3 Universal Algebra for Propositional Logic

The material discussed herein can be found in greater detail in any standard text on universal algebra such as [3]. When discussing the algebraic theory of a class of algebras, it is convenient to specify which kinds of basic operations we are considering on each algebra and to know which operation corresponds to which from algebra to algebra. To this end, we define a *type* of algebras to be a set τ of operation symbols of specified arity. For example, the type of lattices is a set consisting of two symbols, say $\{\vee, \wedge\}$, that are both binary. It is important here to think of $\vee$ and $\wedge$ as symbols and not as operations on a set. We will denote the class of all abstract algebras of type τ by $\mathfrak{A}_\tau$. Given a class K of algebras of some fixed type τ and a non-empty set X, the free K-algebra, freely generated by X, denoted by $\mathcal{F}_K(X)$, is the algebra (not necessarily in K) satisfying the property that there is a mapping $X \longrightarrow \mathcal{F}_K(X)$ so that given any mapping $X \longrightarrow A$, with $A \in \mathfrak{A}_\tau$, there is a unique homomorphic extension to $\mathcal{F}_K(X)$. That is, there is a unique homomorphism $\mathcal{F}_K(X) \longrightarrow A$ so that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{F}_K(X) \arrow[dl] \\ A & \end{tikzcd}$$

commutes. In our setting, K will contain arbitrarily large algebras, and in this case it follows from the definition above that the map $X \longrightarrow \mathcal{F}_K(X)$ is an injection and that $\mathcal{F}_K(X)$ is generated by the image of X. It can also be shown that $\mathcal{F}_K(X)$ exists for any class K and that it is unique up to isomorphism. Given a set X, the free algebra generated by X in $\mathfrak{A}_\tau$ is called the *absolutely free algebra* of type τ over the set X. This algebra is also the term algebra of type τ over X. We denote it by $\mathcal{T}_\tau(X)$ and call its elements *terms*. To illustrate what the terms look like, we describe them inductively within the set of all strings over the alphabet $X \cup \tau$.

1. If $x \in X$ then x is a term;

2. If t_1, $t_2, ..., t_n$ are terms and $f \in \tau$ and f is n-ary, then the string $ft_1t_2 \cdots t_n$ obtained by concatenation is again a term.

The set of terms is the smallest set of strings satisfying both of these properties. Essentially, they can be thought of as 'polynomials' of the given type in the variables in X. This inductive definition may seem similar to the one for well-formed formulas. And in fact this is a generalization of that definition. If τ is the type of our algebra of truth values, then $\mathcal{T}_\tau(X)$ is exactly the algebra $\mathcal{P}$ of well-formed formulas described above with the minor typographical change of writing $(\alpha \wedge \beta)$ as $\wedge\alpha\beta$ (this saves having to include a lot of parentheses and allows for the definition to apply to operations of greater arity than two).