EXPLICIT CONSTRUCTIONS OF CONCENTRATORS

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In the solution of certain problems of switching and coding theory it is required to synthesize structures similar to what in the present article are called concentrators. While the existence of concentrators is easily proved on probabilistic grounds, their explicit construction proves difficult. The theory of group representations is used to solve the problems of the explicit construction of concentrators.

1. Statement of the Problem

1.1. Let it be required to synthesize a structure possessing a certain property. The following approach is one of the most widely used in information theory and related problems: first construct a certain class of structures, and then show that almost all structures in that class are (in some sense) "good" (i.e., have the required property). It frequently happens, however, that while there may be many "good" structures, it is exceedingly difficult to explicitly construct at least one of them. The problem to which we now address ourselves is specifically related to such constructions.

1.2. Here we give the definitions required for the formal statement of the problem. Let A and B be finite sets. We call the elements of A inputs, and the elements of B outputs. Let every input be directly connected to a certain number of outputs. We call any graph so constructed a connector. If the number of elements in the set A is equal to the number of elements in the set B in a connector, the latter is called a uniform connector. Let H be a uniform connector. The capacity of H is the number of elements in A (or, of course, in B), the weight is the number of edges in the graph H, and the density is the ratio of the weight to the capacity. Hereinafter for any uniform connector H we denote by c(H), w(H), and d(H), respectively, the capacity, weight, and density of that connector.

Let H be a uniform connector, and let X be a subset of the set A. We then denote by X_H the subset of B formed by outputs, and only those outputs, that are connected to at least one input in the subset X.

For any subset X of A (or of B) we denote by $\tau(X)$ the number of elements in X, and we call the ratio $c(X)/c(A)$ the density of the subset X. Now let c and α be positive numbers, $c > 1$, $\alpha < 1$. Then a uniform connector H is called a (c, α)-concentrator if the following condition holds: for any subset X of A whose density is less than α the following inequality is satisfied:

$$\frac{c(X_H)}{c(X)} > c.$$

1.3. Formal Statement of the Problem. For any positive numbers c and α such that $c > 1$, $\alpha < 1$, and $c\alpha < 1$ it is required to construct an infinite sequence $H_1, \ldots, H_i, \ldots$ of uniform connectors such that the following conditions are satisfied:

1) for any i $(1 \le i < \infty)$ H_i is a (c, α)-concentrator;

2) as i tends to infinity the capacity of H_i tends to infinity, i.e., $\lim_{i\to\infty} c(H_i) = \infty$;

3) the densities of all the connectors H_i are bounded in the aggregate by a constant D, i.e., for any i $(1 \le i < \infty)$ the inequality $d(H_i) < D$ holds, where D is independent of i.

Despite the comparative ease of proving the existence of the required sequence of uniform connectors on the basis of probabilistic considerations, the explicit construction of such a sequence proves troublesome.

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1.4. Problems similar to that stated above arise in a number of theoretical switching and cooling problems. One might be interested, for example (see Pinsker [1]), in constructing a sequence of connectors $H_1, \ldots, H_i, \ldots$ for which the number of inputs is proportional with a certain constant k to the number of outputs and which are (c, α)-concentrators [for nonuniform connectors the concept of the (c, α)-concentrator is defined the same as for uniform connectors]. It is essential, of course, to demand observance of the inequality $c\alpha < 1/k$. We show how to solve this problem for integer-valued k and $\alpha < 1/k$ once the problem stated in subsection 1.3 has been solved. Thus, inasmuch as the number of inputs, i.e., $c(A)$, is proportional to the number of outputs, i.e., $c(B)$, by an integer-valued constant k, we can partition A into k disjoint subsets $A_1, \ldots, A_k$ such that the power of each subset is equal to the power of B, i.e., $c(A_i) = c(B)$ for any $1 \le i \le k$. Then for any i $(1 \le i \le k)$ we construct a $(c, \alpha k)$-concentrator for which A_i is the set of inputs and B is the set of outputs (using the solution of the problem from subsection 1.3). We then amalgamate all of the resulting k connectors into one. To show that the connector H so obtained is a (c, α)-concentrator it is sufficient to note that for any $X \subset A$ the inequality $c(X \cap A_i)/c(A_i) > k(c(X)/c(A))$ holds for at least one i and then to make use of the fact that H consists of $(c, \alpha k)$-concentrators.

For practical problems in which the problem stated in 1.3 above is encountered it is important that the constant D be sufficiently small. The constructions presented below clearly satisfy this condition. Unfortunately, the author has not been able to prove the fact. We can only prove the existence of D.

2. The Construction; Formulation of Propositions Implying that the Resulting Construction is the One Required

2.1. Notation. For any positive integer m we denote by Z_m the ring of residues modulo m. For any X and Y we denote by $X \times Y$ the direct product of X and Y.

2.2. The Construction. Let m be any positive integer. We put $A_m = Z_m \times Z_m$ and $B_m = Z_m \times Z_m$. Thus, the elements of both sets A_m and B_m are pairs (x, y), where $x \in Z_m$ and $y \in Z_m$. We now synthesize the connector H_m as follows: every element (x, y) of A_m is connected to the following five elements of B_m: 1) (x, y); 2) $(x + 1, y)$; 3) $(x, y + 1)$; 4) $(x, x + y)$; 5) $(-y, x)$.

2.3. THEOREM. A positive constant d independent of m exists such that for any positive integer m the connector H_m constructed in subsection 2.2 is a $(1 + d(1-\alpha), \alpha)$-concentrator for any α satisfying the inequality $0 < \alpha < 1$.

The proof of the theorem will be given in Sec. 3.

2.4. Let H_1 and H_2 be two uniform connectors with the same capacity, i.e., the same number of inputs (and outputs) for H_1 and H_2. Then a one-to-one correspondence can be set up between the outputs of H_1 and the inputs of H_2. In other words, we can assume that the outputs of H_1 are the inputs of H_2. We now define the product $H_2 \circ H_1$ of connectors H_2 and H_1 as follows:

1) the inputs of $H_2 \circ H_1$ are the inputs of H_1, and the outputs of the product are the outputs of H_2;

2) an input x of $H_2 \circ H_1$ is connected to an output of $H_2 \circ H_1$ if and only if there is an output z of H_1 (being also an input of H_2) that is connected to both x and to y.

2.5. The following is an immediate consequence of the definition of the product of connectors and the definition of a (c, α)-concentrator:

LEMMA. If H_1 is a (c_1, α)-concentrator and H_2 is a $(c_2, c_1\alpha)$-concentrator, then $H_2 \circ H_1$ is a $(c_1 c_2, \alpha)$-concentrator.

Remark. Strictly speaking, the definition of the product of connectors H_2 and H_1 depends on the mode by which the outputs of H_1 are associated with the inputs of H_2. Nonetheless, the lemma stated above is true independently of that mode.

2.6. For any connector H and any positive integer k we denote by H^k the product $\overbrace{H \circ \ldots \circ H}$. It then follows at once from the definition of the connector H_m and the definition of the product connector that for any positive integers m and k every input of H_m^k is connected at most to 5^k outputs of that connector. The following is true.

LEMMA. For any positive integers m and k the density of H_m^k is not greater than 5^k.

2.7. The following is a consequence of Theorem 2.3 and Lemma 2.5.

LEMMA. Let c and α be positive numbers such that $\alpha < 1$, $c > 1$, and $c\alpha < 1$. Then there is a k such that for any m the connector H_m^k is a (c, α)-concentrator.

2.8. Remark. Lemmas 2.6 and 2.7 solve the problem stated in subsection 1.3.

3. Proof of Theorem 2.3

3.1. There are two parts to the present section. In the first part (subsections 3.2 through 3.8) Theorem 2.3 is reduced to Lemma 3.7. In the second part (subsections 3.9 through 3.23) Lemma 3.7 is proved by the methods of group representation theory.

3.2. We retain the notation of 2.2. We now define transformations T_1, T_2, T_3, and T_4 of the set $A_m = Z_m \times Z_m$ by the formulas

$$T_1(x, y) = (x+1, y),$$
$$T_2(x, y) = (x, y+1),$$
$$T_3(x, y) = (x+y, y),$$
$$T_4(x, y) = (-y, x),$$

in which $x \in Z_m$ and $y \in Z_m$.

3.3. As before, for any X we denote by $c(X)$ the number of elements in X. Also, for any subset X of A and any i $(1 \le i \le 4)$ we denote by $T_i(x)$ the image of X under the transformation T_i, i.e., $T_i(X) = \bigcup_{x \in X} T_i(x)$.

LEMMA. A positive constant d exists such that for any positive integer m and any α satisfying the inequality $0 < \alpha < 1$ the following is true: if $X \subset A_m = Z_m \times Z_m$ and $c(X) < \alpha c(A_m) = \alpha m^2$, then for a certain i $(1 \le i \le 4)$ the inequality $c(T_i(X) \cup X) > (1 + d(1-\alpha))c(X)$ holds.

It is apparent from the definitions of H_m and the transformations T_1, T_2, T_3, and T_4 that Theorem 2.3 is an immediate consequence of Lemma 3.3 stated above.

3.4. In the space of complex-valued functions on A_m we define the scalar product by the formula

$$(f_1, f_2) = \sum_{a \in A_m} f_1(a)\overline{f_2(a)},$$

where f_1 and f_2 are functions and the bar over $f_2(a)$ indicates the complex conjugate. It is readily seen that the space of complex-valued functions on A_m with the scalar product defined above is a finite-dimensional Euclidean space, which we denote by $L^2(A_m)$. We equip $L^2(A_m)$ with a norm, namely for $f \in L^2(A_m)$ we say that $\|f\| = \sqrt{(f, f)}$.

3.5. Let T be an arbitrary one-to-one mapping of the set A_m onto itself. We associate with T a linear operator $\tilde{T}$ in the space $L^2(A_m)$ as follows: if $f \in L^2(A_m)$, then for any $a \in A_m$

$$\tilde{T}f(a) = f(T^{-1}a),$$

where T^{-1} is the inverse of the mapping T. It follows from the one-to-one character of T and the definition of the scalar product in $L^2(A_m)$ that $\tilde{T}$ is a unitary operator in $L^2(A_m)$ [for any f_1, $f_2 \in L^2(A_m)$ the equality $(\tilde{T}f_1, \tilde{T}f_2) = (f_1, f_2)$ holds].

3.6. We denote by $S(A_m)$ the subspace of $L^2(A_m)$ formed by all functions f for which $\sum_{a \in A_m} f(a) = 0$. In other words, $S(A_m)$ if the orthogonal complement to a one-dimensional space of constants (functions constant on the entire set A_m). It is readily seen that $S(A_m)$ is a subspace of $L^2(A_m)$ of codimension 1, i.e., $\dim L^2(A_m) - \dim S(A_m) = 1$. We also note that for any one-to-one mapping T of A_m into itself the space $S(A_m)$ is invariant under T.

3.7. Let T_1, T_2, T_3, and T_4 have the same meaning as in 3.2 and 3.3.

LEMMA. A positive constant d independent of m exists such that for any $f \in S(A_m)$ there is an i $(1 \le i \le 4)$ such that

$$\frac{(\tilde{T}_i f, f)}{(f, f)} < 1 - d.$$

3.8. The proof of Lemma 3.7 is given below. We show for now that it leads to Lemma 3.3.

Let X by any* subset of the set A_m. We define the function f_X on A_m as follows:

$$f_X(a)=c(A_m)-c(X) \quad \text{if} \quad a\in X,$$
$$f_X(a)=-c(X) \quad \text{if} \quad a\notin X,$$

where c(A) and c(X), as before, denote the numbers of elements in A and X. It is at once obvious from the definition of f_X that $f_X\in S(A_m)$. Therefore, according to Lemma 3.7, there is an i $(1 \le i \le 4)$ satisfying the inequality

$$\frac{(\tilde{T}_i f_X, f_X)}{(f_X, f_X)}<1-d. \tag{1}$$

From the definition of the function f_X and the operator $\tilde{T}$ we infer the following equalities by direct computations (which we omit for their triviality):

$$(f_X, f_X)=c(X)c(A_m)(c(A_m)-c(X)) \tag{2}$$

and

$$(T_i f_X, f_X)=c(A_m)[c(X\cap T_iX)c(A_m)-c^2(X)]. \tag{3}$$

It follows from (1)-(3) that

$$\frac{c(X\cap T_iX)c(A_m)-c^2(X)}{c(X)(c(A_m)-c(X))}<1-d. \tag{4}$$

Since $c(X) > 0$ and $c(A_m-c(X) > 0$ (because $X \ne \phi$ and $X \ne A_m$), we infer from (4), denoting $c(X)/c(A_m)$ by α_X, that

$$\begin{aligned} c(X\cap T_iX) &< \frac{(1-d)c(X)(c(A_m)-c(X))+c^2(X)}{c(A_m)} \\ &= \frac{c(X)c(A_m)-c^2(X)-dc(X)c(A_m)+dc^2(X)+c^2(X)}{c(A_m)} \\ &= c(X)-dc(X)+c(X)d\alpha_X = c(X)(1-d+d\alpha_X). \end{aligned} \tag{5}$$

Inasmuch as T_i is a one-to-one mapping, we have $c(T_iX) = c(X)$. Therefore,

$$c(X\cup T_iX)=c(X)+c(T_iX)-c(X\cap T_iX)=2c(X)-c(X\cap T_iX). \tag{6}$$

It follows from (5) and (6) that

$$c(X\cup T_iX)>c(X):[1+d(1-\alpha_X)]. \tag{7}$$

Lemma 3.3 follows at once from inequality (7).

3.9. This and the ensuing subsections are devoted to the proof of Lemma 3.7. The proof is based on the methods of group representation theory or, more precisely, on methods associated with the property of T investigated by Kazhdan (see [2, 3]). We shall not give any definitions from the actual theory of group representations, because to do so would take up too much space. We merely point out that all the required definitions may be found in [2-4].

3.10. Notation. We denote by H the group of all unimodular† matrices of the form $\begin{pmatrix}1&0&u\\0&1&v\\0&0&1\end{pmatrix}$, and by S the group of matrices of the form $\begin{pmatrix}a&b&u\\c&d&v\\0&0&1\end{pmatrix}$, where a, b, c, d, u, and v run through the real number field. We denote by $H_Z \subset H(S_Z \subset S)$ the subgroup consisting of all matrices belonging to H(S) with integer-valued coefficients.

3.11. For any integer a and any element x of the ring Z_m we denote by ax the product of a and x; if $a > 0$, then ax is defined as $\overbrace{x+\ldots+x}^{a\text{-fold}}$; if $a < 0$, then $ax = -[(-a)x]$; and if $a = 0$, then $ax = 0$.

Let m be an arbitrary positive integer. We now associate with each element $g = \begin{pmatrix}a&b&u\\c&d&v\\0&0&1\end{pmatrix}$ of the group

*We exclude cases in which X is the empty set and in which $X = A_m$.
†A matrix is called unimodular if its determinant is equal to 1.

H_Z a transformation $T_m(g)$ of the set $A_m = Z_m \times Z_m$ as follows: if $(x, y) \in Z_m \times Z_m$, then $T_m(g)(x, y) = (ax + by + u, cx + dy + v)$. It is verified by direction computation that the following equality holds for any $g_1, g_2 \in H_Z$:

$$T_m(g_1g_2) = T_m(g_1)T_m(g_2). \tag{8}$$

3.12. Inasmuch as H_Z is a group, it follows from (8) that the transformation $T_m(g)$ is one-to-one for any $g \in H_Z$. Consequently, it is possible to associate with $T_m(g)$ (see subsection 3.5) a linear unitary operator in $L^2(A_m)$. We denote that operator by $\tilde{T}_m(g)$. It follows from (8) and the definition of the operators $\tilde{T}_m(g)$ that the following equality holds for any $g_1, g_2 \in H_Z$:

$$\tilde{T}_m(g_1g_2) = \tilde{T}_m(g_1)\tilde{T}_m(g_2) \tag{9}$$

(it is necessary here to rely on the easily verified fact that the equality $\tilde{T}_1 \circ \tilde{T}_2 = \widetilde{T_1 \circ T_2}$ is true for any two one-to-one mappings T_1 and T_2).

It follows from Eq. (9) and the unitarity of the operators $\tilde{T}_m(g)$ that $\tilde{T}_m$ is a unitary representation of the group H_Z.

3.13. We pick the following four elements in H_Z:

$$g_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_4 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We infer the following equality directly from the definition of $T_m(g)$ (subsection 3.11) and the definitions of T_1, T_2, T_3, and T_4 (subsection 3.2):

$$T_m(g_i) = T_i \tag{10}$$

for any i $(1 \le i \le 4)$. From Eq. (10), the definition of $\tilde{T}_i$ $(1 \le i \le 4)$ (subsection 3.5), and the definition of the representations $\tilde{T}_m$ (subsection 3.12) we deduce the equality

$$\tilde{T}_m(g_i) = \tilde{T}_i \tag{11}$$

for any i $(1 \le i \le 4)$.

<u>3.14. Definition.</u> We call a unitary representation T of H_Z in the space X essentially nontrivial on S_Z if X does not contain a nonnull vector invariant under $T(S_Z)$, i.e., if for any $x \in X$ $(x \ne 0)$ there is an element $s \in S_Z$ such that $T(s)x \ne x$.

<u>Definition.</u> We call a unitary representation T of H in the space X essentially nontrivial on S if X does not contain a nonnull vector invariant under $T(S)$.

<u>3.15. LEMMA.</u> A positive constant d exists such that if T is any unitary representation of H_Z in the Hilbert space L, essentially nontrivial on S_Z, and x is any nonnull element of L, then there is an i $(1 \le i \le 4)$ such that

$$\frac{(T(g_i)x, x)}{(x, x)} < 1 - d,$$

where x, y denotes the scalar product in L.

3.16. The proof of Lemma 3.15 will be given below. We show for now that Lemma 3.7 follows from Lemma 3.15.

As we mentioned in subsection 3.6, for any one-to-one mapping of the set T into itself the space $S(A_m)$ is invariant under $\tilde{T}$. Consequently, $S(A_m)$ is an invariant subspace for the representation $\tilde{T}_m$. We denote by T'_m the restriction of $\tilde{T}_m$ onto the subspace $S(A_m)$. Inasmuch as $\tilde{T}_m$ is a unitary representation (see subsection 3.12), T'_m is also a unitary representation. We now show that T'_m is essentially nontrivial on S_Z. Indeed, if T'_m were not essentially nontrivial on S_Z, it would be readily apparent that $S(A_m)$ contained a nonnull function f invariant under $T'_m(S_Z)$. But then, by the definition of the representation T'_m, the following would hold for any $x, y, u, v \in Z_m$:

$$f(x+u, y+v) = f(x, y). \tag{12}$$

Since x, y, $v \in Z_m$ are arbitrary, it follows from (12) that f is constant on A_m. But $f \in S(A_m)$, i.e., $\sum_{a \in A_m} f(a) = 0$. Consequently, $f \equiv 0$. We have thus shown that T_m is essentially nontrivial on S_Z. It follows from the latter assertion, the unitarity of T_m, and Lemma 3.15 that for any nonnull function $f \in S(A_m)$ there is an i $(1 \le i \le 4)$ such that

$$\frac{(T_m'(g_i)f, f)}{f, f} < 1 - d. \tag{13}$$

Inasmuch as $T_m(g_i)f = \tilde{T}_m(g_i)f$ for any i $(1 \le i \le 4)$ and $f \in S(A_m)$ (by the definition of T_m), Lemma 3.7 therefore follows from (13) and (11).

3.17. Let G be a locally compact group. We denote by $\tilde{G}$ the set of unitary representations of G. Let $T \in \tilde{G}$, $\varepsilon > 0$, let K be a compactum in G, and let X be a vector in T representation space. We denote by $V(X, K, \varepsilon)$ the set of all $T' \in \tilde{G}$ such that T representation space contains a vector Y such that the inequality $((T'(g)Y, Y) - (T(g)X, X)) < \varepsilon$ holds for any $g \in K$. We now define in G a topology for which the sets $V(X, K, \varepsilon)$ form a basis of neighborhoods of T.

3.18. LEMMA. For the unit representation of H_Z there is a neighborhood U such that any unitary representation of H_Z essentially nontrivial on S_Z is not in U.

3.19. The proof of Lemma 3.18 will be given below. We show for now that Lemma 3.15 reduces to Lemma 3.18. The reduction is based on the following lemma, which will be proved in subsection 3.20.

LEMMA. The elements g_1, g_2, g_3, and g_4 in subsection 3.13 are generators of the group H_Z.

Reduction of Lemma 3.15 to Lemma 3.18. Inasmuch as H_Z is a discrete group, any compactum $K \subset H_Z$ is a finite subset. We therefore infer from Lemma 3.18 the existence of an $\varepsilon > 0$ and of a finite subset $K \subset H_Z$ such that for any unitary representation T' of H_Z essentially nontrivial on S_Z the following is true: if s is an element of T' representation space, then for any $h \in K$ the following inequality holds:

$$\frac{(T'(h)x, x)}{(x, x)} < 1 - \varepsilon. \tag{14}$$

It follows from the unitarity of the operator $T'(h)$ and the Pythagorean theorem that inequality (14) is equivalent to the inequality

$$\frac{\|x - T'(h)x\|}{\|x\|} > \sqrt{2\varepsilon - \varepsilon^2}. \tag{15}$$

We give the following simple proposition without proof:

Proposition. Let A_1 and A_2 be two unitary operators acting in the same Hilbert space L, and let $x \in L$. Then $\|x - T_1T_2x\| \le \|x - T_1x\| + \|x - T_2x\|$ (the unitarity of T_1 and T_2 is essential).

Inasmuch as K is a finite subset in H_Z and as g_1, g_2, g_3, and g_4 are generators of H_Z, a positive integer m exists such that any element $h \in K$ is representable in the form of a word from g_1, g_2, g_3, g_4 whose length is not greater than* m. It follows from this result, inequality (15), and the above-stated proposition that for any x in T' representation space there is an i such that

$$\frac{\|T'(g_i)x - x\|}{\|x\|} < \frac{\sqrt{2\varepsilon - \varepsilon^2}}{m}. \tag{16}$$

It follows from the unitarity of $T'(g_i)$ and the Pythagorean theorem that inequality (16) is equivalent to the inequality

$$\frac{(T'(g_i)x, x)}{(x, x)} < \sqrt{1 - \frac{2\varepsilon - \varepsilon^2}{m^2}}. \tag{17}$$

We have now merely to put d equal to $1 - \sqrt{1 - (2\varepsilon - \varepsilon^2)/m^2}$ and to note that Lemma 3.15 follows from (17).

3.20. Proof of Lemma 3.19. We denote by A_Z the group consisting of integer-valued unimodular matrices of the form

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*In other words, $h = g_{i_1}^{m_1} g_{i_2}^{m_2} \ldots g_{i_l}^{m_l}$, where $1 \le i_j \le 4$ and $\sum_{j=1}^{l} |m_j| < m$ (m_j can be negative).

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We denote by f a mapping of H_Z into A_Z taking the matrix $\begin{pmatrix} & & \\ c & d & v \\ & & \end{pmatrix}$ into the matrix $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$. It is verified by direct computations that the following equality holds for any $g \in H_Z$:

$$g = f(g) \cdot s, \tag{18}$$

where $s \in S_Z$. We now inject the following two remarks:

Remark 1. It is readily inferred from [5], Sec. 2 (see the remark in [5] following Theorem 1 in Sec. 2) that the group A_Z is generated by elements g_3 and g_4 (see subsection 3.13 for the definition of g_3 and g_4).

Remark 2. It is directly verifiable that the following equality holds for any integers m and n:

$$\begin{pmatrix} 1 & 0 & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} = g_1^m g_2^n, \tag{19}$$

where g_1 and g_2 are given in subsection 3.13. It follows from (19) that g_1 and g_2 are generators of the group S_Z.

Lemma 3.19 now follows from Remarks 1 and 2 in conjunction with Eq. (18).

3.21. Let μ be a right-invariant Haar measure* on the group H. Then μ naturally induces on the factor space H/H_Z a measure which we denote by $\tilde{\mu}$.

LEMMA. The measure of the entire factor space H/H_Z under measure $\tilde{\mu}$ is finite, i.e., $\tilde{\mu}(H/H_Z) < \infty$.

Proof. We note first of all that it is necessary to review the treatise [6] with regard to the definitions used in the proof. We then inject the following remarks:

Remark 1. It follows at once from the definition of an arithmetic subgroup of an algebraic group that H_Z is an arithmetic subgroup of H.

Remark 2. We denote by A the subgroup of H consisting of real unimodular matrices of the form $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$. It is readily seen that H is the semidirect product of A and S (see subsection 3.10 for the definition of S). On the other hand: 1) since A is readily perceived to be isomorphic to SL(2, R) (the group of real second-order unimodular matrices) and SL(2, R) is, of course (see, e.g., [8]) a simple group, A is therefore a simple group; 2) S is a unipotent group. It follows from the last two propositions that H does not have nontrivial rational characters.

The following lemma is a consequence of Remarks 1 and 2 and Theorem 9.4 in [6].

3.22. LEMMA. For the unit representation of H there is a neighborhood V such that any unitary representation of H nontrivial on S is not in V.

Proof. It follows directly from Lemmas 2 and 3 of [2] that for the unit representation of H there is a neighborhood V such that any irreducible unitary representation of H nontrivial on S is not in V. It follows from this result, a theorem (see [9]) on the decomposition of any unitary representation into a direct integral of irreducible representations, and the definition of a representation of H nontrivial on S (see subsection 3.14) that the lemma is true.

3.23. Proof of Lemma 3.18. As before, for any locally compact group G we denote by $\tilde{G}$ the space of unitary representations with topology described in 3.17 above.

Consider the mapping $\varphi: \tilde{H}_Z \to \tilde{H}$, which is an induction in the sense of Frobenius (or, in the terminology of [3], an induction in the sense of Mackey). Lemma 3.18 follows directly from two properties of this mapping:

*See Pontryagin [7] for the definition of Haar measure.

a) φ is continuous;

b) if $p \in \tilde{H}_Z$ is essentially nontrivial on S_Z, then $\varphi(p)$ is essentially nontrivial on S.

Property a) is easily derived from Lemma 3.19 and the results of Fell [10] (see also [2, 3]).

Property b) follows at once from the definition of φ.

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