Microeconomic Theory I

2. Utility

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The Utility Function representation of Preferences

The Utility Function

- a way to represent preferences that simplifies the derivation of the demand function.
- Intuitively: assign to each $x \in X$ a number. Call this number $U(x)$, and say that, whenever $U(x) \geq U(x')$, then $x \succeq x'$ (and vice versa).

Definition

$U: X \rightarrow \mathbb{R}$ is a **utility function representing** $\succeq$ if $\forall x, y \in X$, $U(x) \geq U(y) \iff x \succeq y$.

- If we have one U, we have infinitely many: any strictly increasing transformation of a utility function is a utility function.
- In this part of the course our main question is: under what conditions on $\succeq$ a utility function representation exists?
  - We limit the analysis to the case $X = \mathbb{R}_{+}^{L}$

Types of Preferences

Notation

$X \gg Y$ means: each component of X is $>$ than each component of Y
$X > Y$ means: each component of X is $\geq$ than each component of Y, with at least one strict inequality.
$X \geq Y$ means: each component of X is $\geq$ than each component of Y

Definitions

$\succeq$ are **strongly monotonic** if $\forall x, y \in \mathbb{R}^L$ $x > y \Rightarrow x \succ y$.
$\succeq$ are **monotonic** if $\forall x, y \in \mathbb{R}^L$ $x \gg y \Rightarrow x \succ y$.

Types of preferences

Definition

$\succeq$ are **locally non-satiated** (LNS) if $\forall x \in X = \mathbb{R}^L_+$ every neighborhood of x contains a bundle y such that $y \succ x$.

- In other words, for any $x \in X$ there is always another point, arbitrary close to x, which is strictly preferred to x
- There is no **satiation point** or **bliss point**

- $\succeq$ LNS $\Rightarrow$ Walras' Law
- $\succeq$ LNS $\Rightarrow$ $\succeq$ no thick indifference curves
- $\succeq$ monotonic $\Rightarrow$ $\succeq$ LNS

Types of preferences

Definitions

$\succeq$ are **(strictly) convex** if $G(x)$ is a (strictly) convex set.

$\succeq$ are **continuous** if $\forall x \in X$ $G(x)$ and $B(x)$ are closed sets.

Existence of a Utility Function Representation

- What preferences can be represented by a utility function?

Proposition

Assume $X = \mathbb{R}^L_+$. If $\succeq$ are rational, strongly monotonic and continuous, there exist a utility function representing those preferences.

Proof.

- Constructive proof: we are going to construct U

1. Take any $x \in X$ and a vector $e = [1, 1, \ldots, 1] \in X$

Representation theorem

2. Show $\exists t_x \in \mathbb{R}$ such that $t_x e \sim x$

- Define

$$w(x) = \{t \in \mathbb{R} \mid te \in B(x)\}$$
$$k(x) = \{t \in \mathbb{R} \mid te \in G(x)\}$$

  both non empty (why?).

- $w(x)$, $k(x)$ are closed because $G(x)$, $B(x)$ are closed
  - Take a sequence $\alpha^n \to \bar{\alpha}$, such that α^n contained in $w(x)$
  - $\alpha^n e$ contained in $B(x)$. Why?
  - $\bar{\alpha} e \in B(x)$ because $B(x)$ is closed
  - $\bar{\alpha} \in w(x)$
  - $w(x)$ is closed

Representation theorem

- $w(x) \cup k(x) = \mathbb{R}$. Why?
  - if $w(x) \cup k(x) \neq \mathbb{R}$ $\quad \exists b$ such that $b \cdot e \not\succsim x$ and $b \cdot e \not\precsim x$ $\quad$ contradiction
- Since $\mathbb{R}$ is connected, $w(x) \cap k(x) \neq \emptyset$
  - Connectedness: "there is no hole"
  - $A \cup B$ connected and A, B closed $\Rightarrow A \cap B \neq \emptyset$
- $\exists t_x \in w(x) \cap k(x) \implies x \sim t_x e$

Representation theorem

3. Show that t_x is unique
   - By contradiction
   - Assume $\exists t'_x,\ t''_x$ such that $t'_x \neq t''_x$
   - $x \sim e \cdot t'_x$
   - $x \sim e \cdot t''_x \quad \Longrightarrow \quad e \cdot t'_x \sim e \cdot t''_x$
   - But we also know that $t'_x > t''_x$ or $t'_x < t''_x$, so (by monotonicity) either $e \cdot t'_x \succ e \cdot t''_x$ or $e \cdot t'_x \prec e \cdot t''_x$
   - contradiction

Representation theorem

4. Define $U(x) \equiv t_x \ \forall x$; show that $U(x)$ is a utility function representing $\succeq$.
   - Take $x, y \in X$ such that $x \succ y$
     - $U(x)e \sim x \succ y \sim U(y)e \implies U(x) > U(y)$ (by monotonicity)

Representation theorem

- Take $x, y \in X$ such that $x \sim y$
  - $U(x)e \sim x \sim y \sim U(y)e \implies U(x) = U(y)$ (by monotonicity)
- Take $x, y \in X$ such that $U(x) > U(y)$
  - $x \sim U(x)e \succ U(y)e \sim y \implies x \succ y$
- Take $x, y \in X$ such that $U(x) = U(y)$
  - $x \sim U(x)e \sim U(y)e \sim y \implies x \sim y$

5. $U(x)$ is also continuous (we do not prove it)

□

************* Math aside: Optimization ****************

Unconstrained Optimization

One-dimensional unconstrained maximization problem:

$$\max_{x \in [a,b]} f(x)$$

with $f(x) : \mathbb{R} \to \mathbb{R}$.

How would you solve it? It depends!

1. continuous and differentiable functions
    - compute the value of the objective function at:
        - The end points a and b
        - In each point where the FOC

          $$f'(x) = 0$$

          and the SOC

          $$f''(x) < 0$$

          is satisfied. In these point the curve is flat and the function is (locally) under the horizontal tangent.
2. discontinuous function, non differentiable functions: there is no simple receipt!

Do not solve optimization problems mechanically!

- Start by drawing a graph.
- Find the optimum *on the graph.*
- Use math to formalize your intuition.

Constrained Optimization

$$\max f(x)$$
$$\text{s.t. } g(x) = y$$

we use the method of Lagrange multipliers:

$$\mathcal{L}(x,\lambda) = f(x) - \lambda\,(g(x) - y)$$

$$\text{FOC} \quad \begin{cases} \frac{\partial f(x)}{\partial x_i} = \lambda \frac{\partial g(x)}{\partial x_i} & \forall i \\ g(x) = y \end{cases}$$

NOTE: we take the **first order condition also with respect to the multiplier!**

- Intuition behind the Lagrangian:
- Thinking of it as a game

$$\underbrace{\max_{x}}_{\text{Player 1}} \left[\underbrace{\min_{\lambda \in \mathbb{R}}}_{\text{Player 2}} \left\{ f(x) - \lambda \left(g(x) - y \right) \right\} \right]$$

***************** End of Math Aside ***************

Utility Maximization and Demand Function

Utility maximization

Derive the demand function as the solution to a constrained maximization problem:

$$\begin{cases} x(p,w) = \arg\max U(x) \\ \text{s.t. } px \leq w \\ x \in X \end{cases}$$

If $\succeq$ are LNS:

$$\begin{cases} x(p,w) = \arg\max U(x) \\ \text{s.t. } px = w \\ x \in X \end{cases}$$

Utility maximization

Does the Utility Maximization problem have a solution?

- The objective function is continuous (at least, under the assumption of the representation theorem)
- The budget set is compact if $p \gg 0$
- **Maximum Theorem:** if the objective function is continuous and the constraint is compact, there is always at least one solution to the maximization problem.

Utility maximization

In case the utility maximization problem is *smooth, continuous, differentiable, well behaved*, ... you can:

- write the Lagrangean
- take the *first order condition*

$$\max U(x) \qquad x \in \mathbb{R}^L$$
$$\text{s.t. } px = w$$

$$\mathcal{L} = U(x) - \lambda \left(px - w\right)$$

Utility Maximization

$$\text{FOC} \begin{cases} \frac{\partial U(x)}{\partial x_i} = \lambda p_i \quad \forall i \in \{1, \dots, L\} \\ px = w \end{cases}$$

$$\underbrace{\frac{\frac{\partial U(x)}{\partial x_i}}{\frac{\partial U(x)}{\partial x_j}}}_{\text{MRS}} = \frac{p_i}{p_j} \quad \forall i, j \in \{1, \dots, L\}$$

Utility Maximization

- $\frac{\frac{\partial U(x)}{\partial x_i}}{\frac{\partial U(x)}{\partial x_j}}$ is known as the marginal rate of substitution (MRS)
- Indifference curve: $x_2(x_1): \; U(x_1, x_2(x_1)) = \bar{U}$
- The slope of the function $x_2(x_1)$ is

$$\frac{\partial U(x)}{\partial x_1} + \frac{\partial U(x)}{\partial x_2} x_2'(x_1) = 0$$

$$x_2'(x_1) = -\frac{\frac{\partial U(x)}{\partial x_1}}{\frac{\partial U(x)}{\partial x_2}} = -\text{MRS}$$

- Important: do **NOT** use this optimality condition automatically, it cannot be applied to all kinds of problems.
    - There may be corner solutions
    - The SOC may be violated

Examples

- $U(x_1, x_2) = x_1^\alpha x_2^\beta$ (Cobb-Duglas utility function)
- $U(x_1, x_2) = \alpha x_1 + \beta x_2$ (linear utility function)
- $U(x_1, x_2) = \min\{\alpha x_1, \beta x_2\}$ (Leontief utility function)
- $U(x_1, x_2) = e^{10(\alpha x_1 + \beta x_2)}$
- $U(x_1, x_2) = x_1^2 + x_2^2$
- $U(x_1, x_2) = \phi(x_2) + x_1$ with ϕ increasing and concave (quasilinear utility function)

Note:

When preferences are strictly convex, there is always a unique bundle that maximizes utility at given p and w. Therefore the **demand is a function** (and not a correspondence).