Derive Optimal Income Tax Rates, *Review of Economic Studies*, 68, 205–29.

7. ★ Seade, J. K. (1977), On the Shape of Optimal Tax Schedules, *Journal of Public Economics*, Elsevier, vol. 7(2), pages 203-235, April.

8. ⋇ Seade, J. K. (1977), On the Sign of the Optimum Marginal Income Tax, *The Review of Economic Studies*, Vol. 49, No. 4 (Oct., 1982), pp. 637-643.

2.2 Dynamic models: "New Dynamic Public Finance"

1. Golosov, Mike, Aleh Tsyvinski and Iván Werning (2007), New Dynamic Public Finance: a User's Guide, in *NBER Macroeconomic Annual 2006* (MIT Press)

2. Kocherlakota, N. (2006) , Advances in Dynamic Optimal Taxation, in the Ninth World Congress of the Econometric Society conference Volume.

2.2.1 Dynamic incentive compatibility

1. ★ Townsend, Robert M. (1988), Information Constrained Insurance : The Revelation Principle Extended, *Journal of Monetary Economics*, Elsevier, vol. 21(2-3), pages 411-450.

2. Myerson, Roger B. (1986), Multistage Games with Communication, *Econometrica*, Econometric Society, vol. 54(2), pages 323-58, March.

2.2.2 Optimal inter-temporal distortions: "Inverse Euler Equation" and optimal wealth taxes

1. Atkinson, A.B. and J.E. Stiglitz (1976), The Design of Tax Structure: Direct vs. Indirect Taxation, *Journal of Public Economics*, 6, 55-75.

2. ⋇ Albanesi, S. (2006), Optimal Taxation of Entrepreneurial Capital with Private Information, NBER Working Paper 12419.

3. ⋇ Albanesi, S. and Sleet, C. 2006. Dynamic Optimal Taxation with Private Information, *Review of Economic Studies* 73, 1–30.

4. Diamond, P. A. & Mirrlees, J. A. (1978), A Model of Social Insurance with Variable Retirement, *Journal of Public Economics*, Elsevier, vol. 10(3), pages 295-336, December.

5. ⋇ Farhi, Emmanuel and Iván Werning (forthcoming), Optimal Savings Distortions with Recursive Preferences, *The Journal of Monetary Economics*, Carnegie Rochester Series, 2007.

6. ★ Golosov, M., Kocherlakota, N., and Tsyvinski, A. (2003), Optimal Indirect and Capital Taxation, *Review of Economic Studies* 70, 569–87.

7. ★ Golosov, M. and Tsyvinski, A. (2006), Designing Optimal Disability Insurance: a Case for Asset Testing, *Journal of Political Economy* 114, 257–79.

8. ★ Grochulski, B. and N. Kocherlakota (2008), Nonseparable Preferences and Optimal Social Security Systems, Minnesota Economic Research Reports 2007-1.

9. ⋇ Grochulski, B. and T. Piskorski (2006), Risky Human Capital and Deferred Capital Income Taxation, Federal Reserve Bank of Richmond Working Paper No. 06-13.

10. ★ Kocherlakota, N. (2005), Zero Expected Wealth Taxes: a Mirrlees Approach to Dynamic Optimal Taxation, *Econometrica* 73, 1587–622.

11. Rogerson, W. (1985), Repeated Moral Hazard, *Econometrica*, 53, 69-76.

2.2.3 Long run properties of Pareto Optima

1. Atkeson, Andrew and Robert E. Lucas Jr. (1992), On Efficient Distribution with Private Information, *Review of Economic Studies*, v59 (3), 427-453.

2. Green, Edward (1987), Lending and the Smoothing of Uninsurable Income, in Contractual Arrangements for Intertemporal Trade, University of Minnesota Press.

3. ★ Farhi, Emmanuel and Iván Werning (2007), Inequality and Social Discounting, *Journal of Political Economy*, 115(3), 365-402.

4. Phelan, Christopher (2006), Opportunity and Social Mobility, *Review of Economic Studies*. vol. 73(2), pp. 487-505.

5. Thomas, Jonathan and Worrall, Tim (1990), Income Fluctuation and Asymmetric Information: An Example of a Repeated Principal-Agent Problem, *Journal of Economic Theory*, Elsevier, vol. 51(2), pages 367-390, August.

2.2.4 Optimal estate taxes

1. ★ Farhi, Emmanuel and Iván Werning (2008), Progressive Estate Taxation, Working Paper, MIT and Harvard University

2.2.5 Optimal labor income taxes

1. ★ Battaglini, Marco and Steve Coate (forthcoming), Pareto Efficient Income Taxation with Stochastic Abilities, *Journal of Public Economics*.

2. ⋇ Bohacek, Radim and Marek Kapicka (2008), Optimal Human Capital Policies, *Journal of Monetary Economics*, vol. 55 (1), pages 1-16, January 2008)

3. ⋇ Kapicka, Marek (2006), Optimal Income Taxation with Human Capital Accumulation and Limited Record Keeping, *Review of Economic Dynamics*, Elsevier for the Society for Economic Dynamics, vol. 9(4), pages 612-639, October.

4. ⋇ Weinzierl, Matt (2008), The Surprising Power of Age-Dependent Taxes, Working Paper, Harvard University.

2.2.6 Other topics (not covered in this course)

Political Economy of Nonlinear Taxation and Mechanisms

1. ⋇ Acemoglu & Michael Golosov & Aleh Tsyvinski (2008), Political Economy of Mechanisms, *Econometrica*, 76(3), pp. 619-641.

2. ⁎ Acemoglu & Michael Golosov & Aleh Tsyvinski (2008), Markets Versus Governments, *Journal of Monetary Economics*, 55(1), pp. 159-189.

3. ⁎ Acemoglu & Michael Golosov & Aleh Tsyvinski (2008), Dynamic Mirrlees Taxation and Political Economy, NBER Working Paper #12224

4. ⁎ Farhi, Emmanuel and Iván Werning (2008), The Political Economy of Nonlinear Capital Taxation, Working Paper, MIT and Harvard University.

5. Roberts, Kevin (1984), The Theoretical Limits to Redistribution, *Review of Economic Studies*, 51, 177-195.

6. ⁎ Sleet, C. and S. Yeltekin (2006), Credibility and Endogenous Societal Discounting, *Review of Economic Dynamics* 9, 410-437.

Monetary Policy

1. ⁎ da Costa, C. and I. Werning. 2005. On the optimality of the Friedman rule with heterogeneous agents and non-linear income taxation. Working paper, MIT.

Persistent Private Information

1. ⁎ Fernandes, Ana and Christopher Phelan (2000), A Recursive Formulation for Repeated Agency with History Dependence, *Journal of Economic Theory*, Elsevier, vol. 91(2), pages 223-247, April.

2. ⁎ Kapicka, Marek (2006), Efficient Allocations in Dynamic Private Information Economies with Persistent Shocks: A First Order Approach, Working Paper, UCSB.

3. ⁎ Williams, Noah (2008), Persistent Private Information, Working Paper, University of Wisconsin.

4. ⁎ Zhang, Yuzhe (2008), Dynamic Contracting with Persistent Shocks, Working Paper, University of Iowa.

3 Risk Sharing, Insurance Markets and the Role of Government

3.1 Risk sharing within generation: Government or Market?

1. ★ Ales, L. and Maziero (2008), Accounting for Private Information, Working Paper.

2. Allen, F. (1985), Repeated Principal-Agent Relationships with Lending and Borrowing, *Economic Letters*, 17, 27-31.

3. ⋇ Arnott, R. and Stiglitz, J.E., 1986, Moral Hazard and Optimal Commodity Taxation, *Journal of Public Economics* 29, 1-24.

4. ⋇ Arnott, R. and Stiglitz, J.E., 1990, The Welfare Economics of Moral Hazard. In H. Louberge ed. ed., *Information and Insurance: Essays in Memory of Karl H. Borch*, Norwell, MA: Kluwer, 91-121.

5. ⋇ Bisin, Alberto and Adriano Rampini (2006), Markets as Beneficial Constraints on the Government, *Journal of Public Economics*, 90, 601-629.

6. ⋇ Bisin, Alberto and Piero Gottardi (2006), Efficient Competitive Equilibria with Adverse Selection, *Journal of Political Economy*, University of Chicago Press, vol. 114(3), pages 485-516, June.

7. ⋇ Chari, V.V. (2000) Limits of Markets and Limits of Governments: An Introduction to a Symposium on Political Economy, *Journal of Economic Theory*, 94, 1-6.

8. Cole, H. and Kocherlakota, N., 2001, Efficient Allocations with Hidden Income and Hidden Storage, *Review of Economic Studies*, 68, 523-542.

9. ★ Golosov, Michael and Aleh Tsyvinski (2007) Optimal Taxation with Endogenous Insurance Markets, *Quarterly Journal of Economics*, 122, 487-534.

10. ⋇ Hammond, P.J., 1987, Markets as Constraints: Multilateral Incentive Compatibility in Continuum Economies. *Review of Economic Studies*, 54, 399-412.

11. ★ Hosseini, R. (2008), Adverse Selection in The Annuity Market and the Role for Social Security, Working Paper.

12. ⋇ Krueger, D., and Perri, F., 2009, Public vs Private Risk Sharing Mimeo. University of Pennsylvania.

13. Prescott, E. C., and Townsend, R., 1984, Pareto Optima and Competitive Equilibria with Adverse Selection and Moral Hazard, *Econometrica* 52, 21-45.

3.2 Risk sharing across generations

1. Aiyagari, S. Rao and Dan Peled (1991), Dominant Root Characterization of Pareto Optimality and the Existence of Optimal Equilibria in Stochastic Overlapping Generations Models, *Journal of Economic Theory*, Elsevier, vol. 54(1), pages 69-83, June.

2. ⋇ Ball, Laurence and Gregory Mankiw (2007), Intergenerational Risk Sharing in the Spirit of Arrow, Debreu, and Rawls, with Applications to Social Security Design, *Journal of Political Economy*, University of Chicago Press, vol. 115, pages 523-547.

3. ⋇ Chattopadhyay, Subir and Piero Gottardi (1999), Stochastic OLG Models, Market Structure, and Optimality, *Journal of Economic Theory*, Elsevier, vol. 89(1), pages 21-67, November.

4. ⋇ Demange, G. (2002), On Optimality of Intergenerational Risk sharing, *Economic Theory*, 20, 1-27.

5. ★ Henriksen, Espen and Stephen Spear (2006), Dynamic Suboptimality of Competitive Equilibrium in Multiperiod Stochastic Overlapping Generations Economies," 2006 Meeting Papers 35, Society for Economic Dynamics.

6. ★ Gottardi, Piero and Felix Kubler (2006), Social Security and Risk Sharing, CESifo Working Paper Series CESifo Working Paper No.1705.

7. ⋇ Peled, D. (1984), Stationary Pareto Optimality of Stochastic Asset Equilibria with Overlapping Generations Model, *Journal of Economics Theory*, 44, 209-213.

8. ⋇ Prescott, Edward C. and José-Víctor Ríos-Rull (2005), On the Equilibrium Concept for Overlapping Generations Organizations, *International Economic Review*,Vol. 46, No. 4, pp. 1065-1080.

3.3 Optimal unemployment insurance (not covered in class)

1. Acemoglu, D. and Rob Shimer (1999), Efficient Unemployment Insurance, *Journal of Political Economy*, 107, pp. 893-928.

2. Atkeson A. and R. Lucas (1995), Efficiency and Equality in a Simple Model of Efficient Unemployment Insurance, *Journal of Economic Theory* 66 (1995), 64-68.

3. ⋇ Hopenhayn, Hugo A., and Juan Pablo Nicolini (1997), Optimal Unemployment Insurance, *Journal of Political Economy*, 105, 2, 412-38. April 1997.

4. ⋇ Mitchell, Matt and Yuzhe Zhang (2008), Unemployment Insurance with Hidden Saving, Working paper, University of Iowa.

5. S. Shavell and L. Weiss (1979), The Optimal Payment of Unemployment Insurance Benefits over Time, *Journal of Political Economy* 87, 1347-1362.

4 Social Security

4.1 Normative Theories

1. Diamond, P. (1977), A Framework for Social Security Analysis, *Journal of Public Economics*, Vol. 8, No. 3, 275-298.

2. Feldstein, M. (1985), The Optimal Level of Social Security Benefits, *Quarterly Journal of Economics*, Vol. 100, No. 2 (May, 1985), pp. 303-320

3. Feldstein, M. (2005), Rethinking Social Insurance, *American Economic Review*, Vol. 95, No. 1 (Mar., 2005), pp. 1-24

4. Mulligan, Casey B. and Sala-i-Martin, Xavier (1999),Social Security in Theory and Practice (I): Facts and Political Theories, NBER Working Paper No. W7118.

5. Mulligan, Casey B. and Sala-i-Martin, Xavier (1999), Social Security in Theory and Practice (II): Efficiency Theories, NBER Working Paper No. W7119.

Part II

Lecture Notes

Lecture Notes in Public Economics*

Roozbeh Hosseini

March 11, 2015

*In preparing these notes I have benefited from: Chari and Kehoe (1998), V.V. Chari and Larry Jone's first year Macro lectures notes and V.V. Chari, Narayana Kocherlakota and Mike Golosov's lectures on Public Economics. These notes are prepared for the Public Economics course at the W.P. Carey School of Business, Arizona State University. They are preliminary and may contain errors. Please use with caution and do not circulate. Comments are welcome.

Contents

1 Ramsey Taxation - Primal Approach

Consider an economy with n types of consumption good that are produced using labor input:

$$F(c_1 + g_1, \dots, c_n + g_n, l) = 0 \tag{1}$$

c_i is private and g_i is public consumption of good i and l is the labor input. F is a constant return to scale technology. Consumers face the following maximization problem

$$\max_{c_1,\dots,c_n,l} U(c_1, \dots, c_n, l)$$

subject to

$$\sum_{i=1}^{n} p_i(1 + \tau_i)c_i = l$$

in which τ_i is the taxed levied on consumption of good i (wage is normalized to 1).

There is a representative firm that produces goods using technology F:

$$\max_{x_1,\dots,x_n,l} \sum_{i=1}^{n} p_i x_i - l$$

$$F(x_1, \dots, x_n, l) = 0$$

Government has to finance its purchase $g = (g_1, \dots, g_n)$ using linear taxes τ_i

$$\sum_{i=1}^{n} p_i g_i = \sum_{i=1}^{n} p_i \tau_i c_i \tag{2}$$

Let's take government purchase as given. *A Competitive Equilibrium* is

- Consumers and producers allocations: (c, x, l)
- prices: $p = (p_1, \dots, p_n)$
- policy: $\pi = (\tau_1, \dots, \tau_n)$

such that

1. Given policy π and prices p, (c, l) solve consumers problem.
2. Given prices, p, (x, l) solves producers problem.

3. Government budget (equation (2)) holds

4. Allocations are feasible (or market clearing if you like!)

$$c_i + g_i = x_i \text{ for } i = 1, \ldots, n \tag{3}$$

Proposition 1 *Any competitive equilibrium allocations must satisfy the resource feasibility constraint*

$$F(c_1 + g_1, \ldots, c_n + g_n, l) = 0 \tag{4}$$

and an implementability constraint

$$\sum_{i=1}^{n} U_i c_i + U_l l = 0. \tag{5}$$

Furthermore, any allocations that satisfy (4) and (6) can be supported as a competitive equilibrium for appropriately constructed polices and prices.

Proof.

Suppose (c, x, l) is a competitive equilibrium allocation. Then the following FOC must hold

$$\frac{U_i}{U_l} = -(1 + \tau_i) p_i \text{ for } i = 1, \ldots, n$$

together with the following budget constraint

$$\sum_{i=1}^{n} p_i (1 + \tau_i) c_i = l.$$

Replacing out for prices (and taxes) from FOC into budget constraint gives the implementability constraint. The feasibility follows by definition of equilibrium.

Now consider allocations (c, x, l) that are feasible (given vector of g) and satisfy (5). Construct prices from the FOC of the firm

$$p_i = -\frac{F_i}{F_l} \text{ for } i = 1, \ldots, n$$

set policy as

$$1 + \tau_i = \frac{U_i}{U_l} \frac{F_l}{F_i} \text{ for } i = 1, \ldots, n$$

You can verify that the policy and prices (as constructed above) together with the allocation (c, x, l) is a competitive equilibrium.

■

We are interested in the problem of choosing the best policy π to maximize the welfare of consumers. One restriction on such a problem is that the resulting allocation be a competitive equilibrium allocation for each given policy. The timing is the following: First, government chooses a policy, Second, private agents makes decision. We are interested in finding the equilibrium of this game.

1.1 Ramsey problem

Suppose the set of feasible policy for government in Π.

Definition 1 *A Ramsey equilibrium is a policy* $\pi = (\tau_1, \cdots \tau_n) \in \Pi$, *allocation rules* $c(\cdot)$, $x(\cdot)$ *and* $l(\cdot)$ *and price function* $p(\cdot)$ *such that*

$$\pi \in \arg\max_{\pi' \in \Pi} U(c(\pi'), l(\pi'))$$

subject to

$$\sum_{i=1}^{n} p_i g_i = \sum_{i=1}^{n} p_i \tau_i c_i$$

and $(c(\pi'), x(\pi'), l(\pi'))$ *together with* $p(\pi')$ *is a competitive equilibrium for every* $\pi' \in \Pi$.

Suppose π, $(c(\cdot), x(\cdot), l(\cdot))$ and $p(\cdot)$ is a *Ramsey equilibrium.* Then we call $(c(\pi), x(\pi), l(\pi))$ a *Ramsey allocation.*

Proposition 2 *Suppose* c^* *and* l^* *are part of a Ramsey allocation. Then*

$$(c^*, l^*) \in \arg\max_{c,l} U(c, l)$$

subject to (5) *and* (4).

Proof.

Follows from the definition of Ramsey allocation.

■

1.2 Elasticities and optimal taxes

Suppose $n = 2$. Consider the following Ramsey problem

$$\max_{c_1,c_2,l} U(c_1, c_2, l)$$

subject to

1. Implementability constraint

$$U_1 c_1 + U_2 c_2 + U_l l = 0 \tag{6}$$

2. Feasibility

$$F(c_1 + g_1, c_2 + g_2, l) = 0 \tag{7}$$

Let λ and γ be multipliers on implementability constraint (equation (6)) and feasibility (equation (7)). First order conditions are

$$U_i + \lambda (U_i + U_{1i} c_1 + U_{2i} c_2 + U_{li} l) = \gamma F_i \qquad i = 1, 2$$

$$U_l + \lambda (U_l + U_{1l} c_1 + U_{2l} c_2 + U_{ll} l) = \gamma F_l$$

We can write these equations as

$$1 + \lambda - \lambda H_l = \gamma \frac{F_l}{U_l}$$

in which, $H_i = -\frac{(U_{1i}c_1 + U_{2i}c_2 + U_{li}l)}{U_i}$ and $H_l = -\frac{(U_{1l}c_1 + U_{2l}c_2 + U_{ll}l)}{U_l}$.

Note that from individual problem we have

$$1 + \tau_i = \frac{U_i}{U_l} \frac{F_l}{F_i}$$

in other words the optimal wedge must satisfy

$$1 + \tau_i = \frac{1 + \lambda - \lambda H_l}{1 + \lambda - \lambda H_i}$$

There you go! If $H_i > H_j$, then it is optimal to tax good i more than good j.

The problem is that, it is not very helpful. Unfortunately, without imposing assumption on U we cannot say much more. Next we consider some special (yet, interesting) cases.

1.2.1 Additive separable utility functions

Suppose U is of the form

$$U(c_1, c_2, l) = u_1(c_1) + u_2(c_2) - v(l)$$

then

$$H_i = -\frac{U_{ii}c_i}{U_i}$$

Our goal to relate H_i to income elasticity of demand for good i. In order to do that, suppose there is a non-wage income m, such that $p_1c_1 + p_2c_2 = l + m$. Consider FOC of consumer (notice that I have ignored taxes for this part)

$$U_i(c_i(p, m)) = p_i\phi(p, m)$$

in which $\phi(p, m)$ is the lagrange multiplier on budget constrain. Let's take derivative w.r.t m

$$U_{ii}\frac{\partial c_i}{\partial m} = p_i\frac{\partial \phi}{\partial m} = \frac{U_i}{\phi}\frac{\partial \phi}{\partial m}$$

or

$$\frac{U_{ii}c_i}{U_i}\frac{m}{c_i}\frac{\partial c_i}{\partial m} = \frac{m}{\phi}\frac{\partial \phi}{\partial m}.$$

Let $\eta_i = \frac{m}{c_i}\frac{\partial c_i}{\partial m}$. Then

$$H_i = -\frac{m}{\phi}\frac{\partial \phi}{\partial m}\frac{1}{\eta_i}$$

Therefore, $H_i > H_j$ if and only if $\eta_j > \eta_i$. Combine this with the above and we get the following:

Result 1 *If preferences are additive separable, necessities should be taxed more than luxuries.*

Example : $U(c_1, c_2, l) = \log(c_1) + \log(c_2 - \bar{c}) - v(l)$

1.2.2 Quasi-linear utility function

Consider the utility function in the previous section and assume that $v(l) = l$. Then there is no income effect and using income elasticities for guiding us about optimal taxes is not useful. However, we use price elasticities. Consider again the FOC of consumer

$$U_i(c_i) = p_i\phi$$

Note that in this case $\phi = 1$ (independent of prices). Take derivative w.r.t p_i

$$U_{ii}\frac{\partial c_i}{\partial p_i} = \phi = \frac{U_i}{p_i}$$

and

$$H_i = \frac{1}{\epsilon_i}$$

Result 2 *If preferences are additive separable and quasi-linear, price-inelastic goods should be taxed more.*

1.2.3 Complementarity with leisure

Sandmo (1987) and Corlett and Hauge (1953-54) argue that goods that are more complement with leisure should be taxed more heavily. The next example shows this

Example : $U(c_1, c_2, l) = c_1^\alpha + c_2^\alpha (1-l)^\beta$

1.3 Uniform commodity taxation

One of the most useful and interesting result in optimal taxation is the *uniform commodity taxation* result. Suppose the preferences are weakly separable in consumption and leisure

$$U(c_1, \ldots, c_n, l) = W(G(c_1, \ldots, c_n), l) \tag{8}$$

furthermore, $G(\cdot)$ is homothetic.

Proposition 3 *Suppose preferences satisfy (8), then it is optimal to tax all goods at the same rate, i.e. $\tau_i = \tau_j$ for all i and j.*

Proof.

Note that the fact that $G(\cdot)$ is homothetic implies that

$$\frac{U_i(\alpha c, l)}{U_j(\alpha c, l)} = \frac{U_i(c, l)}{U_j(c, l)}$$

or

$$U_i(\alpha c, l) = \frac{U_i(c, l)}{U_j(c, l)} U_j(\alpha c, l).$$

Differentiate w.r.t $\alpha$and set $\alpha = 1$ we get

$$\frac{\sum_{k=1}^{n} U_{ik} c_k}{U_i} = \frac{\sum_{k=1}^{n} U_{jk} c_k}{U_j}$$

Also, note that $U_l = W_l$, $U_{li} = W_{lg} G_i$ and $U_i = W_g G_i$. Therefore,

$$H_i = -\frac{\sum_{k=1}^{n} U_{ik} c_k}{U_i} - \frac{U_{il} l}{U_i} = -\frac{\sum_{k=1}^{n} U_{ik} c_k}{U_i} - \frac{W_{lg} l}{W_g} = H_j$$

■

This can be generalized to utility functions of the form

$$u(c_1, \ldots, c_k, G(c_{k+1}, \ldots, c_n), l)$$

in which, $G(\cdot)$ is homothetic. Then the result is that commodities $(c_{k+1}, \ldots, c_n)$ should be taxed at uniform rate.

Exercise: Suppose consumer is endowed with y unit of good one that cannot be taxed away. Does the uniform commodity taxation still hold? what if the utility function is additive separable?

Exercise: Suppose government is restricted to setting taxed on c_1 to zero. How would modify the Ramsey problem? Does the uniform commodity taxation hold?

1.4 Intermediate good taxation

Another powerful and important result in Ramsey taxation is that intermediate good shall not be taxed.

Suppose there are two sectors. One sector produces commodity x_1 that is consumed by private agent, c_1 and by government, g. Commodity x_1 is produced using intermediate good z and labor l_1 as input according to the following production function

$$f(x_1, z, l_1) = 0.$$

The other sector, uses labor l_2 as input to produce good x_2 that can be used as input in production of good x_1 (that is z) or it can be consumed (c_2 and g_2). The technology is the following

$$h(x_2, l_2) = 0.$$

- *Private agents* solves

$$\max_{c,l} U(c_1, c_2, l_1 + l_2)$$

subject to

$$p_1(1 + \tau_1)c_1 + p_2(1 + \tau_2)c_2 \leq l_1 + l_2.$$

- *Producer of good* x_1 solves

$$\max_{x_1,z,l_1} p_1x - l_1 - p_2(1 + \tau_z)z$$

subject to

$$f(x_1, z, l_1) = 0.$$

The FOC for this problem implies

$$\frac{f_z}{f_l} = p_2(1 + \tau_z).$$

- Producer of good x_2 solves

$$\max_{x_2,l_2} p_2x_2 - l_2$$

subject to

$$h(x_2, l_2) = 0.$$

and FOC implies

$$\frac{h_x}{h_l} = -p_2.$$

Combining the FOC condition for two sector we get

$$\frac{h_x}{h_l}(1 + \tau_z) = -\frac{f_z}{f_l}.$$

- Government budget constraint is

$$\tau_1p_1c_1 + \tau_2p_2c_2 + \tau_zp_2z = p_1g_1 + p_2g_2$$

- Finally, feasibility and market clearing

$$\begin{aligned} c_1 + g_1 &= x_1 \\ c_2 + g_2 + z &= x_2 \\ f(x_1, z, l_1) &= 0 \\ h(x_2, l_2) &= 0 \end{aligned}$$

The Ramsey problem is

$$\max U(c_1, c_2, l_1 + l_2)$$

subject to

$$\begin{aligned} U_1 c_1 + U_2 c_2 + U_l(l_1 + l_2) &= 0 \qquad \lambda \\ f(c_1 + g_1, z, l_1) &= 0 \qquad \phi_1 \\ h(c_2 + g_2 + z, l_2) &= 0 \qquad \phi_2 \end{aligned}$$

FOC w.r.t z

$$\phi_1 f_z = -\phi_2 h_x$$

FOC w.r.t l_1 and l_2

$$U_l + \lambda(U_{ll}(l_1 + l_2) + U_l + U_{cl} c) = f_l \phi_1$$

$$U_l + \lambda(U_{ll}(l_1 + l_2) + U_l + U_{cl} c) = h_l \phi_2$$

and therefore,

$$f_l \phi_1 = h_l \phi_2.$$

This implies that

$$\frac{h_x}{h_l} = -\frac{f_z}{f_l}$$

It means that it is optimal to set $\tau_z = 0$ and not distort production efficiency. For more on intermediate good taxation and production efficiency see Diamond and Mirrlees (1971).

2 Optimal Fiscal Policy-Dynamic Ramsey Taxation

The main focus of this section is the derivation of Chamley-Judd result (Chamley (1986) and Judd (1985)). We are only going to consider deterministic environment. See Chari et al. (1994) and Chari and Kehoe (1998) for stochastic environment and optimal policy over business cycle.

The environment is the following. There are infinitely lived identical consumers. Government has to finance expenditure g_t every period and levies distortionary taxes (or subsidies) on consumption, investment, labor and capital income. It can also issue debt.

Consumer's problem: consumers are endowed with k_0 unit of capital and b_0 unit of government debt

$$\max_{c_t,l_t,x_t,k_{t+1},b_{t+1}} \sum_{t=0}^{\infty} \beta^t U(c_t, l_t)$$

subject to

$$\begin{aligned}
(1+\tau_{ct})c_t + (1+\tau_{xt})x_t + b_{t+1} &\leq (1-\tau_{lt})w_t l_t + (1-\tau_{kt})r_t k_t + R_{bt} b_t \quad ; \lambda_t \\
k_{t+1} &\leq (1-\delta)k_t + x_t \\
-b_{t+1} &\leq M
\end{aligned}$$

$$k_0, b_0 \text{ given}$$

in which M is some large positive number.

The FOC's are

$$\beta^t U_{ct} = \lambda_t (1+\tau_{ct}) \tag{9}$$

$$-\beta^t U_{lt} = \lambda_t w_t (1-\tau_{lt}) \tag{10}$$

$$(1+\tau_{xt})\lambda_t = \lambda_{t+1}\left[(1-\tau_{xt+1})(1-\delta) + (1-\tau_{kt+1})r_{t+1}\right] \tag{11}$$

$$\lambda_t = \lambda_{t+1} R_{bt+1} \tag{12}$$

Government Budget:

$$g_t + R_{bt} b_t = b_{t+1} + \tau_{xt} x_t + \tau_{ct} c_t + \tau_{lt} w_t l_t + \tau_{kt} r_t k_t \tag{13}$$

Feasibility:

$$c_t + g_t + k_{t+1} = F(k_t, l_t) + (1-\delta)k_t \tag{14}$$

Competitive pricing implies that

$$\begin{aligned} r_t &= F_k(k_t, l_t) \\ w_t &= F_l(k_t, l_t) \end{aligned} \tag{15}$$

A competitive equilibrium is: the sequence of allocations $x = \{c_t, l_t, b_{t+1}, k_{t+1}, x_t\}_{t=0}^{\infty}$, prices $\{r_t, w_t, R_{bt}\}_{t=0}^{\infty}$, policy $\pi = \{\tau_{ct}, \tau_{lt}, \tau_{xt}, \tau_{kt+1}\}_{t=0}^{\infty}$ such that, the allocations solve consumer problem, given prices and policy, prices are competitive, government budget holds and allocations are feasible.

A Ramsey Equilibrium is a policy π, an allocation rule $x(\cdot)$ and price rules $r(\cdot)$, $w(\cdot)$ and $R_b(\cdot)$ such that:

$$\pi \in \arg\max \sum_{t=0}^{\infty} \beta^t U(c_t, l_t)$$

subject to 12 and $x(\pi)$ be a competitive equilibrium, and

for any policy π', allocation $x(\pi')$ and prices $(r(\pi'), w(\pi'), R_b(\pi'))$ be a competitive equilibrium.

We next derive the implementability condition. Note that if conditions of Ekeland and Scheinkman (1986) and/or Weitzman (1973) are satisfied, then the equilibrium allocations should also satisfy the following Transversality conditions

$$\lim_{t\to\infty} \lambda_t b_{t+1} = 0 \tag{16}$$

$$\lim_{t\to\infty} \lambda_t k_{t+1} = 0 \tag{17}$$

Now multiply consumer's budget constraint by λ_t and sum over t and use (16)-(17)

$$\sum_{t=0}^{\infty} \lambda_t \left[(1+\tau_{ct})c_t + (1+\tau_{xt})(k_{t+1} - (1-\delta)k_t) + b_{t+1}\right] = \sum_{t=0}^{\infty} \lambda_t \left[(1-\tau_{lt})w_t l_t + (1-\tau_{kt})r_t k_t + R_{bt} b_t\right].$$

Now use (9)-(12) and we get

$$\sum_{t=0}^{\infty} \lambda_t \left[(1+\tau_{ct})c_t - (1-\tau_{lt})w_t l_t\right] = \lambda_0 \left\{\left[(1+\tau_{x0})(1-\delta) + (1-\tau_{k_0})r_0\right] k_0 + R_{b0} b_0\right\}.$$

Now replace (9)-(10) and we arrive at the implementability constraint

$$\sum_{t=0}^{\infty} \beta^t \left[U_{ct}c_t + U_{lt}l_t\right] = U_0 \left\{\left[(1+\tau_{x0})(1-\delta) + (1-\tau_{k_0})r_0\right] k_0 + R_{b0}b_0\right\} \tag{18}$$

Proposition 4 *A feasible allocation $x = \{c_t, l_t, b_{t+1}, k_{t+1}, x_t\}_{t=0}^{\infty}$ is a competitive equilibrium allocation if and only it satisfies the implementability constraint (18) (for some period zero policies).*

Proof.

Suppose x is the competitive equilibrium allocation, then following the steps outlines above we can show that it should satisfy the implementability constraint (18). Now suppose an allocation x^* is feasible and satisfy (18) for some proof zero policies.

Note that in any competitive equilibrium, the bond holding must satisfy

$$b_{t+1} = \sum_{s=t+1}^{\infty} \beta^{t-s} \frac{[U_{cs}c_s + U_{ls}l_s]}{U_{ct}} - k_{t+1} \tag{19}$$

in other words, any sequence of c_t^*, l_t^* and k_{t+1}^* uniquely identifies a sequence of b_t that is a part of competitive equilibrium. Candidate wage and rate of return on capital is given by (15). Therefore, from the FOC (9)-(12) we have

$$\begin{aligned} \frac{1-\tau_{lt}}{1+\tau_{ct}} &= -\frac{U_{lt}^*}{F_{lt}^* U_{ct}^*} \\ (1+\tau_{xt})\frac{U_{ct}^*}{1+\tau_{ct}} &= \beta \frac{U_{ct+1}^*}{1+\tau_{ct+1}} \left[(1-\tau_{xt+1})(1-\delta) + (1-\tau_{kt+1})F_{kt+1}^*\right] \\ \frac{U_{ct}^*}{1+\tau_{ct}} &= \beta \frac{U_{ct+1}^*}{1+\tau_{ct+1}} R_{bt+1} \end{aligned} \tag{20}$$

any two of the four taxes can be chosen such that the above conditions hold.

■

2.1 Ramsey problem

The Ramsey problem is the following

$$\max_{c_t, k_{t+1}, l_t} \sum_{t=0}^{\infty} \beta^t U(c_t, l_t)$$

subject to

$$\sum_{t=0}^{\infty} \beta^t \left[U_{ct}c_t + U_{lt}l_t\right] = U_0 \left\{\left[(1+\tau_{x0})(1-\delta) + (1-\tau_{k_0})r_0\right] k_0 + R_{b0}b_0\right\} \quad ; \lambda$$

$$c_t + g_t + k_{t+1} = F(k_t, l_t) + (1-\delta)k_t \quad ; \phi_t$$

Define function $W(\cdot,\cdot,\cdot)$ as

$$W(c,l,\lambda) = U(c,l) + \lambda \left[U_c c + U_l l\right].$$

Now we can rewrite the Ramsey problem as

$$\max_{c_t, k_{t+1}, l_t} \sum_{t=0}^{\infty} \beta^t W(c_t, l_t, \lambda)$$

subject to

$$c_t + g_t + k_{t+1} = F(k_t, l_t) + (1-\delta)k_t \quad ; \phi_t$$

Take first order conditions

$$\frac{W_{lt}}{W_{ct}} = -F_{lt} \tag{21}$$

$$\frac{W_{ct}}{W_{ct+1}} = \beta(1-\delta+F_{kt}) \quad \text{for } t \geq 1 \tag{22}$$

2.2 Chamley-Judd result

Proposition 5 *If the solution to the Ramsey problem converges to a steady state, then at the steady state, the tax rate on capital income is zero.*

Proof.

In (22) at the steady state we have

$$\beta(1-\delta+F_{kt+1}) = 1.$$

This implies that at the steady state there is no inter-temporal distortion. Compare with (20) we have

$$\frac{(1+\tau_{xt})(1+\tau_{ct+1})}{(1+\tau_{ct})(1+\tau_{xt+1})} = \beta\left[1-\delta+\left(\frac{1-\tau_{kt+1}}{1+\tau_{xt+1}}\right)F_{kt+1}\right]$$

Note that any feasible allocation that satisfies (18) can be implemented by two of the four taxes (that is we only need two of the τ_c, τ_l,τ_x and τ_k to implement the same allocations). This in turn implies that

$$\begin{aligned} \tau_{kt} &= 0 \\ \frac{1+\tau_{ct}}{1+\tau_{xt}} &= \text{constant} \end{aligned}$$

■

2.2.1 Heterogeneous consumers

Suppose there are two type of consumers $i = 1, 2$ with preferences

$$\sum_{t=0}^{\infty} \beta^t U^i(c_{it}, l_{it})$$

The resources constraint for the economy is

$$c_{1t} + c_{2t} + k_{t+1} = F(k_t, l_{1t}, l_{2t}) + (1-\delta)k_t \tag{23}$$

implementability constraint for consumer i is

$$\sum_{t=0}^{\infty} \beta^t \left[U_{ct}^i c_{it} + U_{lt}^i l_{it}\right] = U_0^i \left\{\left[(1+\tau_{x0})(1-\delta) + (1-\tau_{k_0})r_0\right] k_0^i + R_{b0} b_0^i\right\} \tag{24}$$

Suppose government puts welfare weights ω_i on consumers of type i. The Ramsey problem is

$$\max \omega_1 \sum_{t=0}^{\infty} \beta^t U^1(c_{1t}, l_{1t}) + \omega_2 \sum_{t=0}^{\infty} \beta^t U^2(c_{2t}, l_{2t})$$

subject to (23) and (24).

Attached multiplier λ_i to implementability constraint of type i and write

$$W(c_1, c_2, l_1, l_2, \lambda_1, \lambda_2) = \sum_{i=1,2} \left[\omega_i U^i(c_i, l_i) + \lambda_i \left(U_c^i c_i + U_l^i l_i\right)\right]$$

$$\max \sum_{t=0}^{\infty} \beta^t W(c_{1t}, c_{2t}, l_{1t}, l_{2t}, \lambda_1, \lambda_2)$$

subject to

$$c_{1t} + c_{2t} + k_{t+1} = F(k_t, l_{1t}, l_{2t}) + (1-\delta)k_t \quad ; \phi_t$$

where W^i is defined the obvious way.

First order conditions imply

$$W_{c_{it}} = \beta W_{c_{it+1}}(1 - \delta + F_{kt+1})$$

and in the steady state

$$1 = \beta(1 - \delta + F_{kt+1})$$

and, therefore, tax on capital should be zero in the steady state.

Capitalists vs Workers (Judd 1985)

Suppose consumer of type 1 does not hold any asset and cannot save, borrow or invest. We call these 'Worker'. Also, assume that all the capital is held by consumer 2 who do not supply any labor. We call these 'Capitalists'. The implementability constraint for 'Worker' is

$$U^1_{ct}c_{1t} + U^1_{lt}l_{1t} = 0 \quad \forall\ t$$

and for 'Capitalist'

$$\sum_{t=0}^{\infty} \beta^t \left[U^2_{ct}c_{2t}\right] = U^2_0 \left\{\left[(1+\tau_{x0})(1-\delta) + (1-\tau_{k_0})r_0\right] k^2_0 + R_{b0}b^2_0\right\} \tag{25}$$

Suppose the welfare weight on 'Worker' utility is 1 and on 'Capitalist' utility is zero.

$$\max \sum_{t=0}^{\infty} \beta^t U^1(c_{1t}, l_{1t})$$

subject to

$$U^1_{ct}c_{1t} + U^1_{lt}l_{1t} = 0 \quad \forall\ t$$

$$\sum_{t=0}^{\infty} \beta^t U^2_{ct}c_{2t} = U^2_0 \left\{\left[(1+\tau_{x0})(1-\delta) + (1-\tau_{k_0})r_0\right] k^2_0 + R_{b0}b^2_0\right\} \tag{26}$$

$$c_{1t} + c_{2t} + k_{t+1} = F(k_t, l_{1t}, l_{2t}) + (1-\delta)k_t \quad ; \phi_t$$

Define

$$W(c_1, c_2, l_1, l_2, \lambda_1, \lambda_2) = U^1(c_1, l_1) + \lambda_i \left(U_c^i c_i + U_l^i l_i\right)$$

First order conditions

$$\lambda\beta^t \left[U_{cct}^2 c_{2t} + U_{ct}^2\right] + \phi_t = 0$$

$$\phi_t = \phi_{t+1}(1 - \delta + F_{kt+1})$$

in steady state $\phi_{t+1} = \beta\phi_t$ and therefore

$$1 = \beta(1 - \delta + F_{kt+1})$$

and again, tax of capital is zero in the steady state.

Exercise: In the above set up we have implicitly assumed that government can levy different taxes on different consumer types. How would you add the following restrictions to the problem

1. Tax on capital income has to be uniform across different types. Does the result hold with this restriction? Under what assumptions?

2. Tax on labor income has to be uniform across different types. Does the result hold? Under what assumptions?

3. Tax on capital income cannot be more than 100 percent. Does the result hold? Under what assumptions?

Dividend Taxes?!!! (an interesting example)

Suppose we write the environment as in McGrattan and Prescott (2005) with corporate taxes and dividend taxes. Consumers can trade share of corporations, s_t, at price v_t. Let d_t be dividend and τ_{dt} be dividend tax. Consumers solve

$$\max_{c_t, s_{t+1}, l_t} \sum_{t=0}^{\infty} \beta^t U(c_t, l_t)$$

subject to

$$\sum_{t=0}^{\infty} p_t \left[c_t + v_t(s_{t+1} - s_t)\right] \leq \sum_{t=0}^{\infty} p_t \left[(1 - \tau_{dt})d_t s_t + (1 - \tau_{lt})w_t l_t\right]$$

$$s_0 = 1$$

FOC implies

$$\frac{U_{ct}}{U_{lt}} = -(1-\tau_{lt})w_t$$

$$p_t v_t = p_{t+1}v_{t+1} + p_{t+1}(1-\tau_{dt+1})d_{t+1}$$

And therefore implementability constraint is

$$\sum_{t=0}^{\infty} \beta^t \left[U_{ct}c_t + U_{lt}l_t\right] = U_{c0}\left[v_0 + (1-\tau_{d0})d_0\right] s_0 \tag{27}$$

There is a corporation that maximizes the present discounted value of owners' dividends and pays taxes τ_t on corporate income.

$$\max \sum_{t=0}^{\infty} p_t(1-\tau_{dt})d_t$$

subject to

$$d_t = f(k_t, l_t) - x_t - w_t l_t - \tau_t\left(f(k_t, l_t) - \delta k_t - w_t l_t\right)$$

$$k_{t+1} = (1-\delta)k_t + x_t$$

First order conditions for the corporation is

$$f_{lt} = w_t$$

$$\frac{p_t(1-\tau_{dt})}{p_{t+1}(1-\tau_{dt+1})} = 1 - (1-\tau_{t+1})(f_{kt+1} - \delta)$$

For this economy the feasibility is

$$\begin{aligned} c_t + k_{t+1} + g_t &= f(k_t, l_t) + (1-\delta)k_t \\ s_t &= 1 \end{aligned}$$

and there is also a government budget constraint

$$\sum_{t=0}^{\infty} p_t g_t = \sum_{t=0}^{\infty} p_t\left[\tau_{dt}d_t s_t + \tau_t\left(f(k_t, l_t) - \delta k_t - w_t l_t\right) + \tau_{lt}w_t l_t\right]$$

Question: What is the appropriate implementability constraint? Is constraint (27) sufficient? In other words, is it true that any feasible allocation that satisfy (27) can be supported in a competitive equilibrium? If not, what other constraints should be added?

2.2.2 Non-Steady State

Proposition 6 *Suppose the utility function is of the form*

$$U(c, l) = \frac{c^{1-\sigma}}{1-\sigma} - v(l),$$

Then Ramsey taxes on capital income is zero for $t \geq 2$.

Proof.

Do it as an exercise.

■

Exercise: Can you establish any connection between this result and uniform commodity taxation?

2.2.3 Werning (QJE, 2007)

Werning (2007) studies a dynamic environment in which individuals are heterogeneous in their skills. Instead of ruling out lump-sum taxation, he allows them. However, he does not allow government to condition the lump-sum tax on individual skill. Instead he allows for a distortionary labor income (and capital income) tax that government can use to redistribute income across people with different skill. In some sense, it is one step away from traditional Ramsey setup, towards rationalizing distortionary taxes.

The environment is the following: let c be consumption and l be the hours worked. Individual with skill θ who works l hours produce $y = \theta l$ efficiency labor unit. If period utility over hours worked and consumption is $U(c, l)$, then we can write it in terms of consumption and efficiency labor unit as $U^i(c, y) = U(c, y/\theta^i)$.

Suppose there are $\theta \in \Theta = \{\theta^i, \ldots, \theta^N\}$. We call the individual of type θ^i, *person* i or *type* i. The fraction of type i is π^i. Assume $\sum_i \pi^i \theta^i = 1$.

Aggregate state of economy is $s_t \in S$ (finite set) and is publicly observable. Denote the history of aggregate shocks by $s^t = (s_0, \ldots, s_t)$. Probability of history s^t is $\Pr(s^t)$.

Consumer problem

Individual of type i solves

$$\max \sum_{t,s^t} \beta^t \Pr(s^t) U^i(c_t(s^t), y(s^t)) \tag{28}$$

sub. to

$$\sum_{t,s^t} p(s^t)\left[c(s^t)+k(s^t)\right] \leq \sum_{t,s^t} p(s^t)\left[w_t(s^t)(1-\tau(s^t))y(s^t)+R(s^t)k(s^{t-1})\right]-T$$

$$k^i(s_0)=k_0^i \text{ is given}$$

in which $R(s^t) = 1+(1-\kappa(s^t))(r_t(s^t)-\delta)$ and $T=\sum_{t,s^t} p(s^t)T(s^t)$ is present value of lump-sum taxes. Note that there is heterogeneity in skills θ^i as well as initial capital holding $k^i(s_0)$.

Feasibility

Let $L(s^t)=\sum_i \pi^i y^i(s^t)$, $C(s^t)=\sum_i \pi^i c^i(s^t)$, $K(s^t)=\sum_i \pi^i k^i(s^t)$. Then feasibility is

$$C(s^t)+K(s^t)+g(s^t)=F(K(s^{t-1}),L(s^t),s^t,t)+(1-\delta)K(s^{t-1}) \tag{29}$$

Govern met

Government has exogenously given sequence of expenditure $g(s^t)$ to finance. It can levy linear tax on capital income $\kappa(s^t)$. It can also levy the following tax on income

$$\tau(s^t)w_t(s^t)y^i(s^t)+T(s^t)$$

Government budget constraint is

$$\sum_{t,s^t} p(s^t)g(s^t) \leq T+\sum_{t,s^t} p(s^t)\left[\tau(s^t)w_t(s^t)L(s^t)+\kappa(s^t)(r_t(s^t)-\delta)K(s^{t-1})\right] \tag{30}$$

Firms

As usual the firm's problem is static and implies marginal product pricing

$$\begin{aligned} r_t(s^t) &= F_k(K(s^{t-1}),L(s^t),s^t,t) \\ w_t(s^t) &= F_L(K(s^{t-1}),L(s^t),s^t,t) \end{aligned} \tag{31}$$

Equilibrium is defined the usual way.

Next we derive the implementability constraints. Werning (2007) develops a methodology that incorporates the fact that labor income taxes are uniform across types (so no extra constraint needs to be added to the optimal taxation problem). Also, he shows implementability constraints can be written only in terms of aggregates.

First observe that in any equilibrium

$$\begin{aligned}\frac{U_y^i(s^t)}{U_c^i(s^t)} &= \frac{U_y^j(s^t)}{U_c^j(s^t)} = -w(s^t)(1-\tau(s^t)) \\ \frac{U_c^i(s^t)}{U_c^i(s_0)} &= \frac{U_c^j(s^t)}{U_c^j(s_0)} = \frac{p(s^t)}{\beta^t \Pr(s^t) p(s_0)} \quad \forall\ i,j\end{aligned} \tag{32}$$

Therefore, given the aggregate consumption and labor output $(C(s^t), L(s^t))$, the assignment of allocation of consumption and labor output $\{c^i(s^t), y^i(s^t)\}$ are efficient. In other words, given any sequence of aggregate output $(C(s^t), L(s^t))$, there are weights $\varphi = \left\{\varphi^1, \dots, \varphi^N\right\}$ such that $\sum_i \pi^i \varphi^i = 1$ and $\{c^i(s^t), y^i(s^t)\}$ is the solution to

$$U^m(C(s^t), L(s^t); \varphi) \equiv \max_{\{c^i, y^i\}} \sum \pi^i \varphi^i U^i(c^i, y^i)$$

sub. to

$$\sum_i \pi^i c^i = C(s^t), \quad \sum_i \pi^i y^i = L(s^t)$$

Denote the solution by

$$c^i = h_c^i(C, L; \varphi), \quad y^i = h_y^i(C, L; \varphi) \tag{33}$$

therefore

$$(c^i(s^t), y^i(s^t)) = h^i(C, L; \varphi)$$

in which $h^i = (h_c^i, h_y^i)$.

Note also that

$$\begin{aligned}U_C^m(C(s^t), L(s^t); \varphi) &= \varphi^i U_c^i(c^i, y^i) \\ U_L^m(C(s^t), L(s^t); \varphi) &= \varphi^i U_y^i(c^i, y^i)\end{aligned} \tag{34}$$

and therefore, in any equilibrium

$$\begin{aligned}\frac{U_L^m(s^t)}{U_C^m(s^t)} &= -w(s^t)(1-\tau(s^t)) \\ \frac{U_c^m(s^t)}{U_c^m(s_0)} &= \frac{p(s^t)}{\beta^t \Pr(s^t) p(s_0)} \quad \forall\ i,j\end{aligned} \tag{35}$$

Now let's look at individual i's implementability constraint

$$\sum_{t,s^t} \beta^t \left[U_c^i(c^i(s^t), y^i(s^t)) c^i(s^t) + U_y^i(c^i(s^t), y^i(s^t)) y^i(s^t)\right] = U_c^i(c^i(s_0), y^i(s_0)) \left[R_0 k_0^i - T\right]$$

Now we can replace individual i's allocations in terms of aggregate allocations using (33) and (34)

$$\sum_{t,s^t} \beta^t \left[U_C^m(C(s^t), L(s^t); \varphi) h_c^i(C(s^t), L(s^t); \varphi) \right. = \quad (36)$$

$$\left. +U_L^m(C(s^t), L(s^t); \varphi) h_y^i(C(s^t), L(s^t); \varphi)\right] = U_c^i(C(s_0), L(s_0); \varphi) \left[R_0 k_0^i - T\right] \forall i (37)$$

Note that (36) is expressed entirely in terms of aggregate allocations, weights φ and initial endowments.

Proposition 7 *Given initial wealth $R_0 k_0^i$, an aggregate allocation $\{C(s^t), L(s^t), K(s^t)\}$ can be implemented in a competitive equilibrium if and only if*

1. *It is feasible*
2. *There exists weights φ and lump-sum T such that implementability constraint (36) holds for all $i = 1, \ldots, N$*

Proof.

Any equilibrium allocation is feasible and we just showed that it satisfy (36) . Suppose there is a feasible aggregate allocation that satisfies (36) for sum weights and lump-sum taxes. Then individual allocations and prices can be constructed using (33) and (35). Then it is immediate that (32) (consumer optimality) holds. The individual allocations constructed as such are also feasible since they satisfy (36) .

■

A Panning Problem

Suppose λ^i is planer's weight on type i. $\sum_i \pi^i \lambda^i = 1$. Consider the following planning problem

$$\max \sum_{t,s^t,i} \lambda^i \pi^i \beta^t \Pr(s^t) U^i(h^i(C(s^t), L(s^t); \varphi))$$

sub. to

$$\sum_{t,s^t} \beta^t \left[U_C^m(C(s^t), L(s^t); \varphi) h_c^i(C(s^t), L(s^t); \varphi)\right.$$

$$\left. +U_L^m(C(s^t), L(s^t); \varphi) h_y^i(C(s^t), L(s^t); \varphi)\right] = U_c^i(C(s_0), L(s_0); \varphi) \left[R_0 k_0^i - T\right] \forall i \quad ; \mu^i \pi^i$$

$$C(s^t) + K(s^t) + g(s^t) = F(K(s^{t-1}), L(s^t), s^t, t) + (1-\delta)K(s^{t-1})$$

Make our usual change of variable

$$\begin{aligned} W(C, L; \varphi, \mu, \lambda) \equiv & \sum_i \pi^i \left(\lambda^i U^i(h^i(C, L; \varphi)) \right. \\ & \left. + \mu^i \left[U_C^m(C, L; \varphi) h_c^i(C, L; \varphi) + U_L^m(C, L; \varphi) h_y^i(C, L; \varphi) \right] \right. \end{aligned}$$

and rewrite the problem as

$$\max \sum_{t, s^t, i} \lambda^i \pi^i \beta^t \Pr(s^t) W(C(s^t), L(s^t); \varphi, \mu, \lambda) - U_c^i(C(s_0), L(s_0); \varphi) \sum_i \pi^i \mu^i \left[R_0 k_0^i - T \right]$$

sub. to

$$C(s^t) + K(s^t) + g(s^t) = F(K(s^{t-1}), L(s^t), s^t, t) + (1-\delta)K(s^{t-1})$$

First order conditions are

$$F_L(K(s^{t-1}), L(s^t), s^t, t) = -\frac{W_C(C(s^t), L(s^t); \varphi, \mu, \lambda)}{W_L(C(s^t), L(s^t); \varphi, \mu, \lambda)}$$

$$W_C(C(s^t), L(s^t); \varphi, \mu, \lambda) = \beta \sum_{s^{t+1}|s^t} W_C(C(s^{t+1}), L(s^{t+1}); \varphi, \mu, \lambda) R^*(s^{t+1}) \Pr(s^{t+1})$$

in which $R^*(s^{t+1}) = 1 + \delta + F_K(K(s^t), L(s^{t+1}), s^{t+1}, t+1)$.

FOC with respect to tax on initial capital

$$\sum_i \mu^i \pi^i k_0^i = 0 \text{ or } R_0 = 0$$

Optimal Taxes

$$\tau^*(s^t) = 1 - \frac{U_L^m(C, L; \varphi)}{W_L(C, L; \varphi, \mu, \lambda)} \frac{W_C(C, L; \varphi, \mu, \lambda)}{U_C^m(C, L; \varphi)}$$

Consumer inter-temporal optimality in equilibrium implies

$$U_C^m(C(s^t), L(s^t); \varphi) = \beta \sum_{s^{t+1}|s^t} U_C^m(C(s^{t+1}), L(s^{t+1}); \varphi) R(s^{t+1}) \Pr(s^{t+1})$$

One way to get this is to set the capital income taxes such that

$$R(s^{t+1}) = R^*(s^{t+1}) \frac{U_C^m(C(s^t), L(s^t); \varphi)}{W_C(C(s^t), L(s^t); \varphi, \mu, \lambda)} \frac{W_C(C(s^{t+1}), L(s^{t+1}); \varphi, \mu, \lambda)}{U_C^m(C(s^{t+1}), L(s^{t+1}); \varphi)}$$

Note that FOC with respect to initial capital implies

$$\sum_{i=1}^{N} \mu^i k_0^i \pi^i = 0$$

Example: Consider the following preferences

$$U^i(c, y) = \frac{c^{1-\sigma}}{1-\sigma} - \alpha \frac{(y/\theta^i)^\gamma}{\gamma}$$

Note that we have $h_c^i(C, L; \varphi) = \omega_c^i C$ and $h_y^i(C, L; \varphi) = \omega_y^i L$, with

$$\omega_c^i = \frac{(\varphi^i)^{1/\sigma}}{\sum_i \pi_i (\varphi^i)^{1/\sigma}} \text{ and } \omega_y^i = \frac{(\theta^i)^{\frac{\gamma}{\gamma-1}} (\varphi^i)^{\frac{-1}{\gamma-1}}}{\sum_i \pi_i (\theta^i)^{\frac{\gamma}{\gamma-1}} (\varphi^i)^{\frac{-1}{\gamma-1}}}$$

and therefore

$$U^m = \Phi_u^m \frac{c^{1-\sigma}}{1-\sigma} - \Phi_v^m \alpha \frac{(y/\theta^i)^\gamma}{\gamma} \text{ and } W = \Phi_u^W \frac{c^{1-\sigma}}{1-\sigma} - \Phi_v^W \alpha \frac{(y/\theta^i)^\gamma}{\gamma}$$

in which Φ_u^m, Φ_v^m, Φ_u^W and Φ_v^W are some constant. Note that this implies

$$\tau^*(C, L) = 1 - \frac{\Phi_v^m}{\Phi_u^m} \frac{\Phi_u^W}{\Phi_v^W}$$

Note also that

$$\frac{U_C^m(C(s^t), L(s^t); \varphi)}{W_C(C(s^t), L(s^t); \varphi, \mu, \lambda)} \frac{W_C(C(s^{t+1}), L(s^{t+1}); \varphi, \mu, \lambda)}{U_C^m(C(s^{t+1}), L(s^{t+1}); \varphi)} = 1$$

and therefore

$$R(s^{t+1}) = R^*(s^{t+1})$$

which implies

$$\kappa(s^t) = 0 \quad \text{for all } t \geq 1$$

This implies that the result for optimal taxes on capital income holds from date zero (not just for $t \geq 1$ as it was the case before). When $k_0^i = k_0$ for all i, taxing initial capital is like a lump-sum tax. But since lump-tax is allowed here, it is not necessary. However, when individuals are heterogeneous in their initial wealth, then taxing wealth for redistribution is desirable.

Example: Now consider the following preferences

$$U^i(c, y) = \alpha \log(c) + (1 - \alpha) \log(1 - \frac{y}{\theta^i})$$

then $h_c^i(C, L; \varphi) = \omega^i C$ and $h_y^i(C, L; \varphi) = \theta^i - \omega^i(1 - L)$ and

$$\omega^i = \frac{\varphi^i}{\sum_i \pi^i \varphi^i}$$

therefore,

$$U^m(C, L; \phi) = \alpha \log(C) + (1 - \alpha) \log(1 - L) + \sum_i \left[\alpha \log\left(\omega^i\right) + (1 - \alpha) \log\left(\omega^i/\theta^i\right)\right].$$

Also we can we can verify that

$$W(C, L) = \Phi_U^W \left(\alpha \log(C) + (1 - \alpha) \log(1 - L)\right) + \Phi_{U_L}^W \frac{(1 - \alpha)}{1 - L}$$

and therefore

$$\tau^*(L) = \frac{1}{(1 - L)\Phi_U^W / \Phi_{U_L}^W + 1}$$

also

$$\kappa(s^t) = 0 \quad \text{for all } t \geq 1$$

2.3 Taxing Capital in Life Cycle Economies (Erosa and Gervais (2002))

Here, I present a 2 period version of Erosa and Gervais (2002). Individuals live 2 periods (born at age 0, die at age 1). Each generation is indexed by its date of birth. For example in period t, the generations alive are $t - 1, t$. Assume no population growth.

Each individual is endowed with one unit of time at each age j and can transform one unit of time into z_j unit of efficient labor. Let $c_{t,j}$ be the consumption of generation t at age j. Other variables follow the same notation.

Consumer's problem is the following (for generation $t > 0$)

$$\max U(c_{t,0}, l_{t,0}) + \beta U(c_{t,1}, l_{t,1})$$

subject to

$$\begin{aligned} (1-\tau^c_{t,0})c_{t,0} + a_{t,1} &\leq (1-\tau^l_{t,0})w_t z_0 l_{t,0} \\ (1-\tau^c_{t,1})c_{t,1} &\leq (1-\tau^l_{t,1})w_t z_1 l_{t,1} + (1 + (1-\tau^k_{t,1})(r_t - \delta))a_{t,1} \end{aligned}$$

There is a constant return to scale technology and

$$\begin{aligned} r_t &= f_k(k_t, l_t) \\ w_t &= f_l(k_t, l_t) \end{aligned}$$

and feasibility requires that

$$\begin{aligned} c_t + k_{t+1} &= f(k_t, l_t) + (1-\delta)k_t \\ c_t &= c_{t,0} + c_{t-1,1} \\ l_t &= l_{t,0} + l_{t-1,1} \\ k_t &= a_{t-1,1} \end{aligned}$$

Government budget constrain is

$$\sum_{t=0}^{\infty} p_t g_t = \sum_{t=0}^{\infty} p_t \left[\sum_{j=0,1} \tau^c_{t-j,j} c_{t-j,j} + \sum_{j=0,1} \tau^l_{t-j,j} w_t z_j l_{t-j,j} + \tau^k_{t-1,1}(r_t - \delta)a_{t-1,1} \right]$$

Let $U^t = U(c_{t,0}, l_{t,0}) + \beta U(c_{t,1}, l_{t,1})$ be the lifetime utility of generation t for a given sequence of consumption and leisure and let $0 < \gamma < 1$ be government's discount factor across generations. Government objective is to maximize

$$\sum_{t=0}^{\infty} \gamma^t U^t$$

Exercise: Show that, in this environment, implementability constraint for generation t is the following

$$U_{c_{t,0}} c_{t,0} + U_{l_{t,0}} l_{t,0} + \beta \left(U_{c_{t,1}} c_{t,1} + U_{l_{t,1}} l_{t,1} \right) = 0 \qquad (38)$$

Exercise: Show that a feasible allocation is implementable if and only if it satisfy (38).

Ramsey problem

Ramsey problem is the following

$$\max \sum_{t=0}^{\infty} \gamma^t \left[U(c_{t,0}, l_{t,0}) + \beta U(c_{t,1}, l_{t,1})\right]$$

subject to

$$U_{c_{t,0}} c_{t,0} + U_{l_{t,0}} l_{t,0} + \beta \left(U_{c_{t,1}} c_{t,1} + U_{l_{t,1}} l_{t,1}\right) = 0 \quad ; \gamma^t \lambda_t$$

$$\begin{aligned} c_t + k_{t+1} &= f(k_t, l_t) + (1-\delta)k_t \quad ; \gamma^t \phi_t \\ c_t &= c_{t,0} + c_{t-1,1} \\ l_t &= l_{t,0} + l_{t-1,1} \\ k_t &= a_{t-1,1} \end{aligned}$$

First order conditions are

$$\begin{aligned} \gamma^t U_{c_{t,0}} + \gamma^t \lambda_t \left(U_{c_{t,0}} + U_{cc_{t,0}} c_{t,0} + U_{lc_{t,0}} l_{t,0}\right) &= \gamma^t \phi_t \\ \gamma^t \beta U_{c_{t,1}} + \gamma^t \beta \lambda_t \left(U_{c_{t,1}} + U_{cc_{t,1}} c_{t,1} + U_{lc_{t,1}} l_{t,1}\right) &= \gamma^{t+1} \phi_{t+1} \end{aligned} \tag{39}$$

$$\begin{aligned} \gamma^t U_{l_{t,0}} + \gamma^t \lambda_t \left(U_{l_{t,0}} + U_{ll_{t,0}} l_{t,0} + U_{lc_{t,0}} c_{t,0}\right) &= \gamma^t \phi_t f_{lt} \\ \gamma^t \beta U_{l_{t,0}} + \gamma^t \beta \lambda_t \left(U_{l_{t,1}} + U_{ll_{t,1}} l_{t,1} + U_{cl_{t,1}} c_{t,1}\right) &= \gamma^{t+1} \phi_{t+1} f_{lt+1} \end{aligned} \tag{40}$$

$$\gamma^t \phi_t = \gamma^{t+1} \phi_{t+1} \left(1 - \delta + f_{kt+1}\right) \tag{41}$$

Combine (39) and (41)

$$\frac{U_{c_{t,0}} + \lambda_t \left(U_{c_{t,0}} + U_{cc_{t,0}} c_{t,0} + U_{lc_{t,0}} l_{t,0}\right)}{U_{c_{t,1}} + \lambda_t \left(U_{c_{t,1}} + U_{cc_{t,1}} c_{t,1} + U_{lc_{t,1}} l_{t,1}\right)} = \beta \left(1 - \delta + f_{kt+1}\right) \tag{42}$$

Steady State: In the steady state $(c_{t,0}, c_{t,1}, l_{t,0}, l_{t,1}, a_{t,1}) = (c_0, c_1, l_0, l_1, a_1)$ and $\lambda_t = \lambda$. Therefore,

$$\frac{U_{c_0} + \lambda \left(U_{c_0} + U_{cc_0} c_0 + U_{lc_0} l_0\right)}{\beta U_{c_1} + \lambda \left(U_{c_1} + U_{cc_1} c_1 + U_{lc_1} l_1\right)} = \beta \left(1 - \delta + f_{kt+1}\right)$$

Note that this in general does not imply zero tax on capital. When profile of labor productivity, z_j, is not flat over lifetime, in general consumption and leisure allocations over lifetime is not flat.

Question: Intuitively, why is it optimal to distort inter-temporal decision in this environment?

We can impose assumptions on preferences (both for government and individuals) to arrive at zero capital taxation result again.

Proposition 8 *Suppose period utility function is of the following form*

$$u(c, l) = \frac{c^{1-\sigma}}{1-\sigma} - v(l)$$

Then the Ramsey problem prescribes no inter-temporal distortions for periods $t \geq 1$, provided that labor income taxes can be age-dependent.

Proof.

Note that equation (42) becomes

$$\frac{U_{c_{0,t}}}{U_{c_{1,t}}} = \beta \left(1 - \delta + f_{kt+1}\right)$$

Individual problems Euler equation is

$$\frac{U_{c_{0,t}}}{U_{c_{1,t}}} = \beta \left(1 + (1 - \tau_{t,1})(f_{kt+1} - \delta)\right)$$

■

This result should be viewed as a consequence of uniform commodity taxation.

Question: Note that we get this result independent of γ. Isn't that surprising? Why is that?

3 Mirrleesian Approach to Optimal Taxation

One criticism of the Ramsey approach is the *ad hoc* assumption of linear distortionary taxes exclusion of Lump-sum taxation. At the same time without imposing these restrictions or without including informational frictions the Lump-taxes are very desirable in these models. We saw Werning (2007) as an attempt to move away from this limitations and expand the set of instruments available to government, i.e., Lump-sum taxes together with uniform distortionary taxes across individuals. In doing that he appeals to informational friction, i.e., the fact that individuals type (skill) is not observable and hence cannot be taxes. But is it the best government can do? Is it possible for government to implement more sophisticated instruments and achieve "better" outcomes (lets agree for now that "better" means, higher welfare given a welfare function)? What are these instruments? How we possibly restrict the set instruments available to government?

In Mirrleesian approach set of instruments is pinned down by information/enforcement limitation of the government. The optimal tax policy is found among those policies that are incentive compatible, given the information/enforcement restrictions. In doing that we proceed in two steps

1. Find a socially optimal allocation given information/enforcement restrictions
2. Devise a tax system that implements this allocation

Environment with finite number of agents

There are N agents, indexed by $n = 1, \dots, N$, who live $T < \infty$ period. The preferences are

$$\sum_{t=1}^{T} \beta^{t-1} \left(u(c_t) - v(l_t)\right), \quad 1 > \beta > 0, \quad u', -u'', v', v'' > 0$$

in which c_t is consumption and is observed. l_t is hours worked (or effort) and it is not observed.

Let Θ be a finite set of skills or ability. Nature makes a draw $\theta_n^T = (\theta_{n1}, \dots, \theta_{nt}) \in \Theta^T = \Theta \times \cdots \times \Theta$ for each agent n. The θ_n^T draws are i.i.d across agents. Let $\pi(\cdot)$ be probability density function over Θ^T draws. Each agent privately learns θ_{nt} at the beginning of period t. Individual who has skill θ_{nt} and works l_{nt} hours in period t can produce y_{nt} unit of output according to

$$y_{nt} = l_{nt} \cdot \theta_{nt}$$

l_{nt} and θ_{nt} are both private information but y_{nt} is observable. In what follows we make a change of variable $l_{nt} = \frac{y_{nt}}{\theta_{nt}}$.

Definition 2 *An allocation is a sequence of functions* $(c_n, y_n)_{n=1}^N$

$$c_n : \left(\Theta^T\right)^N \longrightarrow \mathbb{R}_+^T$$

$$y_n : \left(\Theta^T\right)^N \longrightarrow \mathbb{R}_+^T$$

such that c_{nt} *and* y_{nt} *are* $(\theta_1^t, \ldots, \theta_N^t) -$ *measurable. Denote the set of feasible allocation by* FA.

Definition 3 *An allocation* $(c_n, y_n)_{n=1}^N$ *is feasible if*

$$\sum_{t=1}^T \sum_{n=1}^N c_{nt}(\theta_1^T, \ldots, \theta_n^T) R^{-t} \leq \sum_{t=1}^T \sum_{n=1}^N y_{nt}(\theta_1^T, \ldots, \theta_n^T) R^{-t}, \quad R > 1$$

for all $(\theta_1^T, \ldots, \theta_n^T)$ *such that* $\pi(\theta_1^T, \ldots, \theta_n^T) > 0$.

So far, we know the information structure, we know what allocations are and we know what allocations are feasible. But how does this guide towards a set of tax policies? In other words the remaining question is, given the private information, what allocations can be implemented?

One period economy

Suppose for now $T = 1$.

Definition 4 *A Game (or a Mechanism) is a set of actions* $(A_1, \ldots, A_N)$ *and outcome functions*

$$(g^c, g^y) : \prod_{n=1}^N A_n \longrightarrow FA$$

The timing is as follows:

- Nature makes a draw θ for each agent n
- Agents privately observe θ and then simultaneously choose an action $a_n \in A_n$

- The outcome is determined according to outcome function

Definition 5 *Let (A, g^c, g^y) be a Mechanism. A Bayesian Nash Equilibrium (BNE) is a collection of strategies $\{\alpha_n^*\}_{n=1}^N$, $\alpha_n^* : \Theta \longrightarrow A_n$ such that*

$$\alpha_n^*(\theta_n) \in \arg\max_{\rho \in A_n} \sum_{\theta_{-n}} \pi(\theta_{-n}) \left(u(g_n^c(\rho, \alpha_{-n}^*(\theta_{-n}))) - v\left(\frac{g_n^y(\rho, \alpha_{-n}^*(\theta_{-n}))}{\theta_n} \right) \right)$$

We call $g_n^c(\alpha_1^*(\theta_1), \ldots, \alpha_N^*(\theta_N))$ and $g_n^y(\alpha_1^*(\theta_1), \ldots, \alpha_N^*(\theta_N))$ $\forall$ $(\theta_1, \ldots, \theta_N)$ equilibrium outcome.

Definition 6 *A feasible allocation $(c_n, y_n)_{n=1}^N$ is implementable if there is a mechanism (A, g^c, g^y) and a BNE $\{\alpha_n^*\}_{n=1}^N$ of that mechanism such that*

$$c_n = g_n^c(\alpha_1^*(\theta_1), \ldots, \alpha_N^*(\theta_N)), \quad y_n = g_n^y(\alpha_1^*(\theta_1), \ldots, \alpha_N^*(\theta_N))$$

So far we have made it clear what exactly do we mean by implementability. But is it helpful? Notice that our setup so far does not impose any restriction on the type of games (mechanisms) considered. Any equilibrium outcome of some game is implementable. Think for a moment about the following problem: we want to find the best implementable allocation. That means we need to search in the space of games, find a game that has a BNE that implements that best allocation as its outcome. This is a very complicated problem.

Good news is that there is a very powerful result that allows us to restrict attention to a very particular game without lose of generality. We need couple of more definitions.

Definition 7 *A Direct Mechanism is a a game such that $A_n = \Theta$ for all n.*

Definition 8 *A truth-telling BNE of a direct mechanism (Θ, g^c, g^y) is $\alpha_n^*(\theta_n) = \theta_n$ for all n such that*

$$\theta_n \in \arg\max_{\rho \in \Theta} \sum_{\theta_{-n}} \pi(\theta_{-n}) \left(u(g_n^c(\rho, \theta_{-n})) - v\left(\frac{g_n^y(\rho, \theta_{-n}))}{\theta_n} \right) \right)$$

A feasible allocation is truthfully implementable if

$$c_n = g_n^c(\theta_1, \ldots, \theta_N), \quad y_n = g_n^y(\theta_1, \ldots, \theta_N)$$

In a direct mechanism, players are basically asked to report their skill type. We are interested in equilibria in which type is revealed truthfully. It turns out there is not loss of generality in doing that.

Proposition 9 *(Revelation Principle)*

A allocation $(c_n, y_n)_{n=1}^N$ is implementable if and only if it is truthfully implementable in a direct mechanism.

Proof.

Suppose allocation $(c_n, y_n)_{n=1}^N$ is implementable as outcome of some mechanism (A, g^c, g^y). We construct a truth-full mechanism $(\Theta, \tilde{g}^c, \tilde{g}^y)$ as the following

$$\tilde{g}^c(\theta_1, \ldots, \theta_N) = g^c(\alpha_1^*(\theta_1), \ldots, \alpha_N^*(\theta_N)), \quad \tilde{g}^y(\theta_1, \ldots, \theta_N) = g^y(\alpha_1^*(\theta_1), \ldots, \alpha_N^*(\theta_N))$$

In which $\{\alpha_n^*\}_{n=1}^N$ is the BNE of the mechanism (A, g^c, g^y). We only need to show that truth-telling is a BNE of $(\Theta, \tilde{g}^c, \tilde{g}^y)$. Suppose not, i.e., suppose there is a type θ_n and a report $\rho \in \Theta$ such that

$$\sum_{\theta_{-n}} \pi(\theta_{-n}) \left(u(\tilde{g}_n^c(\rho, \theta_{-n})) - v\left(\frac{\tilde{g}_n^y(\rho, \theta_{-n}))}{\theta_n} \right) \right) \quad >$$
$$\sum_{\theta_{-n}} \pi(\theta_{-n}) \left(u(\tilde{g}_n^c(\theta_n, \theta_{-n})) - v\left(\frac{\tilde{g}_n^y(\theta_n, \theta_{-n}))}{\theta_n} \right) \right) \quad =$$
$$\sum_{\theta_{-n}} \pi(\theta_{-n}) \left(u(g_n^c(\alpha_n^*(\theta_n), \alpha_{-n}^*(\theta_{-n}))) - v\left(\frac{g_n^y(\alpha_n^*(\theta_n), \alpha_{-n}^*(\theta_{-n}))}{\theta_n} \right) \right)$$

in which the last inequality follows from definition of $(\tilde{g}^c, \tilde{g}^y)$. This implies that there must exist $\alpha = \alpha_n^{*-1}(\rho) \in A_n$ such that

$$\sum_{\theta_{-n}} \pi(\theta_{-n}) \left(u(g_n^c(\alpha, \alpha_{-n}^*(\theta_{-n}))) - v\left(\frac{g_n^y(\alpha, \alpha_{-n}^*(\theta_{-n}))}{\theta_n} \right) \right) \quad >$$
$$\sum_{\theta_{-n}} \pi(\theta_{-n}) \left(u(g_n^c(\alpha_n^*(\theta_n), \alpha_{-n}^*(\theta_{-n}))) - v\left(\frac{g_n^y(\alpha_n^*(\theta_n), \alpha_{-n}^*(\theta_{-n}))}{\theta_n} \right) \right)$$

this is a contradiction. Therefore, $(c_n, y_n)_{n=1}^N$ can be implemented by

$$c_n = \tilde{g}^c(\theta_1, \ldots, \theta_N), \quad y_n = \tilde{g}^y(\theta_1, \ldots, \theta_N)$$

■

Using Revelation Principle we can restrict attention to direct mechanism and allocations that are truthfully revealing. This means that the set of implementable allocations are the

the ones that satisfy the following incentive compatibility constraints

$$\sum_{\theta_{-n}} \pi(\theta_{-n}) \left(u(c_n(\theta_n, \theta_{-n})) + v\left(\frac{y_n(\theta_n, \theta_{-n})}{\theta_n}\right)\right) \geq \sum_{\theta_{-n}} \pi(\theta_{-n}) \left(u(c_n(\theta', \theta_{-n})) + v\left(\frac{y_n(\theta', \theta_{-n})}{\theta_n}\right)\right)$$

$\forall n, \theta_n, \theta' \in \Theta$.

We are going to primarily focus on environment with unit measure of agents.

Environment with infinite number of agents

Consider the same environment as before (for general $T < \infty$) except that now there are unit mass of agents. Nature makes a draw $\theta^T = (\theta_1, \ldots, \theta_t) \in \Theta^T = \Theta \times \cdots \times \Theta$ for each agent. The θ^T draws are i.i.d across agents. Let $\pi(\cdot)$ is probability density function over Θ^T draws. There is no aggregate uncertainty, therefore $\pi(\theta^T)$ is also the mass of people who have the draw θ^T. Let $D \equiv \left\{\theta^T | \pi(\theta^T) > 0\right\}$.

Define allocation as $\theta^t -$ measurable functions

$$c_t : D \longrightarrow \mathbb{R}_+^T$$

$$y_t : D \longrightarrow \mathbb{R}_+^T$$

Allocation is feasible if

$$\sum_{\Theta \in D} \sum_{t=1}^{T} R^{-t} c_t(\theta^T)\pi(\theta^T) \leq \sum_{\Theta \in D} \sum_{t=1}^{T} R^{-t} y_t(\theta^T)\pi(\theta^T)$$

Define a mechanism as set of actions $A \subset \prod_{t=1}^T X_t$ and outcome functions

$$g : A \times \Delta(A) \longrightarrow \mathbb{R}_+^{2T}$$

and $g_t(a, \mu) = g_t(a', \mu')$ if $a^t = a'^t$ and

$$\sum_{(a_{t+1}, \ldots, a_T)} \mu(\bar{a}^t, a_{t+1}, \ldots, a_T) = \sum_{(a'_{t+1}, \ldots, a'_T)} \mu'(\bar{a}^t, a'_{t+1}, \ldots, a'_T) \quad \forall \bar{a}^t$$

in which $\mu, \mu' \in \Delta(A)$ are measure of actions chosen.

A BNE is a strategy $\alpha^* : D \longrightarrow A$ (α_t is θ^t – measurable) such that

$$\sum_{t=1}^{T} \beta^{t-1}\pi(\theta^T)\left[u(g_t^c(\alpha_t^*(\theta^T),\mu^*)) - v\left(\frac{g_t^c(\alpha_t^*(\theta^T),\mu^*)}{\theta_t}\right)\right] \geq$$
$$\sum_{t=1}^{T} \beta^{t-1}\pi(\theta^T)\left[u(g_t^c(\alpha_t'(\theta^T),\mu^*)) - v\left(\frac{g_t^c(\alpha_t'(\theta^T),\mu^*)}{\theta_t}\right)\right]$$

for all $\alpha' : D \longrightarrow A$ (α_t' is θ^t – measurable) and $\mu^*(a) = \sum_{\{\theta^T|\alpha(\theta^T)=a\}} \pi(\theta^T)$. An allocation is implementable if there exist a mechanism (A, g^c, g^y) and a BNE α^* such that

$$(c_t(\theta^T), y_t(\theta^T)) = g_t(\alpha^*(\theta^T), \mu^*)$$

A mechanism is a direct mechanism if $A = \Theta^T$. A strategy α is truth-full if $\alpha_t(\theta^T) = \theta_t$ for all $\theta^T \in \Theta^T$. An allocation is truthfully implementable if there exists a BNE of a direct mechanism, i.e., if

$$\sum_{t=1}^{T} \beta^{t-1}\pi(\theta^T)\left[u(g_t^c(\theta^T,\mu^*)) - v\left(\frac{g_t^y(\theta^T,\mu^*)}{\theta_t}\right)\right] \geq$$
$$\sum_{t=1}^{T} \beta^{t-1}\pi(\theta^T)\left[u(g_t^c(\alpha_t'(\theta^T),\mu^*)) - v\left(\frac{g_t^y(\alpha_t'(\theta^T),\mu^*)}{\theta_t}\right)\right]$$

for all $\alpha' : D \longrightarrow \Theta^T$ (α_t' is θ^t – measurable) $\mu^*(a) = \sum_{\{\theta^T|\theta^T=a\}} \pi(\theta^T) = \pi(a)$. And

$$(c_t(\theta^T), y_t(\theta^T)) = g_t(\theta^T, \pi)$$

Revelation Principle: An allocation (c, y) is implementable only if it is truthfully implementable.

Note then that implementable allocation can be characterized by the following incentive compatibility constraints

$$\sum_{t=1}^{T}\sum_{\theta^T\in D} \pi(\theta^T)\left[u(c_t(\theta^T)) - v\left(\frac{y_t(\theta^T)}{\theta_t}\right)\right] \geq \sum_{t=1}^{T}\sum_{\theta^T\in D} \pi(\theta^T)\left[u(c_t(\alpha_t'(\theta^T))) - v\left(\frac{y_t(\alpha_t'(\theta^T))}{\theta_t}\right)\right]$$

for all $\alpha' : D \longrightarrow D$ (α_t' is θ^t – measurable).

Example : one period problem

Suppose $T = 1$ and there are only two types, θ_H and θ_L with $\theta_H > \theta_L$. Consider the utilitarian planer's problem

$$\max \pi(\theta_H)\left[u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_H}\right)\right] + \pi(\theta_L)\left[u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_L}\right)\right]$$

sub. to.

$$\begin{aligned} u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_H}\right) &\geq u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_H}\right) \\ u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_L}\right) &\geq u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_L}\right) \end{aligned}$$

$$\pi(\theta_H)\left[c(\theta_H) - y(\theta_H)\right] + \pi(\theta_H)\left[c(\theta_H) - y(\theta_H)\right] = 0$$

For a moment suppose there is no private information. Then the optimal allocation must satisfy (note that $v(\cdot)$ is convex):

$$\begin{aligned} u'(c(\theta_H)) &= u'(c(\theta_L)) = \lambda \Rightarrow c(\theta_H) = c(\theta_L) \\ \frac{1}{\theta_H} v\left(\frac{y(\theta_H)}{\theta_H}\right) &= \frac{1}{\theta_L} v'\left(\frac{y(\theta_L)}{\theta_L}\right) = \lambda \Rightarrow y(\theta_H) > y(\theta_L) \end{aligned}$$

Note that there is no distortion

$$u'(c(\theta)) = \frac{1}{\theta} v'\left(\frac{y(\theta)}{\theta}\right)$$

We will show that this allocation does not satisfy I.C. constraints. Note that $y(\theta_H) > y(\theta_L)$, therefore

$$u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_H}\right) = u(c(\theta_H)) - v\left(\frac{y(\theta_L)}{\theta_H}\right) > u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_H}\right)$$

the I.C. for type H is violated.

So we know that when individuals have private information about their type, at least of the the I.C. constraints is binding at the optimal solution.

Consider a relaxed planning problem with only type H's I.C. constraint

$$\max \pi(\theta_H)\left[u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_H}\right)\right] + \pi(\theta_L)\left[u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_L}\right)\right]$$

sub. to.

$$u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_H}\right) \geq u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_H}\right) \quad ;\pi(\theta_H)\mu$$

$$\pi(\theta_H)\left[c(\theta_H) - y(\theta_H)\right] + \pi(\theta_H)\left[c(\theta_H) - y(\theta_H)\right] = 0 \quad ;\lambda$$

we will characterize the solution to this problem and then we will verify that at the solution the I.C. constraint of type L is slack.

$$\begin{aligned} (1+\mu)u'(c(\theta_H)) &= \lambda \\ \left(1 - \mu\frac{\pi(\theta_H)}{\pi(\theta_L)}\right)u'(c(\theta_L)) &= \lambda \end{aligned}$$

$$\begin{aligned} (1+\mu)\frac{1}{\theta_H}v'\left(\frac{y(\theta_H)}{\theta_H}\right) &= \lambda \\ \frac{1}{\theta_L}v'\left(\frac{y(\theta_L)}{\theta_L}\right) - \mu\frac{\pi(\theta_H)}{\pi(\theta_L)}\frac{1}{\theta_H}v'\left(\frac{y(\theta_L)}{\theta_H}\right) &= \lambda \end{aligned}$$

First, note that there is no distortion for type H

$$u'(c(\theta_H)) = \frac{1}{\theta_H}v'\left(\frac{y(\theta_H)}{\theta_H}\right)$$

Also, observe that

$$c(\theta_H) > c(\theta_L)$$

note that incentive compatibility implies

$$v\left(\frac{y(\theta_H)}{\theta_H}\right) - v\left(\frac{y(\theta_L)}{\theta_H}\right) = u(c(\theta_H)) - u(c(\theta_L)) > 0$$

and therefore

$$y(\theta_H) > y(\theta_L)$$

Next we check that under these allocations, the I.C. constraints for type L is slack.

$$\begin{aligned} v\left(\frac{y(\theta_H)}{\theta_L}\right) - v\left(\frac{y(\theta_L)}{\theta_L}\right) &= \int_{y(\theta_L)}^{y(\theta_H)} v\left(\frac{y}{\theta_L}\right) dy > \int_{y(\theta_L)}^{y(\theta_H)} v\left(\frac{y}{\theta_H}\right) dy \\ &= v\left(\frac{y(\theta_H)}{\theta_H}\right) - v\left(\frac{y(\theta_L)}{\theta_H}\right) \\ &= u(c(\theta_H)) - u(c(\theta_L)) \end{aligned}$$

rearrange terms

$$u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_L}\right) > u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_L}\right)$$

So, we know that our characterization is valid. Note that under this characterization

$$\begin{aligned} \frac{1}{\theta_L} v'\left(\frac{y(\theta_L)}{\theta_L}\right) - u'(c(\theta_L)) &= \mu \frac{\pi(\theta_H)}{\pi(\theta_L)} \left[\frac{1}{\theta_H} v'\left(\frac{y(\theta_L)}{\theta_H}\right) - u'(c(\theta_L))\right] \\ &< \mu \frac{\pi(\theta_H)}{\pi(\theta_L)} \left[\frac{1}{\theta_H} v'\left(\frac{y(\theta_H)}{\theta_H}\right) - u'(c(\theta_H))\right] \\ &< 0 \end{aligned}$$

and hence

$$\frac{1}{\theta_L} v'\left(\frac{y(\theta_L)}{\theta_L}\right) < u'(c(\theta_L))$$

there is distortion for the low type.

Suppose we want to implement this allocation with a nonlinear tax function $T(y)$. Let $T(y) = y - c$ if $y \in \{y(\theta_H), y(\theta_L)\}$ and $T(y) = y$ otherwise. Consider the consumer's problem

$$\max u(c) - v\left(\frac{y}{\theta}\right)$$

sub. to.

$$c = y - T(y)$$

FOC

$$u'(c)(1 - T'(y)) = \frac{1}{\theta} v\left(\frac{y}{\theta}\right)$$

discussion above implies that

$$T'(y(\theta_L)) > 0, T'(y(\theta_H)) = 0$$

Implementation in one period problem

How should we design a tax function that implements the efficient allocations characterized above? There are many ways to do this. Here is an example. Consider to the following tax function:

$$T(y) = \begin{cases} T_0 + \tau_L y & \text{for } y \leq y_{kink} \\ T_0 + \tau_H y_{kink} + \tau_H (y - y_{kink}) & \text{for } y > y_{kink} \end{cases}$$

Here is how the parameters of the above tax function can be chosen. Let $(c^*(\theta), y^*(\theta))$ be the efficient allocation characterized above for $\theta \in \{\theta_L, \theta_H\}$. Define $u^*(\theta)$

$$u^*(\theta) = u(c^*(\theta)) - v\left(\frac{y^*(\theta)}{\theta}\right) \qquad \text{for } \theta \in \{\theta_L, \theta_H\}$$

Now let

$$\begin{aligned} \tau_L &= 1 - \frac{v'(y^*(\theta_L)/\theta_L)}{\theta_L c^*(\theta_L)} \\ \tau_H &= 0 \end{aligned}$$

$$T_0 = (1 - \tau_L)\, y^*(\theta_L) - u^{-1}\left(u^*(\theta_L) + v\left(\frac{y^*(\theta_L)}{\theta_L}\right)\right)$$

$$y_{kink} = \frac{y^*(\theta_H) - T_0 - u^{-1}\left(u^*(\theta_H) + v\left(\frac{y^*(\theta_H)}{\theta_H}\right)\right)}{\tau_L}$$

Exercise: Show that the tax function above implements the efficient allocation $(c^*(\theta), y^*(\theta))$.

3.1 Optimal Taxes in Static Model with Many Types

Extending the above two period example to a model with many (finite) types is straight forward. The zero-optimal-tax-at-the-top result is robust to that extension. However, beyond that we cannot say much about the properties of the tax function. In particular, the tax function is not related to the properties of the utility function in a systematic way. Another problem is that our analysis is entirely conducted in terms of distribution of types, which is unobserved.

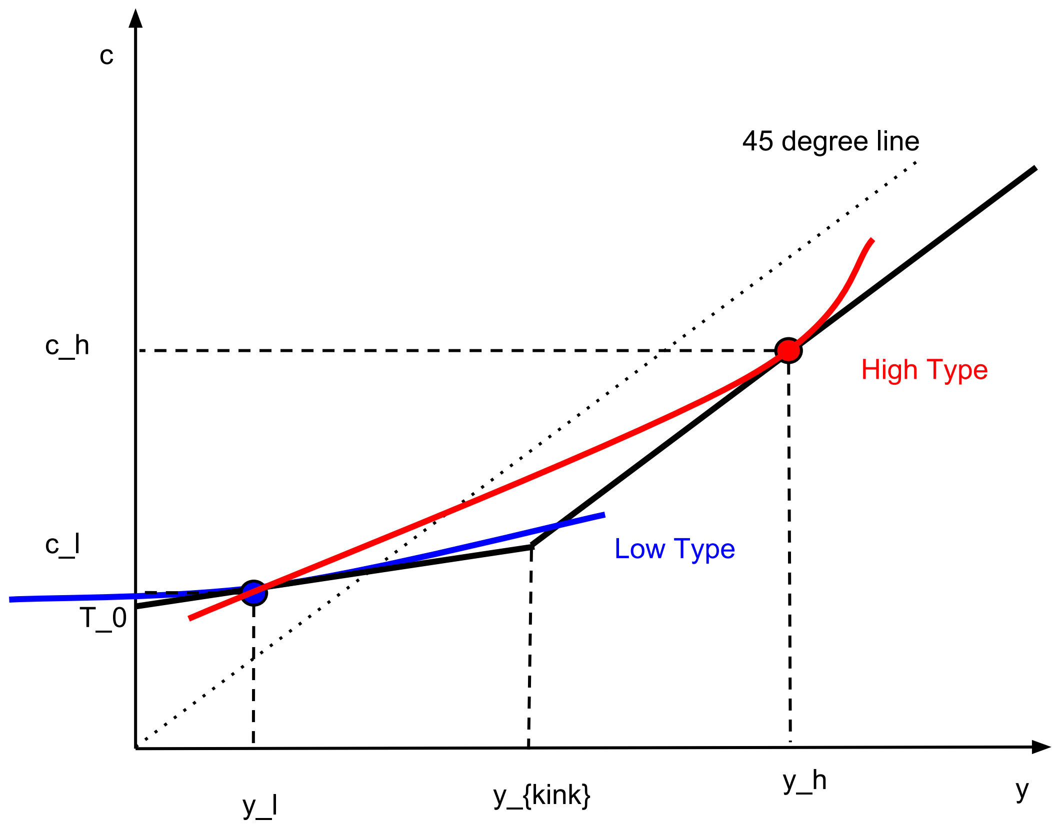

Figure 1: Two Type Example

Saez (2001) has showed that some properties of the tax function can be linked to elasticities of labor supply (which is a property of the preferences). He also, derives tax formula that depends on the distribution of earnings (which are observable). Finally, he demonstrates that the zero-optimal-tax-at-the-top result is a local result. Next, we study a very simple version of his analysis (many details are skipped!)

Suppose there are continuum of types $0 \leq \underline{\theta} \leq \theta \leq \bar{\theta} \leq \infty$. Let $F(\theta)$ be the distribution of types withe density θ. Let $g(\theta)$ be the weight that the planner assigns to a worker of type θ. We are interested in the following planning problem

$$\max \int g(\theta) \left[u(c(\theta)) - v\left(\frac{y(\theta)}{\theta}\right)\right] dF(\theta)$$

subject to

$$u(c(\theta)) - v\left(\frac{y(\theta)}{\theta}\right) \geq u(c(\theta')) - v\left(\frac{y(\theta')}{\theta}\right) \quad \text{for all } \theta, \theta'$$

$$\int [c(\theta) - y(\theta)]\, dF(\theta) = 0$$

Define

$$U(\theta) = \max_{\theta'} \left[u(c(\theta')) - v\left(\frac{y(\theta')}{\theta}\right)\right].$$

This is the utility form truth-telling when facing the allocation $(c(\theta), y(\theta))$. Using this notation, the incentive compatibility constraints can be written as

$$U(\theta) \geq u(c(\theta')) - v\left(\frac{y(\theta')}{\theta}\right)$$

These are infinite number of constraints. We can replace them with one single equation. The envelope condition for the above maximization imply (after replacing $l(\theta = y(\theta)/\theta)$

$$U'(\theta) = \frac{l(\theta)}{\theta} v'(l(\theta))$$

This is called local incentive comparability. You can show that for the preferences we are considering here (separable in consumption and leisure), any allocation that satisfy local incentive compatibility constraints also satisfy the global incentive compatibility constraints. So we can rewrite the planning problem as

$$\max_{U,c,l} \int g(\theta)\, U(\theta) dF(\theta)$$

subject to

$$U'(\theta) = \frac{l(\theta)}{\theta} v'(l(\theta)); \quad \mu(\theta)$$

$$U(\theta) = u(c(\theta)) - v(l(\theta)); \quad \eta(\theta)$$

$$\int [c(\theta) - \theta l(\theta)]\, dF(\theta) = 0; \quad \lambda$$

The second constraint is promise keeping. It guarantees that the the chosen allocation deliver the right amount of utils to each type. This constraint will also show up later when we study the recursive formulation of dynamic models.

We can rewrite the problem as (note, I am replacing $dF(\theta)$with $f(\theta)\,d\theta$)

$$\max_{U,c,l} \int \left[\left(g(\theta) U(\theta) + \lambda \left(\theta l(\theta) - c(\theta) \right) \right) f(\theta) + \left(u(c(\theta)) - v(l(\theta)) - U(\theta) \right) \eta(\theta) \right] d\theta$$

subject to

$$U'(\theta) = \frac{l(\theta)}{\theta} v'(l(\theta)); \quad \mu(\theta)$$

This looks a lot like an optimal control problem with state variable $U(\theta)$ and co-state variable $\mu(\theta)$. We can use calculus of variation to solve it. Let's write down the Hamiltonian

$$H = \left(g(\theta) U(\theta) + \lambda \left(\theta l(\theta) - c(\theta) \right) \right) f(\theta) + \left(u(c(\theta)) - v(l(\theta)) - U(\theta) \right) \eta(\theta) + \mu(\theta) \frac{l(\theta)}{\theta} v'(l(\theta))$$

Take first order conditions

- With respect to $c(\theta)$

$$-\lambda f(\theta) + u'(c(\theta))\,\eta(\theta) = 0 \tag{43}$$

- With respect to $l(\theta)$

$$\lambda f(\theta)\,\theta - v'(l(\theta))\,\eta(\theta) + \mu(\theta) \left(\frac{v'(l(\theta))}{\theta} + \frac{l(\theta)\,v''(l(\theta))}{\theta} \right) = 0 \tag{44}$$

- And

$$\mu'(\theta) = -\left(g(\theta) f(\theta) - \eta(\theta) \right) \tag{45}$$

Also, the following boundary conditions must hold

$$\mu(\underline{\theta}) = \mu\left(\bar{\theta}\right) = 0$$

Combine the first order conditions to get

$$\theta u'(\theta) = v'(l(\theta)) - \frac{\mu(\theta)}{\eta(\theta)} \left(\frac{v'(l(\theta))}{\theta} + \frac{l(\theta)\,v''(l(\theta))}{\theta} \right) \tag{46}$$

This immediately implies that there is no distortion at the boundaries ($\theta = \underline{\theta}$ and $\theta = \bar{\theta}$). Meaning that marginal tax rate for the least productive and the most productive must be zero.

But if there is no least or most productive? For example, suppose $\bar{\theta} = \infty$, what is the limiting value of marginal tax for very productive individuals?

Before answering this question, we derive a formula for optimal marginal taxes. Recall that, a tax system that implements the efficient allocation must be such that

$$\frac{v'(l(\theta))}{\theta u'(c(\theta))} = 1 - T'$$

or

$$\frac{T'}{1-T'} = \frac{\theta u'(c(\theta)) - v'(l(\theta))}{v'(l(\theta))}$$

From equation (46) we have

$$\frac{\theta u'(c(\theta)) - v'(l(\theta))}{v'(l(\theta))} = -\frac{\mu(\theta)}{\eta(\theta)}\frac{1}{\theta}\left(1 + \frac{l(\theta)v''(l(\theta))}{v'(l(\theta))}\right)$$

Recall for general utility function $U(c,l)$ the Frisch elasticity of labor supply is

$$\epsilon = \frac{U_l}{l\left(U_{ll} - \frac{U_{cl}^2}{U_{cc}}\right)}$$

Therefore, for our separable utility function, Frisch elasticity of labor supply is $\epsilon = \frac{v'(l)}{lv''(l)}$. Replace this in the formula above

$$\frac{T'}{1-T'} = -\frac{\mu(\theta)}{\eta(\theta)}\frac{1}{\theta}\left(1 + \frac{1}{\epsilon}\right) \tag{47}$$

To simplify the notation let $u_c(\theta) = u'(c(\theta))$. From equation 43 we have

$$\eta(\theta) = \lambda\frac{f(\theta)}{u_c(\theta)}.$$

Replace this into equation 45

$$\mu'(\theta) = \lambda\frac{f(\theta)}{u_c(\theta)} - g(\theta)f(\theta)$$

Now, note that

$$\begin{aligned}
-\mu(\theta) &= \int_{\theta}^{\bar{\theta}} \mu'(x)\,dx \\
&= \int_{\theta}^{\bar{\theta}} \left(\lambda \frac{1}{u_c(x)} - g(x)\right) f(x)\,dx \\
&= \lambda \int_{\theta}^{\bar{\theta}} \frac{1}{u_c(x)} \left(1 - \frac{g(x)}{\lambda} u_c(x)\right) f(x)\,dx
\end{aligned}$$

Therefore,

$$\begin{aligned}
-\frac{\mu(\theta)}{\eta(\theta)}\frac{1}{\theta} &= \frac{u_c(\theta)}{\theta f(\theta)\lambda} \lambda \int_{\theta}^{\bar{\theta}} \frac{1}{u_c(x)} \left(1 - \frac{g(x)}{\lambda} u_c(x)\right) f(x)\,dx \\
&= \frac{1 - F(\theta)}{\theta f(\theta)} \int_{\theta}^{\bar{\theta}} \frac{u_c(\theta)}{u_c(x)} \left(1 - \frac{g(x)}{\lambda} u_c(x)\right) \frac{f(x)}{(1 - F(\theta))} dx.
\end{aligned}$$

Now we can write down the Diamond-Saez tax formula

$$\frac{T'}{1 - T'} = \left(1 + \frac{1}{\epsilon}\right) \left(\frac{1 - F(\theta)}{\theta f(\theta)}\right) \left(\int_{\theta}^{\bar{\theta}} \frac{u_c(\theta)}{u_c(x)} \left(1 - \frac{g(x)\,u_c(x)}{\lambda}\right) \frac{f(x)}{(1 - F(\theta))} dx\right) \tag{48}$$

To see how this formula can be written in terms of distribution of earning (that means distribution of y) see Saez (2001).

The first term in the formula captures the effect of elasticity of labor supply on optimal taxes. The second term is about the upper tail of the ability distribution. Finally, the last term captures the redistribution motives of the government.

Note that it is obvious that marginal taxes are higher if labor supply is less elastic. Also, if $\bar{\theta} < \infty$, the $F\left(\bar{\theta}\right) = 1$ and $T'\left(\bar{\theta}\right) = 0$.

What if distribution of ability is unbounded? In that case the answer depends on the properties of tail of the ability distribution.

Example (Diamond (1998)**):** Suppose $\bar{\theta} = \infty$ and planner does not care about the welfare of the most productive types. That means that $g(\theta)$ converges to zeros. What is the optimal marginal tax at the top that maximized the tax revenue collected form the most productive types?

Suppose labor supply elasticity is constant. Also assume that $F(\theta)$ is a pareto distribution with

$$F(\theta) = 1 - \left(\frac{\theta_m}{\theta}\right)^{\alpha}$$

and

$$f(\theta) = \frac{\alpha \theta_m^\alpha}{\theta^{\alpha+1}}$$

Then

$$\frac{1 - F(\theta)}{\theta f(\theta)} = \frac{1}{\alpha}$$

If we further assume that $u(c) = c$ (or alternatively assume that $\frac{u_c(\theta)}{u_c(x)} \approx 1$ for $x > \theta$ and θ sufficiently large). Then

$$\left(\int_\theta^{\bar{\theta}} \frac{u_c(\theta)}{u_c(x)} \left(1 - \frac{g(x)\, u_c(x)}{\lambda}\right) \frac{f(x)}{(1 - F(\theta))} dx \right) \to 1$$

and

$$\frac{T'}{1 - T'} = \frac{1 + \epsilon}{\epsilon \alpha}.$$

Therefore, optimal tax at the top is

$$T' = \frac{1}{1 + \epsilon\alpha/(1 + \epsilon)}$$

The revenue maximizing marginal tax rate at the top is higher if ability distribution has a fatter tail (that means lower α). It is also higher if labor supply is less elastic (that means low ϵ).

3.2 The New Dynamic Public Finance

So far we have characterized the set of achievable allocations by any mechanism. The goal of the planner is to find the best achievable allocation.

$$\sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in D} \pi(\theta^T)\omega(\theta_1) \left[u(c_t(\theta^T)) - v\left(\frac{y_t(\theta^T)}{\theta_t}\right) \right] \tag{49}$$

sub. to.

$$\sum_{t=1}^{T} \sum_{\theta^T} \pi(\theta^T) \left[u(c_t(\theta^T)) - v\left(\frac{y_t(\theta^T)}{\theta_t}\right) \right] \geq \sum_{t=1}^{T} \sum_{\theta^T} \pi(\theta^T) \left[u(c_t(\alpha_t'(\theta^T))) - v\left(\frac{y_t(\alpha_t'(\theta^T))}{\theta_t}\right) \right]$$

for all $\alpha' : D \longrightarrow D$ (α'_t is $\theta^t -$ measurable).

$$\sum_{\Theta \in D} \sum_{t=1}^{T} c_t(\theta^T)\pi(\theta^T)/R^{t-1} \leq \sum_{\Theta \in D} \sum_{t=1}^{T} y_t(\theta^T)\pi(\theta^T)/R^{t-1}$$

$$(c_t, y_t) \text{ are } \theta^T\text{-measurable}$$

$$c_t(\theta^T), y_t(\theta^T) \geq 0 \quad \forall t, \theta^T \in D$$

Note that we allow for allocation to depend on date on realization of θ.

The goal of this section is to characterize the properties of constraint efficient allocation (i.e. the solution to the above planning problem). In particular we are interested in identifying the optimal inter-temporal distortions.

To start we look at the full information problem.

Full information optima

Suppose θ_t is public information. Then the planning problem is the same, except that there will no incentive compatibility constraint. Let λ be multiplier on feasibility.

$$\sum_{\theta^T} \pi(\theta^T)\omega(\theta_1)u'(c_t(\theta^T))\beta^{t-1} = \lambda/R^{t-1} \sum_{\theta^T} \pi(\theta^T)$$

Note that this is the FOC with respect to a $c_t(\theta^T)$ at a particular draw θ^T. But we know that $c_t(\theta^T)$ is $\theta^t -$ measurable, therefore we don't need to sum over all $\theta^T \in D$, but only those that contain the particular history θ^t

$$\sum_{\theta^T|\theta^t} \pi(\theta^T)\omega(\theta_1)u'(c_t(\theta^T))\beta^{t-1} = \lambda/R^{t-1} \sum_{\theta^T|\theta^t} \pi(\theta^T)$$

by measurability of $c_t(\theta^T)$.

$$u'(c_t(\theta^T))\beta^{t-1}\omega(\theta_1) = \lambda/R^{t-1}$$

Note: This implies that optimal $c_t(\theta^T)$ is actually θ_1-measurable, i.e., it is independent from θ_t for $t > 1$. In other words there is full insurance.

Note: The following Euler equation hold

$$u'(c_t(\theta^T)) = \beta R \mathbb{E}\left[u'(c_{t+1}(\theta^T))|\theta^t\right]$$

planner is happy to allow access to outside trade.

Note: Another Euler equation also holds.

$$\begin{aligned}\frac{1}{u'(c_t(\theta^T))} &= \lambda^{-1}\beta^{t-1}R^{t-1}\omega(\theta_1) \\ &= \frac{1}{\beta R}\mathbb{E}\left[\frac{1}{u'(c_{t+1}(\theta^T))}|\theta^t\right]\end{aligned}$$

3.2.1 Inverse Euler Equation

Consider again the original planning problem (49)(with incentive constraints). Let (c^*, y^*) be the solution to this problem. Now consider the following perturbation around (c^*, y^*)

$$y' = y^*$$

$$c'_s = c^*_s \quad \text{for all } s \neq t, t+1 \text{(for fixed } t\text{)}$$

for all histories θ^t

$$u(c'_t(\theta^T)) + \beta u(c'_{t+1}(\theta^T)) \quad = k + u(c^*_t(\theta^T)) + \beta u(c^*_{t+1}(\theta^T)) \quad \text{for all } \theta^{t+1} \text{ such that } \pi(\theta^{t+1}|\theta^t) > 0$$

$$\sum_{\theta^T|\theta^t} \pi(\theta^T)\left[c'_t(\theta^T) + c'_{t+1}(\theta^T)/R\right] = \sum_{\theta^T|\theta^t} \pi(\theta^T)\left[c^*_t(\theta^T) + c^*_{t+1}(\theta^T)/R\right]$$

Note: (c', y') is feasible and incentive compatible.

Note: What we are doing is perturbing $u(c_t(\theta^T))$ by some amount and then make an appropriate perturbation in every immediate history following θ^t so that incentive compatibility is preserved. If (c^*, y^*) is the solution to (49), this perturbation cannot improve welfare. One implication of this is that (c^*, y^*) solves the following maximization problem and $k = 0$ at the optimal solution.

$$\max_{k, c'_t(\theta^T), c'_{t+1}(\theta^T)} k$$

sub. to

$$u(c'_t(\theta^T)) + \beta u(c'_{t+1}(\theta^T)) = k + u(c^*_t(\theta^T)) + \beta u(c^*_{t+1}(\theta^T)) \qquad \text{for all } \theta^t, \theta^{t+1} \text{ such that } \pi(\theta^{t+1}|\theta^t) > 0$$

$$\sum_{\theta^T|\theta^t} \pi(\theta^T)\left[c'_t(\theta^T) + c'_{t+1}(\theta^T)/R\right] = \sum_{\theta^T|\theta^t} \pi(\theta^T)\left[c^*_t(\theta^T) + c^*_{t+1}(\theta^T)/R\right]$$

let $\eta(\theta^{t+1})$ and λ be multipliers.

Let's write the FOC

$$\sum_{\theta^{t+1}|\theta^t} \eta(\theta^{t+1}) u'(c'_t(\theta^T)) = \lambda \sum_{\theta^T|\theta^t} \pi(\theta^T) = \lambda \pi(\theta^t)$$

for all θ^t, θ^{t+1} such that $\pi(\theta^{t+1}|\theta^t) > 0$.[1]

$$\beta u'(c'_t(\theta^{t+1}))\eta(\theta^{t+1}) = \lambda/R \sum_{\theta^T|\theta^{t+1}} \pi(\theta^T) = \lambda \pi(\theta^{t+1})/R$$

Substitute for $\eta(\theta^{t+1})$

$$\sum_{\theta^{t+1}|\theta^t} \frac{\lambda \pi(\theta^{t+1})/R}{\beta u'(c'_t(\theta^{t+1}))} u'(c'_t(\theta^T)) = \lambda \pi(\theta^t)$$

Cancel terms and evaluate this the solution $c' = c^*$

$$\frac{\beta R}{u'(c^*_t(\theta^T))} = \sum_{\theta^{t+1}|\theta^t} \frac{\pi(\theta^{t+1})}{\pi(\theta^t)} \frac{1}{u'(c^*_{t+1}(\theta^T))} \tag{50}$$

Note: The Intertemporal condition only depends on consumption which is observable.

Note: The additive separability assumption was key in deriving this result. Look at Farhi and Werning (2008) for a version of this result that is derived for more general class of utility functions.

Note: This result does not hold if private information affect the marginal utility of consumption (for example in Atkeson and Lucas (1992) taste shock model).

This result implies that it is not desirable for planer to allow access to saving. To see this

[1] Note that I am abusing notation here. $\pi(\theta^t)$ means the probability of history θ^t.

look at the following Euler equation (which has to hold if there is access to saving)

$$u'(c_t(\theta^T)) = \beta R \sum_{\theta^{t+1}|\theta^t} \frac{\pi(\theta^{t+1})}{\pi(\theta^t)} u'(c_{t+1}(\theta^T)) \tag{51}$$

But let's look back at Inverse Euler Equation (50)

$$\begin{aligned} u'(c_t(\theta^T)) &= \beta R \frac{1}{\sum_{\theta^{t+1}|\theta^t} \frac{\pi(\theta^{t+1})}{\pi(\theta^t)} \frac{1}{u'(c^*_{t+1}(\theta^T))}} \\ &> \beta R \frac{1}{\frac{1}{\sum_{\theta^{t+1}|\theta^t} \frac{\pi(\theta^{t+1})}{\pi(\theta^t)} u'(c^*_{t+1}(\theta^T))}} \\ &= \beta R \sum_{\theta^{t+1}|\theta^t} \frac{\pi(\theta^{t+1})}{\pi(\theta^t)} u'(c^*_{t+1}(\theta^T)) \end{aligned}$$

Note that at the efficient allocation the individuals are "saving constrained". In other words, if individuals can privately save, they will choose to do so and it is desirable for planer to prevent them from doing that.

Another way of seeing this is the following: suppose (51) holds. Then we must have

$$\frac{\beta R}{u'(c^*_t(\theta^T))} < \sum_{\theta^{t+1}|\theta^t} \frac{\pi(\theta^{t+1})}{\pi(\theta^t)} \frac{1}{u'(c^*_{t+1}(\theta^T))}$$

Now suppose the planer wants to increase utility at time t by ϵ and decrease it at time $t+1$ by $\beta^{-1}\epsilon$. The cost of increase of utility in period t is $u'(c_t(\theta^T))/\epsilon$. On the other hand planer hands in $u'(c_{t+1}(\theta^T))/\epsilon$ less at each θ^{t+1} that follows θ^t. Therefore it can free up resources.

On dynamics of consumption

Consider again the full information optimal allocation

$$u'(c_t(\theta^T))\beta^{t-1}\omega(\theta_1) = \lambda/R^{t-1}$$

Suppose for simplicity $\beta R = 1$. Then

1. Allocation is independent of history (expect possibility θ^1)

2. There is no mobility in short-run or long-run

3. Inequality is constant

Now consider private information optimal allocations. Assume θ_t is i.i.d. Consider two different history θ^t and $\bar{\theta}^t$

$$u'(c_t(\theta^T|\theta^t)) = \beta R \sum_{\theta^{t+1}|\theta^t} \frac{\pi(\theta^{t+1})}{\pi(\theta^t)} u'(c_{t+1}(\theta^T|\theta^t))$$

$$u'(c_t(\theta^T|\bar{\theta}^t)) = \beta R \sum_{\theta^{t+1}|\theta^t} \frac{\pi(\theta^{t+1})}{\pi(\theta^t)} u'(c_{t+1}(\theta^T|\bar{\theta}^t))$$

note that $\pi(\theta^T|\bar{\theta}^t) = \pi(\theta^T|\theta^t)$. Now suppose $u'(c_t(\theta^T|\theta^t)) > u'(c_t(\theta^T|\bar{\theta}^t))$, then there exist a history θ^{t+1} such that $\pi(\theta^{t+1}|\theta^t) = \pi(\theta^{t+1}|\bar{\theta}^t)$ and

$$u'(c_{t+1}(\theta^T|\theta^t)) > u'(c_{t+1}(\theta^T|\bar{\theta}^t))$$

good shocks up to period t has persistent effect on period $t+1$ allocations.

Next we consider inequality. Assume, $u(c) = \log(c)$, then $\frac{1}{u'(c)} = c$. Start from inverse Euler equation ($\beta R = 1$)

$$\frac{1}{u'(c_t(\theta^T))} = \mathbb{E}\left[\frac{1}{u'(c_{t+1}(\theta^T))}|\theta^t\right]$$

We want to know what happens to variance of consumption over time

$$\begin{aligned} Var\left(\frac{1}{u'(c_t(\theta^T))}\right) &= Var\left(\mathbb{E}\left[\frac{1}{u'(c_{t+1}(\theta^T))}|\theta^t\right]\right) \\ &= Var\left(\frac{1}{u'(c_{t+1}(\theta^T))}\right) - \mathbb{E}\left[Var\left(\frac{1}{u'(c_{t+1}(\theta^T))}|\theta^t\right)\right] \end{aligned}$$

If $Var\left(\frac{1}{u'(c_{t+1}(\theta^T))}|\theta^t\right) > 0$ for some θ^t, then

$$Var\left(\frac{1}{u'(c_t(\theta^T))}\right) < Var\left(\frac{1}{u'(c_{t+1}(\theta^T))}\right)$$

and therefore

$$Var\left(c_t(\theta^T)\right) < Var\left(c_{t+1}(\theta^T)\right)$$

So inequality grows. And it is efficient.

How about mobility? In short-run there is mobility. What about long-run?

Note that $\frac{1}{u'(c_t)}$ is a martingale ($\beta R = 1$). We also know that (by feasibility) $\mathbb{E}\left[\frac{1}{u'(c_{t+1})}\right] < \infty$

Martingale Convergence Theorem: If $\{x_t\}_{t=1}^{\infty}$ is stochastic process adapted to filteration $\{\mathcal{F}_t\}_{t=1}^{\infty}$ such that $x_t = \mathbb{E}\left[x_{t+1}|\mathcal{F}_t\right]$ and $\mathbb{E}\left[x_t\right] < \infty$ for all t, then

$$\lim_{t\to\infty} x_t \overset{a.s.}{=} x_\infty < \infty$$

where x_∞ is a random variable with $\mathbb{E}[x_\infty] < \infty$.

Therefore, $\frac{1}{u'(c_t)}$ converges to a finite number and hence there is no mobility in the long-run.

3.2.2 Long-run properties of efficient allocations

This part is mostly based on Farhi and Werning (2010, 2007, 2005), PHELAN (2006) and Atkeson and Lucas (1992).

In what follows we maintain the following assumptions:

- $T = \infty$
- $\Theta = \{\theta_H, \theta_L\}$
- θ_t i.i.d over time

Very important note: In what follows I skip some detailed steps. Most of the arguments are heuristic and have loose ends. For more rigorous proofs please look at the references above.

Immiseration result

For this part assume $\beta R = 1$. Consider the following planning problem

$$w_0 = \max \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[u(c_t(\theta^t)) - v\left(\frac{y_t(\theta^t)}{\theta_t}\right)\right] \tag{52}$$

sub. to.

$$\sum_{t=1}^{T}\sum_{\theta^t} \pi(\theta^t) \left[u(c_t(\theta^t)) - v\left(\frac{y_t(\theta^t)}{\theta_t}\right)\right] \geq \sum_{t=1}^{T}\sum_{\theta^t} \pi(\theta^t) \left[u(c_t(\alpha_t'(\theta^t))) - v\left(\frac{y_t(\alpha_t'(\theta^t))}{\theta_t}\right)\right]$$

for all $\alpha' : D \longrightarrow D$ (α_t' is θ^t − measurable).

$$\sum_{\theta^t}\sum_{t=1}^{T} \pi(\theta^t) \left[c_t(\theta^t) - y_t(\theta^t)\right] / R^{t-1} \leq 0$$

The solution to this problem must also be the solution to the following dual problem

$$K(w_0) = \min \sum_{\theta^t} \sum_{t=1}^{T} \pi(\theta^t) \left[c_t(\theta^t) - y_t(\theta^t) \right] / R^{t-1} \tag{53}$$

sub. to

$$\sum_{t=1}^{T} \sum_{\theta^t} \pi(\theta^t) \left[u(c_t(\theta^t)) - v\left(\frac{y_t(\theta^t)}{\theta_t} \right) \right] \geq \sum_{t=1}^{T} \sum_{\theta^t} \pi(\theta^t) \left[u(c_t(\alpha_t'(\theta^t))) - v\left(\frac{y_t(\alpha_t'(\theta^t))}{\theta_t} \right) \right]$$

$$\sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[u(c_t(\theta^t)) - v\left(\frac{y_t(\theta^t)}{\theta_t} \right) \right] \geq w_0$$

In which $K(U_0)$ is the cost of delivering ex-ante utility U_0 to everyone.

We want to write this recursively. Consider an allocation sequence $(c_t(\theta^t), y_t(\theta^t))$. Consider a history $\bar{\theta}^t$. Then define the ex ante utility of an agent with history $\bar{\theta}^t$ under this plan as

$$w_t(\bar{\theta}^t) = \sum_{s=t}^{\infty} \sum_{\theta^s | \bar{\theta}^t} \pi(\theta^s) \beta^{s-t} \left[u(c_s(\theta^s)) - v\left(\frac{y_t(\theta^s)}{\theta_s} \right) \right]$$

we call $w_t(\bar{\theta}^t)$ the "promised utility" after history $\bar{\theta}^t$.

Let $(c_t^*(\theta^t), y_t^*(\theta^t))$ be the solution to problem (53) $w_t^*(\bar{\theta}^t)$ be promised utility after history $\bar{\theta}^t$. We can show that $(c_t^*(\bar{\theta}^t, \theta_{t+1}), y_t^*(\bar{\theta}^t, \theta_{t+1}), w_t^*(\bar{\theta}^t, \theta_{t+1}))$ solve the following Bellman Equation at $w = w_t^*(\bar{\theta}^t)$ (note: I imposed the $\beta R = 1$ assumption here)

$$K(w) = \min_{c,y,W} \sum_{\theta} \pi(\theta) \left[c(\theta, w) - y(\theta, w) + \beta K(w'(\theta, w)) \right]$$

sub. to

$$u(c(\theta, w)) - v\left(\frac{y(\theta, w)}{\theta} \right) + \beta w'(\theta, w) \geq u(c(\theta', w)) - v\left(\frac{y(\theta', w)}{\theta} \right) + \beta w'(\theta', w) \quad \forall \theta, \theta'$$

$$\sum_{\theta} \pi(\theta) \left[u(c(\theta, w)) - v\left(\frac{y(\theta, w)}{\theta} \right) + \beta w'(\theta, w) \right] \geq w$$

The second constraint is called "promise keeping" constraint.

Proposition 10 *$K(w)$ is strictly increasing and strictly convex (assumptions on $v(\cdot)$ is needed). Also. Let $\underline{w}$ and $\bar{w}$ be the lowest and highest possible values for promised utility. Then,* $\lim_{w \to \underline{w}} K'(w) = 0$ *and* $\lim_{w \to \bar{w}} K'(w) = \lim_{w \to \bar{w}} K(w) = 0$.

Let $\mu(\theta,\theta')$ be multiplier on incentive constraint and ϕ the multiplier on promise keeping. First order condition with respect to $c(\theta,U)$ is

$$u'(c(\theta,w))\left[\sum_{\theta'}\mu(\theta,\theta')-\sum_{\theta'}\mu(\theta',\theta)+\pi(\theta)\phi\right]=\pi(\theta)$$

and with respect to $w'(\theta,U)$

$$\left[\sum_{\theta'}\mu(\theta,\theta')-\sum_{\theta'}\mu(\theta',\theta)+\pi(\theta)\phi\right]=\pi(\theta)K'(w'(\theta,w)) \tag{54}$$

and therefore

$$K'(w'(\theta,w))=\frac{1}{u'(c(\theta,w))}$$

this implies the following lemma

Lemma 1 *Given any $w\in[\underline{w},\bar{w}]$, if $w'(\theta,w)=w'(\theta',w)$ for some $\theta,\theta'\in\Theta$, then $c(\theta,w)=c'(\theta,w)$*

Sum the equation (54) over all θ

$$\sum_{\theta}\pi(\theta)K'(w'(\theta,w))=\sum_{\theta}\sum_{\theta'}\mu(\theta,\theta')-\sum_{\theta}\sum_{\theta'}\mu(\theta',\theta)+\phi=\phi$$

also from envelope condition we have

$$K'(w)=\phi$$

therefore

$$K'(w)=\sum_{\theta}\pi(\theta)K'(w'(\theta,w))$$

Start from a given w_0, construct a stochastic process w_t as

$$w_{t+1}=w'(\theta_t,w_t)$$

then

$$K'(w_t)=\mathbb{E}_t\left[K'(w_{t+1})\right]$$

hence w_t is a martingale. By martingale convergence theorem there must exist a w_∞ such that $w_t \underrightarrow{a.s.} w_\infty$. Suppose $K'(w_\infty) > 0$. Note that convergence implies that

$$w'(\theta, w_\infty) = w'(\theta', w_\infty) \quad \forall \theta, \theta'$$

and therefore

$$c(\theta, w_\infty) = c(\theta', w_\infty) \quad \forall \theta, \theta'$$

and then incentive compatibility implies

$$y(\theta, w_\infty) = y(\theta', w_\infty) \quad \forall \theta, \theta'$$

but we know from our two type static example that the planer can do better by differentiating various θ types. Therefore, this is a contradiction. Hence $K'(w_\infty) = 0$ and $w_\infty = \underline{w}$.

No Immiseration result

Consider the following planning problem in which planer values future consumption more than agent ($\hat{\beta} > \beta$)

$$w_0 = \max \sum_{t=1}^{T} \hat{\beta}^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[u(c_t(\theta^t)) - \hat{v}\left(\frac{y_t(\theta^t)}{\theta_t} \right) \right] \tag{55}$$

sub. to.

$$\sum_{t=1}^{T} \sum_{\theta^t} \pi(\theta^t) \left[u(c_t(\theta^t)) - \hat{v}\left(\frac{y_t(\theta^t)}{\theta_t} \right) \right] \geq \sum_{t=1}^{T} \sum_{\theta^t} \pi(\theta^t) \left[u(c_t(\alpha'_t(\theta^t))) - \hat{v}\left(\frac{y_t(\alpha'_t(\theta^t))}{\theta_t} \right) \right]$$

for all $\alpha' : D \longrightarrow D$ (α'_t is θ^t – measurable).

$$\sum_{\theta^t} \sum_{t=1}^{T} \pi(\theta^t) \left[c_t(\theta^t) - y_t(\theta^t) \right] / R^{t-1} \leq 0$$

assume the following (in addition to the assumption mentioned at the beginning of the discussion)

- $\hat{\beta} R = 1$
- $\hat{v}\left(\frac{y}{\theta}\right) = \frac{v(y)}{\theta}$

- $u()$ is unbounded below, hence $\underline{w} = -\infty$
- $\mathbb{E}\left[\frac{1}{\theta}\right] = 1$.

Suppose the above problem has a solution and let $\hat{\lambda}$be the multiplier on resources constraint. Then the solution to the above problem must also solve the following

$$P(w_0) = \max \sum_{t=1}^{T} \hat{\beta}^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} - \hat{\lambda} c_t(\theta^t) + \hat{\lambda} y_t(\theta^t) \right] \tag{56}$$

sub. to.

$$\sum_{t=1}^{T} \sum_{\theta^t} \pi(\theta^t) \left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} \right] \geq \sum_{t=1}^{T} \sum_{\theta^t} \pi(\theta^t) \left[u(c_t(\alpha'_t(\theta^t))) - \frac{v(y_t(\alpha'_t(\theta^t)))}{\theta_t} \right]$$

for all $\alpha' : D \longrightarrow D$ (α'_t is $\theta^t -$ measurable).

$$\sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} \right] \geq w_0$$

Again, we can show that the solution to the above problem also solves the following Bellman equation (after any history)

$$P(w) = \max_{c,y,w'} \sum_{\theta} \pi(\theta) \left[u(c(\theta, w)) - \frac{v(y(\theta, w))}{\theta} - \hat{\lambda} c(\theta, w) + \hat{\lambda} y(\theta, w) + \hat{\beta} P(w'(\theta, w)) \right] \tag{57}$$

sub. to

$$u(c(\theta, w)) - \frac{v(y(\theta, w))}{\theta} + \beta w'(\theta, w) \geq u(c(\theta', w)) - \frac{v(y(\theta', w))}{\theta} + \beta w'(\theta', w) \quad \forall \theta, \theta'$$

$$\sum_{\theta} \pi(\theta) \left[u(c(\theta, w)) - \frac{v(y(\theta, w))}{\theta} + \beta w'(\theta, w) \right] \geq w$$

We want to show that in this problem in the long-run the promised utility cannot be at misery ($w_\infty > -\infty$). We use two lemmas to show this

Lemma 2 *The value function $P(w)$ is strictly concave and continuously differentiable on*

$(-\infty, \bar{w})$. *Moreover,*

$$\lim_{v \to -\infty} P(w) = \lim_{v \to \bar{w}} P(w) = \lim_{v \to \bar{w}} P'(w) = -\infty$$

and

$$\lim_{v \to -\infty} P'(w) = 1$$

Proof.

We are not going to prove concavity and differentiability. We take them as given.

Warning: This proof is not complete! There are some steps that needs to be filled in or reformulated. I present it to provide the core idea of the proof as I understand it.

Define value function $P_{FI}(w)$ as

$$P_{FI}(w) = \max \sum_{t=1}^{T} \hat{\beta}^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} - \hat{\lambda} c_t(\theta^t) + \hat{\lambda} y_t(\theta^t) \right]$$

sub. to.

$$\sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} \right] \geq w$$

$P_{FI}(w)$ is the value to the planner from delivering utility w to individual, if we ignore incentive constraint. Note that $P_{FI}(w) > P(w)$. Also, $P_{FI}(w)$ is strictly concave and differentiable.

Next consider the following maximization problem

$$m = \max_{c,y,\theta} u(c) - \frac{v(y)}{\theta} - \hat{\lambda} c + \hat{\lambda} y$$

The above problem has a solution ($u'(c) = \hat{\lambda}$,$v'(y) = \hat{\lambda}\theta$, and θ belongs to a compact set). Next, note that

$$\begin{aligned}
P_{FI}(w) &= \sum_{t=1}^{T} \hat{\beta}^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} - \hat{\lambda} c_t(\theta^t) + \hat{\lambda} y_t(\theta^t) \right] \\
&= \sum_{t=1}^{T} \hat{\beta}^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} - \hat{\lambda} c_t(\theta^t) + \hat{\lambda} y_t(\theta^t) \right] \\
&+ \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} - \hat{\lambda} c_t(\theta^t) + \hat{\lambda} y_t(\theta^t) \right] \\
&- \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} - \hat{\lambda} c_t(\theta^t) + \hat{\lambda} y_t(\theta^t) \right] \\
&= w + \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[-\hat{\lambda} c_t(\theta^t) + \hat{\lambda} y_t(\theta^t) \right] + \\
&\quad \sum_{t=1}^{T} (\hat{\beta}^{t-1} - \beta^{t-1}) \sum_{\theta^T \in D} \pi(\theta^t) \left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} - \hat{\lambda} c_t(\theta^t) + \hat{\lambda} y_t(\theta^t) \right] \\
&\leq w + \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[-\hat{\lambda} c_t(\theta^t) + \hat{\lambda} y_t(\theta^t) \right] + m \left(\frac{1}{1-\hat{\beta}} - \frac{1}{1-\beta} \right) \\
&\leq w - \hat{\lambda} \tilde{K}(w) + m \left(\frac{1}{1-\hat{\beta}} - \frac{1}{1-\beta} \right)
\end{aligned}$$

in which

$$\tilde{K}(w) = \min \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[c_t(\theta^t) - y_t(\theta^t) \right]$$

sub. to.

$$\sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in D} \pi(\theta^t) \left[u(c_t(\theta^t)) - \frac{v(y_t(\theta^t))}{\theta_t} \right] \geq w$$

Note that $\tilde{K}(w)$ is strictly convex and differentiable and $\lim_{w \to -\infty} \tilde{K}'(w) = 0$.

Let $P_{max}(w) = w - \tilde{K}(w) + m$. Then $P_{max}(w) \geq P_{FI}(w)$ and both are strictly concave. Also, $\lim_{w \to -\infty} P_{FI}(w) \leq \lim_{w \to -\infty} P_{max}(w) = -\infty$. Therefore, $\lim_{w \to -\infty} P'_{FI}(w) \leq \lim_{w \to -\infty} P'_{max}(w) = 1$.

Also, Note that $\lim_{w \to -\infty} P(w) \leq \lim_{w \to -\infty} P_{FI}(w) = -\infty$ and therefore (since both are strictly concave)

$$\lim_{w \to -\infty} P'(w) \leq \lim_{w \to -\infty} P'_{FI}(w) = 1.$$

$$\lim_{w\to-\infty} P'(w) \leq 1$$

Next, consider allocations $(c(w_0,\theta^t), y(w_0,\theta^t))$ that solve the original problem. Suppose they attain the value $P(w_0)$. Define new allocations $(\tilde{c}(w,\theta^t), \tilde{y}(w,\theta^t))$ for $w \leq w_0$ as

$$\begin{aligned}
\tilde{c}(w,\theta^t) &= c(w_0,\theta^t) \quad \forall\theta^t\forall t \\
\tilde{y}(w,\theta^t) &= y(w_0,\theta^t) \quad \forall\theta^t\forall t > 1 \\
\tilde{y}(w,\theta_1) &= v^{-1}(v(y(w_0,\theta_1)) + w_0 - w)
\end{aligned}$$

Now define $P_m(w)$ for $w \leq w_0$ as

$$\begin{aligned}
P_m(w) &= \sum_{t=1}^{T} \hat{\beta}^{t-1} \sum_{\theta^T\in D} \pi(\theta^t)\left[u(\tilde{c}(w,\theta^t)) - \frac{v(\tilde{y}(w,\theta^t))}{\theta_t} - \hat{\lambda}\tilde{c}(w,\theta^t) + \hat{\lambda}\tilde{y}(w,\theta^t)\right] \\
&= \sum_{\theta_1} \pi(\theta_1)\left[u(c(w_0,\theta_1)) - \frac{v(y(w_0,\theta_1)) + w_0 - w}{\theta_1} - \hat{\lambda}c(w_0,\theta_1) + \hat{\lambda}\tilde{y}(w_0,\theta_1)\right] + \\
&\quad \sum_{t=2}^{T} \hat{\beta}^{t-1} \sum_{\theta^T\in D} \pi(\theta^t)\left[u(c(w_0,\theta^t)) - \frac{v(c(w_0,\theta^t))}{\theta_t} - \hat{\lambda}c(w_0,\theta^t) + \hat{\lambda}y(w_0,\theta^t)\right] \\
&= \sum_{\theta_1} \pi(\theta_1)\left[\frac{w - w_0}{\theta_1} + \hat{\lambda}\tilde{y}(w,\theta_1)\right] + \sum_{\theta_1} \pi(\theta_1)\left[u(c(w_0,\theta_1)) - \frac{v(y(w_0,\theta_1))}{\theta_1} - \hat{\lambda}c(w_0,\theta_1)\right] \\
&\quad + \sum_{t=2}^{T} \hat{\beta}^{t-1} \sum_{\theta^T\in D} \pi(\theta^t)\left[u(c(w_0,\theta^t)) - \frac{v(c(w_0,\theta^t))}{\theta_t} - \hat{\lambda}c(w_0,\theta^t) + \hat{\lambda}y(w_0,\theta^t)\right]
\end{aligned}$$

Note that $P_m(w)$ is strictly concave, $P_m(w) \leq P(w)$. Also, note that only the first term depends on w. Therefore

$$P'_m(w) = 1 - \hat{\lambda}\mathbb{E}\left[\frac{1}{v'(v(y(w,\theta_0)) + w_0 - w)}\right]$$

and $\lim_{w\to-\infty} P'_m(w) = 1$.

Therefore, $\lim_{w\to-\infty} P'(w) \geq \lim_{w\to-\infty} P'_m(w) = 1$. Hence, we proved that $\lim_{w\to-\infty} P'(w) = 1$.

■

In the next lemma we show that $1 - P(w'(w,\theta))$ can be bounded above and bellow for all θ. Let's rewrite the problem (57) again

$$P(w) = \max_{c,y,w'} \sum_{\theta} \pi(\theta) \left[u(c(\theta,w)) - \frac{v(y(\theta,w))}{\theta} - \hat{\lambda} c(\theta,w) + \hat{\lambda} y(\theta,w) + \hat{\beta} P(w'(\theta,w)) \right] \quad (58)$$

sub. to

$$u(c(\theta,w)) - \frac{v(y(\theta,w))}{\theta} + \beta w'(\theta,w) \geq u(c(\theta',w)) - \frac{v(y(\theta',w))}{\theta} + \beta w'(\theta',w) \quad \forall \theta, \theta'$$

$$\sum_{\theta} \pi(\theta) \left[u(c(\theta,w)) - \frac{v(y(\theta,w))}{\theta} + \beta w'(\theta,w) \right] \geq w$$

Suppose only high type's incentive constraint binds. Let μ and ϕ be multipliers on the IC and promise keeping (and let $\pi(\theta_H) = \pi$)

FOC w.r.t $c(w,\theta)$

$$\begin{aligned} \pi \left(1 - \hat{\lambda} \frac{1}{u'(c(w,\theta_H))} \right) + \phi\pi + \mu &= 0 \\ (1-\pi) \left(1 - \hat{\lambda} \frac{1}{u'(c(w,\theta_L))} \right) + \phi\pi - \mu &= 0 \end{aligned}$$

FOC w.r.t $w'(w,\theta)$

$$\begin{aligned} \pi \hat{\beta} P'(w,\theta_H) + \phi\pi\beta + \beta\mu &= 0 \\ (1-\pi)\hat{\beta} P'(w,\theta_L) + \phi(1-\pi)\beta - \beta\mu &= 0 \end{aligned}$$

FOC w.r.t $y(w,\theta)$

$$\begin{aligned} -\frac{\pi}{\theta_H} + \pi\hat{\lambda} \frac{1}{v'(y(w,\theta_H))} - \phi \frac{\pi}{\theta_H} - \frac{\mu}{\theta_H} &= 0 \\ -\frac{(1-\pi)}{\theta_L} + (1-\pi)\hat{\lambda} \frac{1}{v'(y(w,\theta_L))} - \phi \frac{(1-\pi)}{\theta_L} + \frac{\mu}{\theta_H} &= 0 \end{aligned}$$

combining these FOC's together with envelope condition $P'(w) = -\phi$ we get

$$\mathbb{E}\left[1 - P'(w'(w,\theta)) \right] = \frac{\beta}{\hat{\beta}} (1 - P'(w)) + 1 - \frac{\beta}{\hat{\beta}}$$

and

$$\hat{\lambda}\mathbb{E}\left[\frac{1}{v'(y(w,\theta))}\right] = (1+\phi)\mathbb{E}\left[\frac{1}{\theta}\right] = 1+\phi$$

Assume that we know $y(w,\theta_H) > y(w,\theta_L)$ and $w'(w,\theta_H) > w'(w,\theta_L)$. Now we can prove the next lemma.

Lemma 3 *The following inequalities hold*

$$(1-P'(w))\frac{\beta}{\hat{\beta}}\left(1+\frac{\theta_H}{\theta_L}-\theta_L\right)+1-\frac{\beta}{\hat{\beta}} \le 1-P'(w'(w,\theta)) \le (1-P'(w))\frac{\beta}{\hat{\beta}}\theta_H+1-\frac{\beta}{\hat{\beta}}$$

Proof.

From the FOC for $y(w,\theta_H)$ we get

$$(1+\phi)\theta_H = \mathbb{E}\left[\frac{\hat{\lambda}}{v'(y(w,\theta))}\right]\theta_H \ge \frac{\hat{\lambda}}{v'(y(w,\theta))} = 1+\phi+\frac{\mu}{\pi}$$

from this we can get the following bound on $\frac{\mu}{\pi}$

$$\frac{\mu}{\pi} \le (1+\phi)(\theta_H-1)$$

Now we use this in the FOC for $w'(w,\theta_H)$ and we get

$$\begin{aligned} P'(w'(w,\theta_H)) &\ge -\phi\frac{\beta}{\hat{\beta}}\theta_H - \frac{\beta}{\hat{\beta}}(\theta_H-1) \\ &= P'(w)\frac{\beta}{\hat{\beta}}\theta_H - \frac{\beta}{\hat{\beta}}(\theta_H-1) \end{aligned}$$

after rearranging terms

$$1-P'(w'(w,\theta_L)) < 1-P'(w'(w,\theta_H)) \le (1-P'(w))\frac{\beta}{\hat{\beta}}\theta_H+1-\frac{\beta}{\hat{\beta}}$$

Using similar argument we can show that

$$\frac{\mu}{1-\pi} \le (1+\phi)(1-\theta_L)\frac{\theta_H}{\theta_L}$$

and

$$\begin{aligned} P'(w'(w,\theta_L) &\leq -\phi\frac{\beta}{\hat{\beta}}\left(1+\frac{\theta_H}{\theta_L}-\theta_L\right)+\frac{\beta}{\hat{\beta}}\left(\frac{\theta_H}{\theta_L}-\theta_L\right) \\ &= P'(w)\frac{\beta}{\hat{\beta}}\left(1+\frac{\theta_H}{\theta_L}-\theta_L\right)+\frac{\beta}{\hat{\beta}}\left(\frac{\theta_H}{\theta_L}-\theta_L\right) \end{aligned}$$

and hence

$$1-P'(w'(w,\theta_H)) > 1-P'(w'(w,\theta_L)) \geq (1-P'(w))\frac{\beta}{\hat{\beta}}\left(1+\frac{\theta_H}{\theta_L}-\theta_L\right)+1-\frac{\beta}{\hat{\beta}}$$

■

Now let $w \to -\infty$, then $P'(w) \to 1$ and

$$\lim_{w\to-\infty} P'(w'(w,\theta)) = \frac{\beta}{\hat{\beta}} < 1$$

So $w'(w,\theta)$ can never stay at misery level.

For more detailed arguments and complete proof of the existence of stationary distribution see Farhi and Werning (2010, 2007, 2005).

3.2.3 Aggregate Risk

In this section we derive inverse Euler equation in an environment in which there is aggregate risk. We need to introduce new notations. Let Θ be the space of individual shocks and Z be the space of aggregate shock. The timing is the following

1. Nature draws $z^T \in Z^T$ according to p.d.f $\pi_z(z^T)$.
2. Nature draw individual shocks $\theta^T \in \Theta^T$ according to $\pi_\theta(\theta^T|z^T)$. These draws are i.i.d across individuals conditional on z^T.

By law of large number, given z^T, the fraction of population with shocks θ^T is $\pi_\theta(\theta^T|z^T)$.

We impose the following restriction:

Assumption 1 *For all $\theta^T \in \Theta^T$, $\pi_\theta(\theta^t|z^T) = \sum_{(\theta_{t+1},\ldots,\theta_T)} \pi_\theta(\theta^t,\theta_{t+1},\ldots,\theta_T|z^T)$ is independent of $z_{t+1},\ldots,z_T$.*

This assumption implies that conditional on z^t, $(\theta_{t+1}, \ldots, \theta_T)$ and $(z_{t+1}, \ldots, z_T)$ are independent.

Remarks: Note that in this setup we are not imposing any restriction on time series properties of θ_t and z_t. However, our assumption implies that by observing history of private shocks up to date t the individuals cannot infer anything about future aggregate shocks.

As before, we assume that agents learn z_t and θ_t at the beginning of date t and θ^t is private information.

There is an initial capital stock $\bar{K}_1$ in the economy.

Definition 9 *An allocation is a sequence of functions* (c, y, K) *such that*

$$c : \Theta^T \times Z^T \to \mathbb{R}^T_+, \quad c_t : (\theta^t, z^t)\text{-}measurable$$

$$y : \Theta^T \times Z^T \to \mathbb{R}^T_+, \quad y_t : (\theta^t, z^t)\text{-}measurable$$

$$K : Z^T \to \mathbb{R}^T_+, \quad K_{t+1} : z^t\text{-}measurable$$

Definition 10 *An allocation is feasible if*

$$\begin{aligned} C_t(z^T) + K_{t+1}(z^T) &= (1-\delta)K_t(z^T) + F(K_t(z^T), Y_t(z^T), z^T) \\ C_t(z^T) &= \sum_{\theta^T \in \Theta^T} \pi_\theta(\theta^T | z^T) c_t(\theta^T, z^T) \\ Y_t(z^T) &= \sum_{\theta^T \in \Theta^T} \pi_\theta(\theta^T | z^T) y_t(\theta^T, z^T) \\ K_1 &= \bar{K}_1 \end{aligned} \tag{59}$$

Definition 11 *An allocation is incentive compatible if*

$$\sum_{z^T \in Z^T} \pi_z(z^T) \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in \Theta^T} \pi_\theta(\theta^T | z^T) \left[u(c_t(\theta^T, z^T)) - v\left(\frac{y_t(\theta^T, z^T)}{\theta_t}\right)\right] \geq \tag{60}$$

$$\sum_{z^T \in Z^T} \pi_z(z^T) \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in \Theta^T} \pi_\theta(\theta^T | z^T) \left[u(c_t(\alpha'_t(\theta^T, z^T))) - v\left(\frac{y_t(\alpha'_t(\theta^T, z^T))}{\theta_t}\right)\right]$$

for all $\alpha' : \Theta \longrightarrow \Theta$ *(*α'_t *is* $(\theta^t, z^t) -$ *measurable).*

The planing problem

$$\max \sum_{z^T \in Z^T} \pi_z(z^T) \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^T \in \Theta^T} \pi_\theta(\theta^T | z^T) \left[u(c_t(\theta^T, z^T)) - v\left(\frac{y_t(\theta^T, z^T)}{\theta_t} \right) \right]$$

sub. to (59) and (60).

The inverse Euler equation

Suppose (c^*, y^*, K^*) is the solution to the above problem. Fix a public history $\bar{z}^t$. We perturb the solution to (c', y^*, K') such that

$$u(c'_t(\theta^t, \bar{z}^t)) = u(c^*_t(\theta^t, \bar{z}^t)) + \beta \sum_{z_{t+1}} \pi_z(\bar{z}^t, z_{t+1}) \delta_{t+1}(\theta^t, \bar{z}^t, z_{t+1}) + \gamma \quad \forall \theta^t \quad (61)$$

$$u(c'_{t+1}(\theta^{t+1}, \bar{z}^t, z_{t+1})) = u(c^*_{t+1}(\theta^{t+1}, \bar{z}^t, z_{t+1})) - \delta_{t+1}(\theta^t, \bar{z}^t, z_{t+1}) \quad (62)$$

$$\sum_{\theta^t} \pi(\theta^t | \bar{z}^t) c'_t(\theta^t, \bar{z}^t) + K'_{t+1}(\bar{z}^t) \leq \sum_{\theta^t} \pi(\theta^t | \bar{z}^t) c^*_t(\theta^t, \bar{z}^t) + K^*_{t+1}(\bar{z}^t) \quad (63)$$

$$\sum_{\theta^{t+1}} \pi_\theta(\theta^{t+1} | \bar{z}^t, z_{t+1}) c'_{t+1}(\theta^{t+1}, \bar{z}^t, z_{t+1}) - K'_{t+1}(\bar{z}^t)(1-\delta) - F(K'_{t+1}(\bar{z}^t), Y^*_t(\bar{z}^t, z_{t+1}), \bar{z}^t, z_{t+1}) \quad (64)$$

$$\leq$$

$$\sum_{\theta^{t+1}} \pi_\theta(\theta^{t+1} | \bar{z}^t, z_{t+1}) c^*_{t+1}(\theta^{t+1}, \bar{z}^t, z_{t+1}) - K^*_{t+1}(\bar{z}^t)(1-\delta) - F(K^*_{t+1}(\bar{z}^t), Y^*_t(\bar{z}^t, z_{t+1}), \bar{z}^t, z_{t+1})$$

The idea is that (c^*, y^*, K^*) must be the solution to the following maximization problem (with $\gamma = 0$)

$$0 = \max_{\delta_{t+1}, c'_t, c'_{t+1}, \gamma, K'_{t+1}} \gamma$$

sub. to (61)-(64).

Let $\eta(\theta^t, \bar{z}^t)$,$\eta(\theta^{t+1}, \bar{z}^t, z_{t+1})$, $\mu(\bar{z}^t)$ and $\mu(\bar{z}^t, z_{t+1})$ be multipliers on (61), (62), (63) and (64).

Write the FOCs

$$
\begin{aligned}
u'(c_t^*(\theta^t, \bar{z}^t))\eta(\theta^t, \bar{z}^t) &= \mu(\bar{z}^t)\pi_\theta(\theta^t|\bar{z}^t) && (65)\\
u'(c_{t+1}^*(\theta^{t+1}, \bar{z}^t, z_{t+1}))\eta(\theta^{t+1}, \bar{z}^t, z_{t+1}) &= \mu(\bar{z}^t, z_{t+1})\pi_\theta(\theta^{t+1}|\bar{z}^t, z_{t+1}) && (66)\\
\beta\pi_z(\bar{z}^t, z_{t+1})\eta(\theta^t, \bar{z}^t) &= \sum_{\theta^{t+1}, z_{t+1}|\theta^t, \bar{z}^t} \eta(\theta^{t+1}, \bar{z}^t, z_{t+1}) && (67)\\
\mu(\bar{z}^t) &= \sum_{z_{t+1}} \mu(\bar{z}^t, z_{t+1})(1 - \delta + F_K(K_{t+1}^*(\bar{z}^t), Y_t^*(\bar{z}^t, z_{t+1}), \bar{z}^t, z_{t+1})) && (68)
\end{aligned}
$$

substitute $\eta(\theta^t)$ and $\eta(\theta^{t+1}, \bar{z}^t, z_{t+1})$ from (65) and (66) into (67)

$$
\beta\pi_z(\bar{z}^t, z_{t+1})\frac{\mu(\bar{z}^t)\pi_\theta(\theta^t|\bar{z}^t)}{u'(c_t^*(\theta^t, \bar{z}^t))} = \sum_{\theta^{t+1}, z_{t+1}|\theta^t, \bar{z}^t} \frac{\mu(\bar{z}^t, z_{t+1})\pi_\theta(\theta^{t+1}|\bar{z}^t, z_{t+1})}{u'(c_{t+1}^*(\theta^{t+1}, \bar{z}^t, z_{t+1}))}
$$

Take $\mu(\bar{z}^t, z_{t+1})$ out of summation and rearrange terms

$$
\frac{\beta\pi_z(\bar{z}^t, z_{t+1})\pi_\theta(\theta^t|\bar{z}^t)}{u'(c_t^*(\theta^t, \bar{z}^t))}\left[\sum_{\theta^{t+1}, z_{t+1}|\theta^t, \bar{z}^t} \frac{\pi_\theta(\theta^{t+1}|\bar{z}^t, z_{t+1})}{u'(c_{t+1}^*(\theta^{t+1}, \bar{z}^t, z_{t+1}))}\right]^{-1} = \frac{\mu(\bar{z}^t, z_{t+1})}{\mu(\bar{z}^t)}
$$

Note that by the Independence assumption we had

$$
\frac{\pi_\theta(\theta^{t+1}|\bar{z}^t, z_{t+1})}{\pi_\theta(\theta^t|\bar{z}^t)} = \frac{\pi_\theta(\theta^{t+1}|\bar{z}^t, z_{t+1})}{\pi_\theta(\theta^t|\bar{z}^t, z_{t+1})} = \pi_\theta(\theta^{t+1}|\bar{z}^t, z_{t+1}, \theta^t)
$$

Let

$$
\lambda(\bar{z}^t, z_{t+1}) \equiv \frac{\mu(\bar{z}^t, z_{t+1})}{\mu(\bar{z}^t)\pi_z(\bar{z}^t, z_{t+1})} = \frac{\beta}{u'(c_t^*(\theta^t, \bar{z}^t))}\left[\sum_{\theta^{t+1}|\bar{z}^t, z_{t+1}, \theta^t} \frac{\pi_\theta(\theta^{t+1}|\bar{z}^t, z_{t+1})}{u'(c_{t+1}^*(\theta^{t+1}, \bar{z}^t, z_{t+1}))}\right]^{-1}
$$

then

$$
1 = \sum_{z_{t+1}} \pi_z(\bar{z}^t, z_{t+1})\lambda(\bar{z}^t, z_{t+1})(1 - \delta + F_K(K_{t+1}^*(\bar{z}^t), Y_t^*(\bar{z}^t, z_{t+1}), \bar{z}^t, z_{t+1}))
$$

or

$$
\lambda(\bar{z}^t, z_{t+1}) = \frac{\beta}{u'(c_t^*(\theta^t, \bar{z}^t))}\left\{\mathbb{E}\left[\frac{1}{u'(c_{t+1}^*(\theta^{t+1}, \bar{z}^t, z_{t+1}))}\middle| \theta^{t+1}, \bar{z}^t, z_{t+1}\right]\right\}^{-1}
$$

$$
1 = \mathbb{E}\left[\lambda(\bar{z}^t, z_{t+1})(1 - \delta + F_K(K_{t+1}^*(\bar{z}^t), Y_t^*(\bar{z}^t, z_{t+1}), \bar{z}^t, z_{t+1}))|\bar{z}^t\right]
$$

Note that (using Jensen's inequality)

$$\begin{aligned}\lambda(\bar{z}^t, z_{t+1}) &= \frac{\beta}{u'(c_t^*(\theta^t, \bar{z}^t))}\left\{\mathbb{E}\left[\frac{1}{u'(c_{t+1}^*(\theta^{t+1}, \bar{z}^t, z_{t+1}))}\right| \theta^{t+1}, \bar{z}^t, z_{t+1}\right]\right\}^{-1} \\ &< \frac{\beta\mathbb{E}\left[u'(c_{t+1}^*(\theta^{t+1}, \bar{z}^t, z_{t+1}))\right| \theta^{t+1}, \bar{z}^t, z_{t+1}\right]}{u'(c_t^*(\theta^t, \bar{z}^t))}\end{aligned}$$

and therefore

$$u'(c_t^*(\theta^t, \bar{z}^t)) < \beta\mathbb{E}\left[u'(c_{t+1}^*(\theta^{t+1}, \bar{z}^t, z_{t+1}))(1-\delta+F_K(K_{t+1}^*(\bar{z}^t), Y_t^*(\bar{z}^t, z_{t+1}), \bar{z}^t, z_{t+1}))\right| \theta^{t+1}, \bar{z}^t, z_{t+1}\right]$$

Example 1: suppose Θ is singleton. Then

$$\lambda(\bar{z}^t, z_{t+1}) = \frac{\beta u'(c_{t+1}^*(\bar{z}^t, z_{t+1}))}{u'(c_t^*(\bar{z}^t))}$$

and therefore

$$1 = \mathbb{E}\left[\frac{\beta u'(c_{t+1}^*(\bar{z}^t, z_{t+1}))}{u'(c_t^*(\bar{z}^t))}(1-\delta+F_K(K_{t+1}^*(\bar{z}^t), Y_t^*(\bar{z}^t, z_{t+1}), \bar{z}^t, z_{t+1}))|\bar{z}^t\right]$$

Example 2: suppose Z is singleton. Then

$$\lambda_{t+1} = \frac{\beta}{u'(c_t^*(\theta^t))}\left\{\mathbb{E}\left[\frac{1}{u'(c_{t+1}^*(\theta^{t+1}))}\right|\theta^{t+1}\right]\right\}^{-1}$$

$$1 = \lambda_{t+1}(1-\delta+F_K(K_{t+1}^*, Y_t^*))$$

and therefore

$$\frac{\beta(1-\delta+F_K(K_{t+1}^*, Y_t^*))}{u'(c_t^*(\theta^t))} = \mathbb{E}\left[\frac{1}{u'(c_{t+1}^*(\theta^{t+1}))}\right|\theta^{t+1}\right]$$

and therefore (using Jensen's inequality)

$$u'(c_t^*(\theta^t)) < \beta(1-\delta+F_K(K_{t+1}^*, Y_t^*))\mathbb{E}\left[u'(c_{t+1}^*(\theta^{t+1}))\right|\theta^{t+1}\right]$$

3.2.4 Inter-temporal optimality with balanced growth preferences

The additive separable preferences that we have assumed so far are not consistent with balanced growth fact (except if $u(c) = \log(c)$). However, the perturbation that we used in deriving the inverse Euler equation depends crucially on additive separability. If the preferences are of the form

$$u(c,l) = \frac{c^{1-\sigma}}{1-\sigma}v(l)$$

then the proof presented will not work. See Farhi and Werning (2008) for derivation of the inter-temporal optimality condition for larger class of preferences (including those that are consistent with balanced growth).

3.3 Implementing Efficient Allocations

3.3.1 Wedges and Taxes

So far we have shown that efficient allocations must satisfy a condition like the following (if there is no aggregate shock)

$$\frac{\beta(1-\delta+F_K(K^*_{t+1},Y^*_t))}{u'(c^*_t(\theta^t))} = \mathbb{E}\left[\left.\frac{1}{u'(c^*_{t+1}(\theta^{t+1}))}\right|\theta^{t+1}\right].$$

We also showed that this implies the following

$$u'(c^*_t(\theta^t)) < \beta\mathbb{E}\left[u'(c^*_{t+1}(\theta^{t+1}))(1-\delta+F_K(K^*_{t+1},Y^*_t))\middle|\,\theta^{t+1}\right].$$

In other words there is an inter-temporal wedge

$$1-\tau_t = \frac{u'(c^*_t(\theta^t))}{\beta\mathbb{E}\left[u'(c^*_{t+1}(\theta^{t+1}))(1-\delta+F_K(K^*_{t+1},Y^*_t))\middle|\,\theta^{t+1}\right]} < 1.$$

But does this mean that the efficient allocations can be implemented by a positive tax on capital?

Two period example

Consider the following example:

- $T = 2$.
- $\Theta = \{0,1\}$, $\pi(\theta^1 = 1) = 1$, $\pi(\theta^2 = (1,1)) = 0.5$ and $\pi(\theta^2 = (1,0)) = 0.5$. (Note that this implies $y_{2h} = y_2((1,1)) = l_{2h}(1,1)$ and $y_{2l} = y_2((1,0)) = 0$)
- $u(c,l) = \log(c) - \frac{l^2}{2}$, $\beta = 1$.

- $F(K, Y) = RK + wY$, $\delta = 1$.
- There is endowment of K_1 in period 1.

The planner's problem is

$$\max_{c_1, c_{2h}, c_{2l}, y_1, y_{2h}, K_2} \log(c_1) - \frac{y_1^2}{2} + 0.5\left[\log(c_{2h}) - \frac{y_{2h}^2}{2}\right] + 0.5\left[\log(c_{2l})\right]$$

sub. to

$$\begin{aligned} c_1 + K_2 &= RK_1 + wy_1 \\ 0.5c_{2h} + 0.5c_{2l} &= RK_2 + 0.5wy_{2h} \end{aligned}$$

$$\log(c_{2h}) - \frac{y_{2h}^2}{2} \geq \log(c_{2l})$$

$$c_1, c_{2h}, c_{2l}, y_1, y_{2h}, K_2 \geq 0$$

Let $(c_1^*, c_{2h}^*, c_{2l}^*, y_1^*, y_{2h}^*, K_2^*)$ be the solution. Then it should satisfy the following FOC

$$\begin{aligned} c_1^* + K_2^* &= RK_1 + wy_1^* \\ 0.5c_{2h}^* + 0.5c_{2l}^* &= RK_2^* + 0.5wy_{2h}^* \end{aligned}$$

$$\log(c_{2h}^*) - \frac{y_{2h}^{*2}}{2} = \log(c_{2l}^*)$$

$$Rc_1^* = 0.5c_{2h}^* + 0.5c_{2l}^*$$

$$\begin{aligned} \frac{w}{c_{2h}^*} &= y_{2h}^* \\ \frac{w}{c_1^*} &= y_1^* \end{aligned}$$

We want to implement these allocations in a competitive equilibrium. We assume the following market structure

- There is a single firm that rents capital and labor.
- Individuals have the same endowment of $k_1 = K_1$ and choose how much to work and how much to save.

- Government taxes savings at rate τ_k.
- Government makes transfers T_{2h} and T_{2l} to individual who produce y_{2h} and zero output in the second period. Transfers can be negative.

Competitive equilibrium is allocations $(c_1, y_1, c_{2h}, y_{2h}, c_{2l}, k_2)$ and policy (τ_k, T_{2h}, T_{2l}) such that:

1. Given the policy, allocations solve consumer problem

$$\max_{c_1, c_{2h}, c_{2l}, y_1, y_{2h}, k_2} \log(c_1) - \frac{y_1^2}{2} + 0.5\left[\log(c_{2h}) - \frac{y_{2h}^2}{2}\right] + 0.5\left[\log(c_{2l})\right]$$

sub. to

$$c_1 + k_2 = Rk_1 + wy_1$$

$$\begin{aligned} c_{2h} &= R(1-\tau_k)k_2 + wy_{2h} + T_{2h}, \text{ if } y_{2h} > 0 \\ c_{2h} &= R(1-\tau_k)k_2 + T_{2l}, \text{ if } y_{2h} > 0 \\ c_{2l} &= R(1-\tau_k)k_2 + T_{2l} \end{aligned}$$

$$c_1, c_{2h}, c_{2l}, y_1, y_{2h}, k_2 \geq 0$$

2. Markets clear

$$\begin{aligned} c_1 + K_2 &= RK_1 + wy_1 \\ 0.5c_{2h} + 0.5c_{2l} &= RK_2 + 0.5wy_{2h} \\ k_1 &= K_1 \\ k_2 &= K_2 \end{aligned}$$

Note that the government budget constraint is satisfied in equilibrium

$$R\tau_k K_2 = 0.5T_{2h} + 0.5T_{2l}$$

We want the tax system be such that $y_{2h} > 0$. Then it is necessary that

$$\log(c_{2h}) - \frac{y_{2h}^2}{2} \geq \log(c_{2l})$$

Let's assume we have such a tax system. Then the consumer FOCs are

$$\frac{1}{c_1} = R(1-\tau_k)\left[\frac{.5}{c_{2h}} + \frac{.5}{c_{2l}}\right]$$

$$\begin{aligned}\frac{w}{c_1} &= y_1 \\ \frac{w}{c_{2h}} &= y_{2h}\end{aligned}$$

Also note that

$$\begin{aligned}c_{2h} &= R(1-\tau_k)k_2 + wy_{2h} + T_{2h} \\ c_{2l} &= R(1-\tau_k)k_2 + T_{2l}\end{aligned}$$

Question: How do we pick taxes to implement efficient allocations $(c_1^*, c_{2h}^*, c_{2l}^*, y_1^*, y_{2h}^*, K_2^*)$ in the equilibrium?

Why don't we try the simplest way to do this. And choose the following taxes

$$1-\tau_k = \frac{[0.5c_{2h}^* + 0.5c_{2l}^*]^{-1}}{\left[\frac{0.5}{c_{2h}^*} + \frac{0.5}{c_{2l}^*}\right]}$$

and

$$\begin{aligned}T_{2h} &= c_{2h}^* - R(1-\tau_k)K_2^* - wy_{2h}^* \\ T_{2l} &= c_{2l}^* - R(1-\tau_k)K_2^*\end{aligned}$$

This way the equilibrium allocations satisfy all of the optimality conditions listed above. They are also feasible and hence should implement the efficient allocations.

What is missing?

The taxes above guarantee that the individual saves the correct amount "if" he plans to tell the truth about his ability in the second period. Also, they guarantee that the individual tells the truth "if" he saves the correct amount.

Question: What if the individual saves more than K_2^* and plan to lie about their ability if they happen to be the able type?

We first show that this double deviation is feasible and improves individuals welfare. Consider three different plan by the agent

PLAN 1:

- Save $K_2^*(\equiv K_1 + wy_1^* - c_1^*)$
- Produce y_{2h}^* if $\theta_2 = 1$ and zero if $\theta_2 = 0$

PLAN 2:

- Save $K_2^*(\equiv K_1 + wy_1^*)$
- Produce zero in the second period

Note first that, because incentive compatibility in planner's problem binds, individual is indifferent between PLAN 1 and PLAN 2. Also,

$$\frac{1}{c_1^*} < R(1-\tau_k)\left[\frac{.5}{c_{2h}^*} + \frac{.5}{c_{2l}^*}\right] < R(1-\tau_k)\frac{1}{c_{2l}^*}$$

PLAN 3:

- Save $K_2^* + \epsilon$.
- Consume $c_{2l}^* + R(1-\tau_k)\epsilon$. and produce zero.

Define function $g(\epsilon)$

$$g(\epsilon) = \log(c_1^* - \epsilon) + \log(c_{2l}^* + R(1-\tau_k)\epsilon) - [\log(c_1^*) + \log(c_{2l}^*)]$$

and note that

$$g'(0) = -\frac{1}{c_1^*} + \frac{R(1-\tau_k)}{c_{2l}^*} > 0$$

and therefore

$$\text{PLAN 3} \succ \text{PLAN 2} \simeq \text{PLAN 1}$$

Therefore under the proposed tax system individuals prefer PLAN 3 to PLAN 1. Hence this implementation fails. In other words the efficient allocation cannot be implemented by a separable tax system in which capital taxes do not depend on individuals income ex-post.

3.3.2 Implemeting Efficient Allocations-General Case

We implement efficient allocations using arbitrary nonlinear taxes on labor income and linear taxes on wealth. The description of envrionment is the following

- Single representative firm that owns the production technology and takes capital rents r_t and wages w_t as given.
- Agenst are endowed with $\bar{K}_1$ in period 1.
- Agents trade supply labor and capital to the firm in a sequence of competitive markets.
- Agents face a *labor tax schedule* $\psi : \mathbb{R}_+^T \times Z^T \to \mathbb{R}$. ψ_t is $(y^t, z^t)-$measurable. An individual with sequence of effectice labor supply $\{y_t\}_{t=1}^T$, pays labor taxes $\psi_t((y_t)_{t=1}^T, z^T)$ is period t.
- Agents face linear tax on wealth $\tau : \mathbb{R}_+^T \times Z^T \to \mathbb{R}$. τ_t is $(y^t, z^t)-$measurable. Note that τ_t may depend on history of effective labor supply. And individula who holds wealth W_t and has sequence of effectice labor supply $\{y_t\}_{t=1}^T$, pays wealth taxes $\tau_t((y_t)_{t=1}^T, z^T)W_t$.

Given taz system (ψ, τ) individual solves

$$\max_{c,k,y} \sum_{z^T \in Z^T} \pi_z(z^T) \sum_{t=1}^T \beta^{t-1} \sum_{\theta^T \in \Theta^T} \pi_\theta(\theta^T|z^T) \left[u(c_t(\theta^T, z^T)) - v\left(\frac{y_t(\theta^T, z^T)}{\theta_t}\right)\right]$$

subject to

$$\begin{aligned} c_t(\theta^T, z^T) + k_{t+1}(\theta^T, z^T) &\leq \left(1 - \tau_t(y(\theta^T, z^T), z^T)\right)(1 - \delta + r_t(z^T))k_t(\theta^T, z^T) \\ &\quad + w(z^T)y_t(\theta^T, z^T) - \psi_t(y(\theta^T, z^T), z^T) \qquad \forall(\theta, z^T) \\ k_1 &\leq \bar{K}_1 \end{aligned}$$

$$(c_t, k_{t+1}, y_t) \text{ is } (\theta^t, z^t) - \text{measurable and nonnegative}$$

Given tax system (ψ, τ), an eqilibrium is allocations rules (c, k, y) and prices (r, w) such that given prices and tx system, the allocation rules solves the individual problem, r and w are such that $r_t(z^T) = F_{kt}(K_t(z^T), Y_t(z^T), z^T)$ and $w_t(z^T) = F_{yt}(K_t(z^T), Y_t(z^T), z^T)$, and alloctions are feasible for all t and all z^T:

$$
\begin{aligned}
C_t(z^T) + K_{t+1}(z^T) &= (1-\delta)K_t(z^T) + F(K_t(z^T), Y_t(z^T), z^T) \\
C_t(z^T) &= \sum_{\theta^T \in \Theta^T} \pi_\theta(\theta^T|z^T) c_t(\theta^T, z^T) \\
Y_t(z^T) &= \sum_{\theta^T \in \Theta^T} \pi_\theta(\theta^T|z^T) y_t(\theta^T, z^T) \\
K_{t+1}(z^T) &= \sum_{\theta^T \in \Theta^T} \pi_\theta(\theta^T|z^T) k_{t+1}(\theta^T, z^T) \\
K_1 &= \bar{K}_1
\end{aligned}
$$

Note that the tax system described only in terms of sequence of lebor supply (and not skill shocks). However, households choose consumption as function of past and present skills (the θ's). We need to make sure consumption depends on past and present skills only trough past and present labor supply. In other words, we need to make sure observing past and present labor supply is enough to determine individual's consumtion. For that we need to impose an assumption:

Suppose (c^*, y^*, K^*) is a socially optimal allocation. Define

$$
\begin{aligned}
DOM_t = \Big\{ (y^T, z^T) \in \mathbb{R}_+^T \times Z^T \mid \exists\, \theta^T \in \Theta^T,\ (y'_s)_{s=t+1}^T \in \mathbb{R}_+^{T-t} \\
\text{and } (z'_s)_{s=t+1}^T \in Z^{T-t} \text{such that} \\
(y^t, (y'_s)_{s=t+1}^T, z^T) = (y^*(\theta^T, z^T), z^T) \Big\}
\end{aligned}
$$

where $z^T = (z^t, (z'_s)_{s=t+1}^T)$ for some $(z'_s)_{s=t+1}^T \in Z^{T-t}$.

(y^T, z^T) is in DOM_t if in socially optimal allocation, there exist a type in Θ^T that receives the effective labor history y^t when public history is z^t.

Claim: $DOM_t \subseteq DOM_{t-1}$.

Now we make the following assumption

Assumption 2 *There exist a sequence of functions* $\widehat{c^*} = \left(\widehat{c_t^*}\right)_{t=1}^T$, *where* $\widehat{c_t^*} : DOM_t \to \mathbb{R}_+$, $\widehat{c_t^*}$ *is* $(y^t, z^t)-$*measurable, and*

$$
\widehat{c_t^*}(y^*(\theta^T, z^T), z^T) = c_t^*(\theta^T, z^T) \quad \forall\, (\theta^T, z^T)
$$

This assumption is satisfied in static case and in i.i.d case. See Kocherlakota (2005) for further discussion and an example for the case in which this assumption is violated.[2]

This assumption simply says it is possible to implement consumption allocation as function of labor supply rather than skill. In this implementation a person who acts like (θ^T, z^T) (have same labor supply as type (θ^T, z^T)), will also sonsume the same allocation as type (θ^T, z^T). Whether this is in fact efficient or not is something we need to show.

Details of the tax system

Given optimal allocation (c^*, y^*, K^*), there exists $\lambda^*_{t+1} : Z^T \to \mathbb{R}_+$ ($z^{t+1}-$measurable) such that

$$\lambda^*_{t+1} = \frac{\beta}{u'(c^*_t)} \left\{ \mathbb{E}\left[\frac{1}{u'(c^*_{t+1}(\theta^{t+1}, \bar{z}^t, z_{t+1}))} \middle| \theta^t, z^{t+1} \right] \right\}^{-1}$$

Define $\tau^*_{t+1} : \mathbb{R}^T_+ \times Z^T \to \mathbb{R}$ by

$$\tau^*_{t+1}(y^T, z^T) = \begin{cases} 1 - \frac{\lambda^*_{t+1}(z^T) u'(\widehat{c^*_t}(y^*(\theta^T, z^T))}{\beta u'(\widehat{c^*_{t+1}}(y^*(\theta^T, z^T))} & \forall (y^T, z^T) \in DOM_{t+1} \\ 1 & \forall (y^T, z^T) \notin DOM_{t+1} \end{cases}$$

Note that τ^*_{t+1} is $(y^{t+1}, z^{t+1})-$measurable.

Let

$$\begin{aligned} MPK^*_t(z^t) &\equiv F_K(K^*_t(z^{t-1}), Y^*_t(z^t), z^t) \\ MPL^*_t(z^t) &\equiv F_Y(K^*_t(z^{t-1}), Y^*_t(z^t), z^t) \end{aligned}$$

Now let $(\psi^{**}, \widehat{k^*}) : DOM_T \to \mathbb{R}^T \times \mathbb{R}^T_+$ be defined so that

$$\begin{aligned} \widehat{c^*_t}(y^T, z^T) + \widehat{k^*_{t+1}}(y^T, z^T) &= (1 - \tau^*_{t+1}(y^T, z^T))(1 - \delta + MPK^*_t(z^T))\widehat{k^*_t}(y^T, z^T) \\ &\quad + MPL^*_t(z^T) y_t - \psi^{**}_t(y^T, z^T) \\ \sum_{\theta^T} \widehat{k^*_{t+1}}(y^*(\theta^T, z^T), z^T) &= K^*_t(z^T) \\ \widehat{k^*_1} &= K^*_1 \end{aligned}$$

Now we define labor taxes as

[2]Page 1601.

$$\psi^*_{t+1}(y^T, z^T) = \begin{cases} \psi^{**}_t(y^T, z^T) & \forall (y^T, z^T) \in DOM_{t+1} \\ 2y_t w_t(z^T) & \forall (y^T, z^T) \notin DOM_{t+1} \end{cases}$$

Proposition 11 *Given prices* (r^*, w^*) *and tax system* (ψ^*, τ^*)*, and given that the typical agent chooses a budget feasible* y'*, his optimal choices of* (c, k) *are* $c'_t(\theta^T, z^T) = \widehat{c^*_t}(y'(\theta^T, y^T), z^T)$ *and* $k'_t(\theta^T, z^T) = \widehat{k^*_t}(y'(\theta^T, y^T), z^T)$

References

Atkeson, A. and J. Lucas, Robert E (1992): "On Efficient Distribution with Private Information," *Review of Economic Studies*, 59, 427–53. 3.2.1, 3.2.2

Chamley, C. (1986): "Optimal Taxation of Capital Income in General Equilibrium with Infinite Lives," *Econometrica*, 54, 607–22. 2

Chari, V., L. Christiano, and P. J. Kehoe (1994): "Optimal fiscal policy in a business cycle model," *Journal of Political Economy*, 102, 617–52. 2

Chari, V. and P. J. Kehoe (1998): "Optimal fiscal and monetary policy," *Minneapolis Fed Staff Report, SR 251*. ∗, 2

Corlett, W. and D. Hauge (1953-54): "Complementarity and the excess burden of taxation," *Review of Economic Studies*, 21, 21–30. 1.2.3

Diamond, P. and J. Mirrlees (1971): "Optimal taxation and public production I: production efficiency," *American Economic Review*, 61, 8–27. 1.4

Diamond, P. A. (1998): "Optimal income taxation: an example with a U-shaped pattern of optimal marginal tax rates," *American Economic Review*, 83–95. 3.1

Ekeland, I. and J. A. Scheinkman (1986): "Transversality Conditions for some infinite horizon discrete time optimization problems," *Mathematics of operation research*, 11. 2

Erosa, A. and M. Gervais (2002): "Optimal Taxation in Life-Cycle Economies," *Journal of Economic Theory*, 105, 338–369. 2.3

Farhi, E. and I. Werning (2005): "Inequality, Social Discounting and Estate Taxation," . 3.2.2, 3.2.2

——— (2007): "Inequality and Social Discounting," *Journal of Political Economy*, 115, 365–402. 3.2.2, 3.2.2

——— (2008): "Optimal Savings Distortions with Recursive Preferences," *The Journal of Monetary Economics*, 55, 21–42. 3.2.1, 3.2.4

——— (2010): "Progressive estate taxation," *The Quarterly Journal of Economics*, 125, 635–673. 3.2.2, 3.2.2

Judd, K. L. (1985): "Redistributive Taxation in a Simple Perfect Foresight Model," *Journal of Public Economics*, 28, 59–83. 2

Kocherlakota, N. R. (2005): "Zero Expected Wealth Taxes: A Mirrlees Approach to Dynamic Optimal Taxation," *Econometrica*, 73, 1587–1621. 3.3.2

McGrattan, E. R. and E. C. Prescott (2005): "Taxes, Regulations, and the Value of U.S. and U.K. Corporations," *Review of Economic Studies*, 72, 767–796. 2.2.1

PHELAN, C. (2006): "Opportunity and Social Mobility," *Review of Economic Studies*, 73, 487–504. 3.2.2

Saez, E. (2001): "Using elasticities to derive optimal income tax rates," *The review of economic studies*, 68, 205–229. 3.1, 3.1

Sandmo, A. (1987): "A Reinterpretation of Elasticity Formulae in Optimum Tax Theory," *Economica*, 54, 89–96. 1.2.3

Weitzman, M. L. (1973): "Duality Theory of Convex Programming for Infinite Horizon Economic Models," *Management Science*, 19. 2

Werning, I. (2007): "Tax Smoothing with Redistribution," *Quarterly Journal of Economics*, 122, 925–67. 2.2.3, 2.2.3, 3