The Unique Readability Theorem for Terms. *Let* $\mathbf{T}$ *be the term algebra for some fixed signature. For any operation symbols* P *and* Q *and any terms* $p_0, \ldots, p_{n-1}$ *and* $q_0, \ldots, q_{m-1}$, *where* n *is the rank of* P *and* m *is the rank of* Q*. if* $P^{\mathbf{T}}(p_0, \ldots, p_{n-1}) = Q^{\mathbf{T}}(q_0, \ldots, q_{m-1})$, *then* $P = Q, n = m$, *and* $p_i = q_i$ *for all* $i < n$.

We offer no proof here, but rather make this the object of the first set of exercises.

Of course, our intention is that just as the operation symbols are to name the basic operations in a structure, so the terms are intended to name certain derived operations in the structure. But even after we make this explicit, our syntax will be lacking everything but certain, perhaps very involved, names. We need more.

Formulas and Sentences

The operation and relation symbols have meanings that differ from structure to structure. The meanings of the logical symbols of our syntactical apparatus, on the other hand, will not vary from structure to structure—apart from, perhaps, a restriction to the universe of each structure. The logical symbols have essentially grammatical roles intended to be the same from structure to structure. We have already seen one sort of logical symbols, namely the variables $x_0, x_1, x_2, \ldots$. There are four remaining logical symbols: $\approx$, which is referred to as the **equality symbol** $\approx$, $\neg$ and $\vee$, which are referred to as the **negation symbol** $\neg$ and the **disjunction symbol** $\vee$, and $\exists$, which is referred to as the **existential quantifier** $\exists$.

Just as operation symbols and variables can be put together to form terms, with the help of terms, relation symbols and the logical symbols, we can build more complicated strings of symbols called formulas.

Atomic formulas are those strings of symbols of the form $\approx st$ where s and t are terms and those of the form $Rt_0, t_1 \ldots t_{r-1}$ where R is a relation symbol and $t_0, \ldots, t_{r-1}$ are terms, where r is the rank of R. The **set of formulas** is the smallest set F of finite strings of symbols such that

- F contains all the atomic formulas,
- if $\varphi, \psi \in F$, then $\neg\varphi \in F$ and $\vee\varphi\psi \in F$, and
- if $\psi \in F$, then $\exists x_i \psi \in F$ for all natural numbers i.

Like the definition of the set of terms, this definition is recursive and it invites proofs by induction. You will observe that we have continued to use prefix notation here. This is our official definition. In practice, we never write things like $\approx st$ or $\vee\varphi\psi$, writing instead $s \approx t$ and $\varphi \vee \psi$.

Just as for terms, there is a Unique Readability Theorem for formulas. We leave not only the proof but even the formulation of this theorem to the eager graduate students.

We need one more notion, that of a variable occurring free in a formula. Let us define a function, FreeVar on the set of formulas, by the following recursion:

- If φ is an atomic formula, then $\operatorname{FreeVar}\varphi$ is the set of variables occurring in φ.
- If φ is $\neg\psi$, then $\operatorname{FreeVar}\varphi = \operatorname{FreeVar}\psi$.
- If φ is $\psi \vee \theta$, then $\operatorname{FreeVar}\varphi = \operatorname{FreeVar}\psi \cup \operatorname{FreeVar}\theta$.