Microeconomic Theory I

10. Social Choice

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Social Choice: How to aggregate preferences, when one option for everybody must be chosen.

Social Choice Theory

We have

- X alternatives
- I agents, $\succeq_i$ over X
- G set of all rational preferences over X
- $(\succeq_1, \succeq_2, \ldots, \succeq_I) \in G^I$
- Social welfare functional (SWF) F:
    - $G^I \to G$
    - $\forall x, y \in X$, $x \underbrace{F(\succeq_1, \succeq_2, \ldots, \succeq_I)}_{\succeq^*} y$
    - We say that x is **socially preferred** to y

Example: two alternatives

- $X = \{x, y\}$
- $G = \alpha \in \{-1, 0, 1\}$
  - $\alpha_i = -1 \Leftrightarrow x \succ_i y$
  - $\alpha_i = 0 \Leftrightarrow x \sim_i y$
  - $\alpha_i = 1 \Leftrightarrow x \prec_i y$
- $(\alpha_1, \alpha_2, \ldots, \alpha_I) \in G^I \equiv \{-1, 0, 1\}^I$

Example: possible social welfare functionals

Majority voting

Dictatorship

Borda count Give 1 point to favorite alternative, 2 points to 2nd, etc. Give the same points to the two alternatives if indifferent.

Condorcet's paradox

- $X = \{x, y, z\}$
- We have 3 agents:

$$x \succ_1 y \succ_1 z$$
$$y \succ_2 z \succ_2 x$$
$$z \succ_3 x \succ_3 y$$

- Majority voting:

$$x \succ_{MR} y$$
$$y \succ_{MR} z$$
$$z \succ_{MR} x$$

- We have a loop – transitivity is violated
- Majority rule is not a social welfare functional in this case

Properties of social welfare functionals

Definition

A SWF is **Paretian** if

$$F(1,1,\ldots,1) = 1$$
$$F(0,0,\ldots,0) = 0$$
$$F(-1,-1,\ldots,-1) = -1$$

- More in general: if $\exists x, y \in X$ such that $\forall i \; x \succ_i y$ then

$$\underbrace{x \underbrace{F(.)}_{\succeq^*} y \qquad y \underbrace{\neg F(.)}_{\not\succeq^*} x}_{x \succ^* y}$$

Properties of social welfare functionals

Definition

A SWF is **anonymous** if

$$x\,F\left(\succeq_1, \succeq_2, \ldots, \succeq_I\right)y \quad \Longleftrightarrow \quad x\,F\left(\succeq_1', \succeq_2', \ldots, \succeq_I'\right)y$$

whenever $\left(\succeq_1', \succeq_2', \ldots, \succeq_I'\right)$ is a permutation of $(\succeq_1, \succeq_2, \ldots, \succeq_I)$.

- E.g. dictatorship is not anonymous

Properties of social welfare functionals

Definition

A SWF satisfies **pairwise independence condition** (or **independence of irrelevant alternatives**) if $\forall \left(\succeq_1, \ldots, \succeq_I\right)$ and $\left(\succeq_1', \ldots, \succeq_I'\right)$ such that $x \succeq_i y \Leftrightarrow x \succeq_i' y$ and $y \succeq_i x \Leftrightarrow y \succeq_i' x$ $\forall i$ then

$$x \, F\left(\succeq_1, \ldots, \succeq_I\right) y \quad \Longleftrightarrow \quad x \, F\left(\succeq_1', \ldots, \succeq_I'\right) y$$

and

$$y \, F\left(\succeq_1, \ldots, \succeq_I\right) x \quad \Longleftrightarrow \quad y \, F\left(\succeq_1', \ldots, \succeq_I'\right) x$$

- E.g. majority voting satisfies the independent of irrelevant alternatives (if it is a SWF!)

Arrow's impossibility theorem

Theorem

Assume $\sharp X \geq 3$. Then every SWF which is Paretian and satisfies the independence of irrelevant alternatives must be dictatorial.

- Interpretation: for a *generic domain*, there is no SWF that is Paretian, satisfies the independence of irrelevant alternatives and not dictatorial.
- But: in some *restricted domains* this may be possible

Restricted domains

- **Assumption:** $X \subset \mathbb{R}$ (outcomes can be ordered)

Definition

Preferences are **single-peaked** if $\forall i \ \exists x_i^*$ such that

$$x_i^* \geq z > y \ \Rightarrow \ z \succ_i y$$

and

$$y > z \geq x_i^* \ \Rightarrow \ z \succ_i y$$

- In other words, preference depends on the distance from a peak in both directions

The median voter theorem

Definition

An agent is called the **median voter** m if both sets $\{i \in I : x_i^* \geq x_m^*\}$ and $\{i \in I : x_i^* \leq x_m^*\}$ have at least $\frac{I}{2}$ elements.

- A median voter always exists but may not be unique (for an even number of voters it is never unique)

Theorem

If $X \subset \mathbb{R}$ and preferences are single-peaked, x_m^* cannot be defeated by any other alternative under majority voting.

- x_m^* is called a **Condorcet winner**

The median voter theorem

Proof.

- Take $y \in X$, assume $y < x_m^*$.
- We need to show that

$$\#\{i \in I : x_m^* \succ_i y\} \geq \#\{i \in I : x_m^* \prec_i y\}$$

- Define $S : \{i \in I : x_i^* \geq x_m^*\}$
- By the assumption of single-peaked preferences, $\forall i \in S \quad x_m^* \succ_i y$
- We also know that $\#S \geq \frac{I}{2}$, which implies that

$$\#\{i \in I : x_m^* \succ_i y\} \geq \frac{I}{2} \geq \#\{i \in I : x_m^* \prec_i y\}$$

□

Majority voting on restricted domain

Proposition

If I is odd, $\succeq$ are single-peaked with I distinct peaks, and $\forall x, y \in X$ if $x \neq y$ then either $x \succ_i y$ or $y \succ_i x$ (no indifferences), then majority voting is a SWF.

Proof.

- Completeness is obvious under majority voting
- Transitivity:
  - Take $x, y, z \in X$, assume $x\, F\,(\succeq_1, \ldots, \succeq_I)\, y$ and $y\, F\,(\succeq_1, \ldots, \succeq_I)\, z$. We need to show $x\, F\,(.)\, z$.
  - $X' = \{x, y, z\}$. Note that preferences are single-peaked over X' since they are single-peaked over X.
  - By the median voter theorem, there exists a Condorcet winner and it must be x (since y is defeated by x and z by y, while a Condorcet winner is not defeated by any alternative). Hence, $x\, F\,(.)\, z$.

□